# Math Upgrade Week: Sets and Functions

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**Exercise sheet 2 with solutions**

1. *Injectivity, surjectivity, bijectivity.*
   Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{A, B, C, D\}$.

   (a) Describe a surjective map from $A$ to $B$.

   **Solution:** For example: $f(1) = A$, $f(2) = B$, $f(3) = C$, $f(4) = D$, $f(5) = A$.

   (b) Describe a map from $A$ to $B$ which is neither surjective nor injective.

   **Solution:** For example: $f(1) = A$, $f(2) = A$, $f(3) = A$, $f(4) = A$, $f(5) = A$.

   (c) Does there exist an injective map from $A$ to $B$? Why?

   **Solution:** No, because if there were such an injective map $f$, then we'd have $\#\mathrm{im}(f) = \#A = 5$ and $\mathrm{im}(f) \subseteq B$, but $B$ does not possess a subset of cardinality $5$ because $\#B = 4$, contradiction.

   (d) Describe an injective map from $B$ to $A$.

   **Solution:** For example: $g(A) = 1$, $g(B) = 2$, $g(C) = 3$, $g(D) = 4$.

   (e) Describe a map from $B$ to $A$ which is neither surjective nor injective.

   **Solution:** For example: $g(A) = 1$, $g(B) = 1$, $g(C) = 1$, $g(D) = 1$.

   (f) Does there exist a surjective map from $B$ to $A$? Why?

   **Solution:** No, because if $g : B \to A$ were surjective, then we would have $4 = \#B \geq \#A = 5$, contradiction.

2. *Domain, image, preimage*

   (a) Let $f$ be a map from $\mathbb{N}$ to $\mathbb{Z}$ defined by $f(n) = n^3$ and $g$ a map from $\mathbb{Z}$ to $\mathbb{N}$ defined by $g(n) = n^2$. Calculate the image of 2 under $f$ and determine $f \circ g$.

   **Solution:** $f(2) = 2^3 = 8$, $f \circ g(n) = f(g(n)) = f(n^2) = (n^2)^3 = n^6$.

   (b) Let $f$ be a map from $E = \{1, 2, 3, 4\}$ to $F = \{0, 1, 3, 5, 7, 10\}$ such that $f(1) = 3$, $f(2) = 5$, $f(3) = 5$ and $f(4) = 0$.

   **Solution:** $f(\{2, 3\}) = \{5\}$, $\mathrm{im}(f) = \{0, 3, 5\}$.
   $f^{-1}(\{5\}) = \{2, 3\}$, $f^{-1}(\{0, 1, 3\}) = \{3, 4\}$, $f^{-1}(\{1, 10\}) = \emptyset$.
   The map $f$ is not injective because $f(2) = f(3)$ and it is not surjective (e.g. $10$ is not in the image), consequently it is not bijective either.

3. *More on injectivity, surjectivity, bijectivity.*

   (a) Find an injective but not bijective map from $\mathbb{N}$ to $\mathbb{N}$.

   **Solution.** For example: $f : \mathbb{N} \to \mathbb{N}$, $n \mapsto 2n$.

   (b) Find a surjective but not bijective map from $\mathbb{N}$ to $\mathbb{N}$.

   **Solution.** For example: $f : \mathbb{N} \to \mathbb{N}$, $n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$

   (c) Find a bijection between $\mathbb{Z}$ and $\mathbb{N}$.

   **Solution.** For example: $f : \mathbb{N} \to \mathbb{Z}$, $n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$

4. *Some proofs concerning maps.*
   Let $A, B, C$ be sets and $f : A \to B$ and $g : B \to C$ maps. Prove:

   (a) If $f$ and $g$ are both injective (resp. surjective, resp. bijective), then $g \circ f$ is injective (resp. surjective, resp. bijective).

   **Solution.** Suppose $f, g$ are injective. Let $a_1, a_2 \in A$ such that $g \circ f(a_1) = g \circ f(a_2)$. This means $g(f(a_1)) = g(f(a_2))$. As $g$ is injective, we infer $f(a_1) = f(a_2)$. Further, as $f$ is injective we arrive at $a_1 = a_2$. Thus we have shown that $g \circ f$ is injective.

   Suppose $f, g$ are surjective. Let $c \in C$. As $g$ is surjective, there is $b \in B$ such that $g(b) = c$. Further, as $f$ is surjective, there is $a \in A$ such that $f(a) = b$. Putting things together we have $g \circ f(a) = g(f(a)) = g(b) = c$, showing that $g \circ f$ is surjective.

   The statement about bijectivity follows immediately from the statements about injectivity and surjectivity.

   (b) If $g \circ f$ is injective, then $f$ is injective.

   **Solution.** Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. We conclude $g(f(a_1)) = g(f(a_2))$, that is, $g \circ f(a_1) = g \circ f(a_2)$. Using now that $g \circ f$ is injective, we infer $a_1 = a_2$. This now means that $f$ is injective.

   (c) If $g \circ f$ is surjective, then $g$ is surjective.

   **Solution.** Let $c \in C$ be given. As $g \circ f$ is surjective, there is $a \in A$ such that $g \circ f(a) = c$. Putting $b := f(a) \in B$ we see $g(b) = c$, showing the surjectivity of $g$.

