Math Prep Camp: Sets and Functions

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Exercise sheet 1 with solutions

1. *Membership (1)*
   Fill in correctly either $\in$ or $\notin$:

   (a) $-3 \notin \mathbb{N}$, $-3 \in \mathbb{Z}$, $-3 \in \mathbb{Q}$, $-3 \in \mathbb{R}$

   (b) $\frac{1}{2} \notin \mathbb{N}$, $\frac{1}{2} \notin \mathbb{Z}$, $\frac{1}{2} \in \mathbb{Q}$, $\frac{1}{2} \in \mathbb{R}$

2. *Sets via enumeration.*

   (a) Define via enumeration the set of even integers between 3 and 9. **Solution:** $\{4, 6, 8\}$.

   (b) Define via enumeration the set of all capital letters from A to G (in alphabetic order). **Solution:** $\{A, B, C, D, E, F, G\}$.

3. *Two important rules.*
   $A := \{a, d, b, a, d\} = D := \{b, a, d, d\} = G := \{d, a, b\}$
   $B := \{a, d, d, a\} = F := \{d, a, a, d\}$
   $C := \{a, d, c, a\}$, $E := \{a, b, c, d\}$

4. *Sets by properties.*

   (a) Define the following set via a property: $\{2, 4, 8, 16, 32, 64, 128, ...\}$.
   Solution: $\{x \in \mathbb{N} \mid \exists\, n \in \mathbb{N} \setminus \{0\} : x = 2^n\}$.

   (b) Define the following set via a property: $\{-2, -1, 0, 1, 2\}$.
   Solution: $\{x \in \mathbb{Z} \mid -2 \le x \le 2\}$.

   (c) Define the emptyset via a property.
   Solution: $\{x \in \mathbb{N} \mid 1 = 2\}$; of course, there are plenty of variations of this (any assertion that is false works).

5. *Subsets and equality of sets*
   If possible, fill in correctly one of the symbols: $=, \subseteq, \supseteq$.

   (a) $\mathbb{N} \subseteq \mathbb{Z}$

   (b) $\{1, 5, 3, 4\} \supseteq \{4, 1, 5\}$

   (c) $\{1, 5, 3\}$ _____ $\{4, 1, 5\}$: none of the symbols fits.

   (d) $\emptyset \subseteq \{0\}$

   (e) $\emptyset = \{n \in \mathbb{N} \mid n < 0\}$.

6. *Operations on sets*
   Let $A := \{a, d, g\}$, $B := \{e, b, f\}$, $C := \{a, b, c\}$. Compute:

   (a) $A \cup B = \{a, b, d, e, f, g\}$

   (b) $A \cup C = \{a, b, c, d, g\}$

   (c) $A \cup B \cup C = \{a, b, c, d, e, f, g\}$

   (d) $A \cap B = \emptyset$

   (e) $A \cap C = \{a\}$

   (f) $A \setminus C = \{d, g\}$

   (g) The union $A \sqcup B$ is disjoint.

7. *Cartesian product*
   Let $A := \{a, d, g\}$, $B := \{e, b, f\}$. Then
   $A \times B = \{(a, e), (a, b), (a, f), (d, e), (d, b), (d, f), (g, e), (g, b), (g, f)\}$.

8. *Power sets (1)*
   $\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.

9. *Membership (2)*
   Fill in correctly either $\in$ or $\notin$:

   Solution:

   (a) Let $E = \{n \in \mathbb{N} \mid 3 \text{ divides } n\}$:
       $3 \in E, 2 \notin E, -3 \notin E, \frac{1}{3} \notin E$

   (b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$.

   (c) $4 \notin \{\{4\}\}$.

10. *Equality of sets*
    Which of the following sets are equal?
    $A := \{1, 5, 9 - 2, 3\} = \{1, 3, 5, 7\}$,
    $B := \{x \in \mathbb{N} \mid 1 \leq x < 7 \wedge (2 \text{ does not divide } x)\} = \{1, 3, 5\}$,
    $C := (\{x \in \mathbb{R} \mid x \geq 1\} \setminus \{2, 4, 6\}) \cap \{x \in \mathbb{R} \mid x < 8\} = [1, 8) \setminus \{2, 4, 6\}$,
    $D := \{x \in \mathbb{N} \mid 1 \leq x \leq 8 \wedge (2 \text{ does not divide } x)\} = \{1, 3, 5, 7\}$,
    $E := \{x \in \mathbb{R} \mid 1 \leq x \leq 8 \wedge x \notin \{n \in \mathbb{N} \mid n \text{ is even }\}\} = [1, 8] \setminus \{2, 4, 6, 8\}$.

    Solution: Thus $A = D$ and $C = E$ and these are the only equalities.

11. *Subsets and equality of sets (2)*
    Consider the following sets:

    $$A = \{1, 2, 5\}, \ B = \{\{1, 2\}, 5\}, \ C = \{\{1, 2, 5\}\}, \ D = \{\emptyset, 1, 2, 5\},$$
    $$E = \{5, 1, 2\}, \ F = \{\{1, 2\}, \{5\}\}, \ G = \{\{1, 2\}, \{5\}, 5\}, \ H = \{5, \{1\}, \{2\}\}.$$

    (a) Which sets are related by equality and which by inclusion?
        Solution: $A = E, A \subseteq D, E \subseteq D, B \subseteq G, F \subseteq G$

    (b) Compute the cardinality of each of these sets.
        Solution: $\#A = 3, \#B = 2, \#C = 1, \#D = 4, \#E = 3, \#F = 2, \#G = 3, \#H = 3$.

    (c) Determine $A \cap B, G \cup H$ and $E \setminus G$.
        Solution: $A \cap B = \{5\}, G \cup H = \{5, \{1\}, \{2\}, \{5\}, \{1, 2\}\}, E \setminus G = \{1, 2\}$.

