Math Upgrade Week: Sets and Functions

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Exercise sheet 1 with solutions

1. *Sets via enumeration.*

   (a) Define via enumeration the set of even integers between 3 and 9.
   Solution: $\{4, 6, 8\}$

   (b) Define via enumeration the set the elements of which are the subsets of $\{1, 2, 3\}$ having 2 elements.
   Solution: $\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \}$

2. *Sets via predicates.*

   (a) Define the following set via a predicate: $\{2, 4, 8, 16, 32, 64, 128, ...\}$.
   Solution: $\{x \in \mathbb{N} \mid \exists\, n \in \mathbb{N} \setminus \{0\} : x = 2^n\}$.
   A slightly more sloppy solution is: $\{2^n \mid n \in \mathbb{N} \setminus \{0\}\}$.

   (b) Define the following set via a predicate: $\{-2, -1, 0, 1, 2\}$.
   Solution: $\{x \in \mathbb{N} \mid -2 \leq x \leq 2\}$.

   (c) Define the emptyset via a predicate.
   Solution: $\{x \in \mathbb{N} \mid 1 = 2\}$; of course, there are plenty of variations of this (any assertion that is always false works).

   (d) Define a set of your choice by a predicate and also enumerate its elements.
   Solution: Your choice, not mine.

3. *Membership*
   Fill in correctly either $\in$ or $\notin$:

   Solution:

   (a) $-3 \in \mathbb{Z}$, $-3 \notin \mathbb{N}$, $-3 \in \mathbb{Q}$

   (b) Let $E = \{n \in \mathbb{N} \mid 3 \text{ divides } n\}$:
   $3 \in E$, $2 \notin E$, $-3 \notin E$, $\frac{1}{3} \notin E$

   (c) $\emptyset \in \{\emptyset, \{\emptyset\}\}$.

   (d) $4 \notin \{\{4\}\}$.

4. *Equality of sets*
Which of the following sets are equal?
$A := \{1, 5, 9 - 2, 3\} = \{1, 3, 5, 7\}$,
$B := \{x \in \mathbb{N} \mid 1 \leq x < 7 \wedge (2 \text{ does not divide } x)\} = \{1, 3, 5\}$,
$C := (\{x \in \mathbb{R} \mid x \geq 1\} \setminus \{2, 4, 6\}) \cap \{x \in \mathbb{R} \mid x < 8\} = [1, 8) \setminus \{2, 4, 6\}$,
$D := \{x \in \mathbb{N} \mid 1 \leq x \leq 8 \wedge (2 \text{ does not divide } x)\} = \{1, 3, 5, 7\}$,
$E := \{x \in \mathbb{R} \mid 1 \leq x \leq 8 \wedge x \notin \{n \in \mathbb{N} \mid n \text{ is even }\}\} = [1, 8] \setminus \{2, 4, 6, 8\}$.

Solution: Thus $A = D$ and $C = E$ and these are the only equalities.

5. *Inclusion and equality of sets*
Consider the following sets:

$$A = \{1, 2, 5\},\ B = \{\{1, 2\}, 5\},\ C = \{\{1, 2, 5\}\},\ D = \{\emptyset, 1, 2, 5\},$$
$$E = \{5, 1, 2\},\ F = \{\{1, 2\}, \{5\}\},\ G = \{\{1, 2\}, \{5\}, 5\},\ H = \{5, \{1\}, \{2\}\}.$$

(a) Which sets are related by equality and which by inclusion?
Solution: $A = E$, $A \subseteq D$, $E \subseteq D$, $B \subseteq G$, $F \subseteq G$

(b) Compute the cardinality of each of these sets.
Solution: $\#A = 3$, $\#B = 2$, $\#C = 1$, $\#D = 4$, $\#E = 3$, $\#F = 2$, $\#G = 3$, $\#H = 3$.

(c) Determine $A \cap B$, $G \cup H$ and $E \setminus G$.
Solution: $A \cap B = \{5\}$, $G \cup H = \{5, \{1\}, \{2\}, \{5\}, \{1, 2\}\}$, $E \setminus G = \{1, 2\}$.