   (d) Suppose that both $f$ and $g$ are bijective with inverses $f^{-1}$ and $g^{-1}$, respectively.
   Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

   **Solution.** We use the fact that we can put brackets as we like and compute:

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ \mathrm{id} \circ g^{-1} = g \circ g^{-1} = \mathrm{id}$$

   and

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ \mathrm{id} \circ f = f^{-1} \circ f = \mathrm{id}.$$

5. *Invertibility*
   Consider the function
$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} |x+1| \text{ if } x < 0 \\ |x-1| \text{ if } x \geq 0. \end{cases}$$
   Make a sketch of $f$.

   (a) Is the function $f$ a bijection?
   **Solution.** No, for instance $f(-1) = f(1) = 0$.

   (b) Find the biggest closed interval $[a, 10] \subseteq \mathbb{R}$ such that $f$ restricted to $[a, 10]$ is injective.
   **Solution.** $[1, 10]$

   (c) Write $g : [a, 10] \to f([a, 10])$ for the restriction of $f$ to $[a, 10]$ with $a$ from (b). Now $g$ is bijective. Describe the inverse of $g$ explicitly.
   **Solution.** On $[1, 10]$, the function $f$ simply is $f(x) = x - 1$. Its inverse is $h : [0, 9] \to [1, 10]$ such that $x \mapsto x + 1$.

6. *Involution*
   Let $E$ be a set and $f : E \to E$ a map such that: $f \circ f = \mathrm{id}_E$.
   Prove that $f$ is bijective.
   What is its inverse?

   **Solution.** Since $f \circ f$ is the identity, it is bijective. From exercises 4(b) and 4(c), we thus obtain that $f$ is injective and surjective, hence bijective. The inverse is simply $f$ itself.

7. *Sine function*
   Let $\sin : \mathbb{R} \to [-1, 1]$ be the sine function (known from school):

   (a) Is $\sin$ bijective?
   **Solution.** No, it is not injective. For instance, $\sin(0) = \sin(\pi) = 0$.

   (b) $\sin^{-1}(\{0\}) = \{n\pi \mid n \in \mathbb{Z}\}$.

   (c) $\sin^{-1}(\{1\}) = \{\frac{\pi}{2} + 2\pi n \mid n \in \mathbb{Z}\}$.

8. *Maps and power sets.*
   Let $E$ be a non-empty set, $\mathcal{P}(E)$ its power set, and $A, B \in \mathcal{P}(E)$. One defines
$$f : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto (A \cap X) \cup (B \cap \overline{X}^E),$$
   where $\overline{X}^E = E \setminus X$ is the complement of $X$ in $E$.
   Analyse the equality $f(X) = \emptyset$.
   Deduce a necessary condition for $f$ to be bijective.

   **Solution.** We have $f(X) = \emptyset$ if and only if $A \cap X = \emptyset$ and $B \cap \overline{X}^E = \emptyset$. The second condition is equivalent to $B \subseteq X$. We thus have proved the equivalence:
$$f(X) = \emptyset \Leftrightarrow A \cap X = \emptyset \wedge B \subseteq X.$$

   If $f$ is bijective, then, in particular, $\emptyset$ is in the image of $f$. This, however, means that a subset $X \subseteq E$ exists such that $X \cap A = \emptyset$ and $B \subseteq X$. This implies that $A \cap B = \emptyset$. That $A$ and $B$ are disjoint is hence a necessary condition for $f$ to be surjective.

9. *Increasing maps.*
Let $I \subseteq \mathbb{R}$ and $J \subseteq \mathbb{R}$ be two intervals in $\mathbb{R}$. Let $f : I \longrightarrow J$ be a strictly increasing function.

   (a) Show that $f$ is injective.

   **Solution.** Strictly increasing means that for any $x_1 < x_2$ one has $f(x_1) < f(x_2)$. This immediately implies injectivity.

   (b) Determine the unique subset $K \subseteq J$ such that $f : I \longrightarrow K$ is bijective.

   **Solution.** The sought for set $K$ is just $\mathrm{im}(f)$.

10. *Maps from* $\mathbb{N}$ *to* $\mathbb{N}$
Consider a map $u : \mathbb{N} \to \mathbb{N}$ and assume that

$$\forall k \in \mathbb{N} : u(k+1) > u(k).$$

   (a) Show rigorously (justifying each step of your argument) that for any $k, l \in \mathbb{N}$ with $k < l$, one has $u(k) < u(l)$.

   **Solution.** This can be proved by induction. More precisely, for any $k \in \mathbb{N}, k \geq 1$ we perform an induction in order to prove the assertion: $\forall\, n \geq 1 : u(k) < u(k+n)$.

   Initialisation: The assertion for $n = 1$ is precisely the assumption.

   Induction step: Suppose the assertion is proved for $n$, we now prove it for $n+1$:

$$u(k) < u(k+n) < u((k+n)+1),$$

   where the second inequality is the assumption applied to $k+n$.

   (b) Is $u$ necessarily injective? Justify your answer by a precise argument or by a counterexample.

   **Solution.** Let $k, l \in \mathbb{N}$ be distinct. By possibly renaming them, we may suppose $k < l$. By (a) we have $u(k) < u(l)$, in particular, $u(k) \neq u(l)$. This proves the injectivity.

   (c) Is $u$ necessarily surjective? Justify your answer by a precise argument or by a counterexample.

   **Solution.** No. A counterexample is provided by $u : \mathbb{N} \to \mathbb{N}$ given by $u(n) = 2n$.