Questions collected at the workshop
Frontiers in Serre's Modularity Conjecture
June 2015

(1) (by Mladen Dimitrov) Consider geometric Hilbert modular forms *à la Wiles* over a totally real field F (and of parallel weights). Fix an ideal $\mathfrak{n} \lhd \mathcal{O}_F$ (the level) and a prime p not dividing $\mathfrak{n}$. We know that there exists $k_0 = k_0(\mathfrak{n}) \in \mathbb{N}$ such that for all $k \geq k_0$ one has

$$\overline{\mathbb{F}}_p \otimes M_k(\mathfrak{n}; \overline{\mathbb{Z}}_p) \cong M_k(\mathfrak{n}; \overline{\mathbb{F}}_p).$$

Question: Make k_0 explicit.

(2) (by Mladen Dimitrov, Gabor Wiese) Consider geometric Hilbert modular forms *à la Wiles* over a totally real field F.

Question: Find a Hecke eigenform $f \in S_1(\mathfrak{n}; \overline{\mathbb{F}}_p)$ which is not the reduction of any holomorphic parallel weight one eigenform.

(3) (Folklore, asked by David Savitt) Let $K/\mathbb{Q}_p$ be a finite extension and $n \in \mathbb{N}$. Given a local Galois representation

$$\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p).$$

Question: Does there exist a lift $r : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ of $\overline{r}$

- which is crystalline?
- which is potentially crystalline?
- which is de Rham?
- at all?

(4) (by Haluk Sengun) Consider Serre type conjectures for $\mathrm{GL}(2)$ beyond totally real fields. There exist implementations that produce systems of Hecke eigenvalues for $\mathrm{GL}(2)$ over the following fields K

- imaginary quadratic,
- more general fields having a unique complex place,
- of signature $(1, 1)$
- of signature $(2, 0)$ (quartic with two complex places).

Question: Is there any demand to see examples of systems of eigenvalues (conjecturally) corresponding to Galois representations

$$\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

having special properties like having non-solvable image or being unramified above p?

(5) (by Florian Herzig, David Savitt) Consider a compact unitary group $U(3)$ over a CM field K, and let p be a prime such that all places above it split in K/K^+. In particular, consider the following situations:

- $K^+ = \mathbb{Q}$,
- $K^+ \neq \mathbb{Q}$ but p is unramified in K^+,
- p ramifies in K^+.

Questions: Compute (non-Eisenstein) mod p eigensystems in various weights. Test/come up with Serre weight recipes for mod p Galois representations. Do the same with $U(4)$.

(6) (by Jack Thorne) Consider the completed Hida cohomology

$$H_{\mathfrak{m}} = \varprojlim_{n \geq 1} H_1(\Gamma_1(\mathfrak{n}p^n), \mathbb{Z}_p)^{\mathrm{ord}}_{\mathfrak{m}}.$$

It carries a natural structure of module over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T(\mathbb{Z}_p)(p)]]$, where $T = \mathrm{Res}^K_{\mathbb{Q}} \mathbb{G}_m$. In fact, our hypothesis implies that there is an isomorphism $H_{\mathfrak{m}} \cong \Lambda/(a)$ of Λ-modules for some non-zero $a \in \Lambda$.

Question: Hida's control theorem implies that the structure of certain quotients of $H_{\mathfrak{m}}$ can be calculated in terms of classical cohomology groups. What can one say about the module $H_{\mathfrak{m}}$, in specific cases, by explicit calculation? For example, is it p-torsion free?

(7) (by Jacques Tilouine) This is a special case of a 'Serre conjecture' for Siegel modular forms. Let

$$\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{F}}_p)$$

be an irreducible odd ($\nu \circ \overline{\rho}(c) = -1$ with ν the multiplier/similitude factor and c a complex conjugation) Galois representation. Assume that the restriction of $\overline{\rho}$ to an inertia group at p has the following shape:

$$\begin{pmatrix} \omega^{i_3} & * & * & * \\ 0 & \omega^{i_2} & * & * \\ 0 & 0 & \omega^{i_1} & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $0 < i_1 < i_2 < i_3 < p-1$ (i.e. $\overline{\rho}$ is *regular*). Also assume that $\overline{\rho}$ is *peu ramifié*.

Question: Do there exist a positive integer N having the same prime divisors as the prime-to-p Artin conductor of $\overline{\rho}$ and a genus 2 Siegel modular eigenform $f \in S_{i_2+1,i_1+1}(N)$ such that $\overline{\rho} \cong \overline{\rho_f}$?