Math Upgrade Week: Sets and Functions

Winter Term 2018/2019

University of Luxembourg
Prof. Dr. Gabor Wiese

Exercise sheet 1

1. *Sets via enumeration.*

   (a) Define via enumeration the set of even integers between 3 and 9.

   (b) Define via enumeration the set the elements of which are the subsets of $\{1, 2, 3\}$ having 2 elements.

2. *Sets via predicates.*

   (a) Define the following set via a predicate: $\{2, 4, 8, 16, 32, 64, 128, ...\}$.

   (b) Define the following set via a predicate: $\{-2, -1, 0, 1, 2\}$.

   (c) Define the emptyset via a predicate.

   (d) Define a set of your choice by a predicate and also enumerate its elements.

3. *Membership*
   Fill in correctly either $\in$ or $\notin$:

   (a) -3 _ $\mathbb{Z}$, -3 _ $\mathbb{N}$, -3 _ $\mathbb{Q}$

   (b) Let $E = \{n \in \mathbb{N} \mid 3 \text{ divides } n\}$:
   3 _ E, 2 _ E, -3 _ E, $\frac{1}{3}$ _ E

   (c) $\emptyset$ _ $\{\emptyset, \{\emptyset\}\}$.

   (d) 4 _ $\{\{4\}\}$.

4. *Equality of sets*
   Which of the following sets are equal?
   $A := \{1, 5, 9 - 2, 3\}$,
   $B := \{x \in \mathbb{N} \mid 1 \leq x < 7 \wedge (2 \text{ does not divide } x)\}$,
   $C := (\{x \in \mathbb{R} \mid x \geq 1\} \setminus \{2, 4, 6\}) \cap \{x \in \mathbb{R} \mid x < 8\}$,
   $D := \{x \in \mathbb{N} \mid 1 \leq x \leq 8 \wedge (2 \text{ does not divide } x)\}$,
   $E := \{x \in \mathbb{R} \mid 1 \leq x \leq 8 \wedge x \notin \{n \in \mathbb{N} \mid n \text{ is even }\}\}$.

5. *Inclusion and equality of sets*
   Consider the following sets:

$$A = \{1, 2, 5\},\ B = \{\{1, 2\}, 5\},\ C = \{\{1, 2, 5\}\},\ D = \{\emptyset, 1, 2, 5\},$$
$$E = \{5, 1, 2\},\ F = \{\{1, 2\}, \{5\}\},\ G = \{\{1, 2\}, \{5\}, 5\},\ H = \{5, \{1\}, \{2\}\}.$$

   (a) Which sets are related by equality and which by inclusion?

   (b) Compute the cardinality of each of these sets.

   (c) Determine $A \cap B$, $G \cup H$ and $E \setminus G$.

6. *Complements*
   Consider the following four subsets of $\mathbb{N}$:

$$I = \{1, 2, 3, 4, 5, 6, 7\},\ J = \{1, 3, 5, 7\},\ K = \{2, 4, 6\}.$$

   (a) Determine $I \setminus J$ and $I \setminus K$ (i.e. the complements of J and K in I).
   (b) The symmetric difference of two sets A and B, denoted by $A\Delta B$, is the set of elements that are either in A or in B, but not in $A \cap B$. Determine $I\Delta J$ and $J\Delta K$.

7. *Intersection, union, complement, etc.*
   Let

$$A = \{n \in \mathbb{N} \mid n \text{ is divisible by } 2\ \} \text{ and } B = \{n \in \mathbb{N} \mid n \text{ is divisible by } 5\ \}.$$

   (a) Describe the intersection $A \cap B$.
   (b) Describe the union $A \cup B$.
   (c) Describe the complement $B \setminus A$.
   (d) Describe the complement $A \setminus B$.
   (e) Give the cardinality of $[12, 27] \cap A$ and of $[12, 27] \cap B$.

8. *Power sets*
   List the elements of:

   (a) $\mathcal{P}(\{0, 1, 2\})$.
   (b) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$.
   (c) $\mathcal{P}(\mathcal{P}(\{1, 2\}))$.

9. *Proofs of equivalences involving sets.*
   Let A and B be sets. Prove:

   (a) $A \subseteq B \iff A = A \cap B \iff B = A \cup B$;
   (b) $A \cap B = \emptyset \iff A \setminus B = A$.

10. *More proofs of equivalences involving sets.*
    Let E be a set and A, B subsets of E. Prove:

    (a) $A \cap B = \emptyset \iff B \subseteq E \setminus A \iff A \subseteq E \setminus B$;
    (b) $A \cup B = E \iff E \setminus A \subseteq B \iff E \setminus B \subseteq A$.