6. *Complements*
Consider the following four subsets of $\mathbb{N}$:

$$I = \{1, 2, 3, 4, 5, 6, 7\},\ J = \{1, 3, 5, 7\},\ K = \{2, 4, 6\}.$$

(a) Determine $I \setminus J$ and $I \setminus K$ (i.e. the complements of J and K in I).
Solution: $I \setminus J = \{2, 4, 6\} = K$, $I \setminus K = \{1, 3, 5, 7\} = J$

(b) The symmetric difference of two sets A and B, denoted by $A\Delta B$, is the set of elements that are either in A or in B, but not in $A \cap B$. Determine $I\Delta J$ and $J\Delta K$.
Solution: $I\Delta J = K$, $J\Delta K = I$.

7. *Intersection, union, complement, etc.*
Let
$$A = \{n \in \mathbb{N} \mid n \text{ is divisible by } 2\ \} \text{ and } B = \{n \in \mathbb{N} \mid n \text{ is divisible by } 5\ \}.$$

(a) $A \cap B = \{n \mid n \in \mathbb{N}, n \text{ is divisible by } 10\ \}$.
Reason: an integer n is divisible by 10 if and only if it is divisible by 2 and 5.

(b) $A \cup B = \{n \mid n \in \mathbb{N}, n \text{ is divisible by } 5 \text{ or by } 2\ \}$.

(c) $B \setminus A = \{n \mid n \in \mathbb{N}, n \text{ is divisible by } 5 \text{ and not divisible by } 10\ \}$.

(d) $A \setminus B = \{n \mid n \in \mathbb{N}, n \text{ is divisible by } 2 \text{ and not divisible by } 5\ \}$.

(e) $[12, 27] \cap A = \{12, 14, 16, 18, 20, 22, 24, 26\}$, hence its cardinality is 8.
$[12, 27] \cap B = \{15, 20, 25\}$, hence its cardinality is 3.

8. *Power sets*

   (a) $\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$.

   (b) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ because $\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$ and $\mathcal{P}(\emptyset) = \{\emptyset\}$.

   (c) We have $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ and thus

$$\begin{aligned}\mathcal{P}(\mathcal{P}(\{1,2\})) = \{&\emptyset, \{\emptyset\}, \{\{1\}\}, \{\{2\}\}, \{\{1,2\}\},\\ &\{\emptyset,\{1\}\}, \{\emptyset,\{2\}\}, \{\emptyset,\{1,2\}\}, \{\{1\},\{2\}\}, \{\{1\},\{1,2\}\}, \{\{2\},\{1,2\}\},\\ &\{\emptyset,\{1\},\{2\}\}, \{\emptyset,\{1\},\{1,2\}\}, \{\emptyset,\{2\},\{1,2\}\}, \{\{1\},\{2\},\{1,2\}\}, \{\emptyset,\{1\},\{2\},\{1,2\}\}\}.\end{aligned}$$

9. *Proofs of equivalences involving sets.*
   Let A and B be sets. Prove:

   (a) $A \subseteq B \iff A = A \cap B \iff B = A \cup B$;
   Proof. There are many ways to prove this. Here is one. We start with the following equivalences:

$$A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B) \Leftrightarrow (x \in A \Leftrightarrow (x \in A \wedge x \in B)) \Leftrightarrow A = A \cap B.$$

   Now consider the following equivalences:

$$A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B) \Leftrightarrow (x \in B \Leftrightarrow (x \in A \vee x \in B)) \Leftrightarrow B = A \cup B.$$

   (b) $A \cap B = \emptyset \iff A \setminus B = A$.
   Proof.

$$A \cap B = \emptyset \Leftrightarrow (x \in A \Rightarrow x \notin B) \Leftrightarrow A \setminus B = \{x \mid x \in A \wedge x \notin B\} = \{x \mid x \in A\} = A.$$

10. *More proofs of equivalences involving sets.*
    Let E be a set and A, B subsets of E. Prove:

    (a) $A \cap B = \emptyset \iff B \subseteq E \setminus A \iff A \subseteq E \setminus B$;
    Proof. First we have the equivalences

$$A \cap B = \emptyset \Leftrightarrow \forall\, x \in E : (x \in A \Rightarrow x \notin B) \Leftrightarrow A \subseteq E \setminus B.$$

    For the rest, it suffices to take the contrapositive of the assertion $\forall\, x \in E : (x \in A \Rightarrow x \notin B)$, which is $\forall\, x \in E : (x \in B \Rightarrow x \notin A)$, and is equivalent to the inclusion $B \subseteq E \setminus A$.

    (b) $A \cup B = E \iff E \setminus A \subseteq B \iff E \setminus B \subseteq A$.
    These equivalences can be proved with similar arguments. It is also possible to apply the rules from the course for working with complements.