12. *Complements*

Consider the following four subsets of $\mathbb{N}$:

$$I = \{1, 2, 3, 4, 5, 6, 7\}, \ J = \{1, 3, 5, 7\}, \ K = \{2, 4, 6\}.$$

(a) Determine $I \setminus J$ and $I \setminus K$ (i.e. the complements of J and K in I).

Solution: $I \setminus J = \{2, 4, 6\} = K$, $I \setminus K = \{1, 3, 5, 7\} = J$

(b) The symmetric difference of two sets A and B, denoted by $A \triangle B$, is the set of elements that are either in A or in B, but not in $A \cap B$. Determine $I \triangle J$ and $J \triangle K$.

Solution: $I \triangle J = K$, $J \triangle K = I$.

13. *Intersection, union, complement, etc.*

Let

$$A = \{n \in \mathbb{N} \mid n \text{ is divisible by } 2\} \text{ and } B = \{n \in \mathbb{N} \mid n \text{ is divisible by } 5\}.$$

(a) $A \cap B = \{n \mid n \in \mathbb{N}, n \text{ is divisible by } 10\}$.

Reason: an integer n is divisible by 10 if and only if it is divisible by 2 and 5.

(b) $A \cup B = \{n \mid n \in \mathbb{N}, n \text{ is divisible by 5 or by 2}\}$.

(c) $B \setminus A = \{n \mid n \in \mathbb{N}, n \text{ is divisible by 5 and not divisible by 10}\}$.

(d) $A \setminus B = \{n \mid n \in \mathbb{N}, n \text{ is divisible by 2 and not divisible by 5}\}$.

(e) $[12, 27] \cap A = \{12, 14, 16, 18, 20, 22, 24, 26\}$, hence its cardinality is 8.

$[12, 27] \cap B = \{15, 20, 25\}$, hence its cardinality is 3.

14. *Power sets (2)*

(a) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ because $\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$ and $\mathcal{P}(\emptyset) = \{\emptyset\}$.

(b) We have $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and thus

$$\mathcal{P}(\mathcal{P}(\{1, 2\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\{2\}\}, \{\{1, 2\}\},$$
$$\{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}, \{\emptyset, \{1, 2\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{1, 2\}\}, \{\{2\}, \{1, 2\}\},$$
$$\{\emptyset, \{1\}, \{2\}\}, \{\emptyset, \{1\}, \{1, 2\}\}, \{\emptyset, \{2\}, \{1, 2\}\}, \{\{1\}, \{2\}, \{1, 2\}\}, \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}\}.$$

15. *Proofs of equivalences involving sets.*

Let A and B be sets. Prove:

(a) $A \subseteq B \iff A = A \cap B \iff B = A \cup B$;

Proof. There are many ways to prove this. Here is one. We start with the following equivalences:

$$A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B) \Leftrightarrow (x \in A \Leftrightarrow (x \in A \land x \in B)) \Leftrightarrow A = A \cap B.$$

Now consider the following equivalences:

$$A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B) \Leftrightarrow (x \in B \Leftrightarrow (x \in A \lor x \in B)) \Leftrightarrow B = A \cup B.$$

(b) $A \cap B = \emptyset \iff A \setminus B = A$.

Proof.

$$A \cap B = \emptyset \Leftrightarrow (x \in A \Rightarrow x \notin B) \Leftrightarrow A \setminus B = \{x \mid x \in A \land x \notin B\} = \{x \mid x \in A\} = A.$$

16. *More proofs of equivalences involving sets.*

 Let E be a set and A, B subsets of E. Prove:

 (a) $A \cap B = \emptyset \Longleftrightarrow B \subseteq E \setminus A \Longleftrightarrow A \subseteq E \setminus B$;

 Proof. First we have the equivalences

 $$A \cap B = \emptyset \Leftrightarrow \forall x \in E : (x \in A \Rightarrow x \notin B) \Leftrightarrow A \subseteq E \setminus B.$$

 For the rest, it suffices to take the contrapositive of the assertion $\forall x \in E : (x \in A \Rightarrow x \notin B)$, which is $\forall x \in E : (x \in B \Rightarrow x \notin A)$, and is equivalent to the inclusion $B \subseteq E \setminus A$.

 (b) $A \cup B = E \Longleftrightarrow E \setminus A \subseteq B \Longleftrightarrow E \setminus B \subseteq A$.

 These equivalences can be proved with similar arguments. It is also possible to apply the rules from the course for working with complements.

17. Let S be a set and let $A \subseteq S$ and $B \subseteq S$ be subsets. In the sequel, let $x \in S$ and let a be the assertion $x \in A$ and b be the assertion $x \in B$.

 (1) $A \sqcup (S \setminus A) = S$

 This statement can be broken up into two statements:

 $$A \cup (S \setminus A) = S \text{ and } A \cap (S \setminus A) = \emptyset.$$

 The first statement translates to: $a \vee (\neg a) = $ true, a tautology.
 The second statement translates to: $a \wedge (\neg a) = $ false, a contradiction.

 (2) $S \setminus (S \setminus A) = A$

 This comes down to $\neg\neg a = a$, double negation.

 (3) $A \subseteq B \Leftrightarrow (S \setminus B) \subseteq (S \setminus A)$

 This translates to: $(a \Rightarrow b) \Leftrightarrow (\neg b \Rightarrow \neg a)$, the contrapositive.

 (4) $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$

 This translates to: $\neg(a \vee b) = (\neg a) \wedge (\neg b)$.

 (5) $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$

 This translates to: $\neg(a \wedge b) = (\neg a) \vee (\neg b)$.