Math Prep Camp: Sets and Functions

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Exercise sheet 2 with solutions

1. *Describing a map.*
   Let $A := \{0, 1, 2\}$ and $B := \{X, Y\}$ be sets.

   Which of the following lines describe a map $g : A \to B$? If not, why?

   (a) $g(2) = X$, $g(0) = Y$, $g(1) = X$: This is a valid description of a map.

   (b) $g(0) = X$, $g(2) = Y$, $g(0) = X$: Invalid because $g(1)$ is not defined.

   (c) $g(0) = X$, $g(2) = Y$, $g(0) = Y$: Invalid because $g(1)$ is not defined (and also because $g(0)$ is assigned to different values).

   (d) $g(0) = Y$, $g(1) = Y$, $g(2) = Y$: This is a valid description of a map.

2. *Image and preimage.*
   Consider the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$.

   (a) $9 = f(3) = 3^2$ is the image of 3 under f.

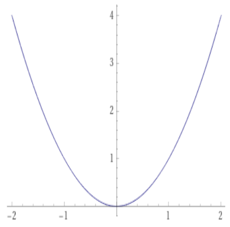
   (b) $(1 + x)^2 = 1 + 2x + x^2 = f(1 + x)$ is the image of $1 + x$ under f for $x \in \mathbb{R}$.

   (c) The interval $[4, 9]$ is the image of the interval $[2, 3]$ under f.

   (d) 2 and -2 are exactly the preimages of 4 under f because they are the only elements in $\mathbb{R}$ the square of which equals 4.

   (e) $[-2, -1] \sqcup [1, 2]$ is the preimage of the interval $[1, 4]$ under f.

3. *Graphs.*

   (a) Sketch of the graph of the function $f : [-2, 2] \to [0, 4], x \mapsto x^2$.

   (b) If we reflect your sketch at the line $x = y$ (i.e. swap x- and y-axis), then it is no longer the graph of a function because for any x, there would be two values for y.

4. *Injectivity, surjectivity, bijectivity.*
   For each of the following functions, state if they are injective, surjective or bijective (or none of these).

   (a) $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$: not injective, for example $f(-1) = 1 = f(1)$; not surjective because -1 is not in the image.

   (b) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto x^2$: injective.

   (c) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$: injective, surjective and bijective.

(d) $f : \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$: not injective, but surjective; not bijective.

Here $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$.

5. *Composition.*

(a) Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ and $g : \mathbb{R} \to \mathbb{R}, x \mapsto x + 1$. For $x \in \mathbb{R}$, we have

$$(f \circ g)(x) = f(g(x)) = (x + 1)^2 = x^2 + 2x + 1$$

and

$$(g \circ f)(x) = g(f(x)) = x^2 + 1.$$

(b) Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$ and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto \sqrt{x}$. For $x \in \mathbb{R}$, we have

$$(f \circ g)(x) = f(g(x)) = (\sqrt{x})^2 = x$$

and

$$(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|.$$

6. *Inverse map.*

(a) The inverse of $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$ is the map $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, x \mapsto \sqrt{x}$.

(b) Let $A := \{0, 1, 2\}$, $b := \{a, b, c\}$ and let $f : A \to B$ be given by $f(0) = b$, $f(1) = a$, $f(2) = c$. Its inverse is the map $g : B \to A$ given by $g(a) = 1$, $g(b) = 0$, $g(c) = 2$.

7. *Injectivity, surjectivity, bijectivity (2).*
Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d\}$.

(a) Describe a surjective map from A to B.

Solution: For example: $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = d$, $f(5) = a$.

(b) Describe a map from A to B which is neither surjective nor injective.

Solution: For example: $f(1) = a$, $f(2) = a$, $f(3) = a$, $f(4) = a$, $f(5) = a$.

(c) Does there exist an injective map from A to B? Why?

Solution: No, because if there were such an injective map f, then we'd have $\#\mathrm{im}(f) = \#A = 5$ and $\mathrm{im}(f) \subseteq B$, but B does not possess a subset of cardinality 5 because $\#B = 4$, contradiction.

(d) Describe an injective map from B to A.

Solution: For example: $g(a) = 1$, $g(b) = 2$, $g(c) = 3$, $g(d) = 4$.

(e) Describe a map from B to A which is neither surjective nor injective.

Solution: For example: $g(a) = 1$, $g(b) = 1$, $g(c) = 1$, $g(d) = 1$.

(f) Does there exist a surjective map from B to A? Why?

Solution: No, because if $g : B \to A$ were surjective, then we would have $4 = \#B \geq \#A = 5$, contradiction.

8. *Domain, image, preimage*

(a) Let f be a map from $\mathbb{N}$ to $\mathbb{Z}$ defined by $f(n) = n^3$ and g a map from $\mathbb{Z}$ to $\mathbb{N}$ defined by $g(n) = n^2$. Calculate the image of 2 under f and determine $f \circ g$.

Solution: $f(2) = 2^3 = 8$, $f \circ g(n) = f(g(n)) = f(n^2) = (n^2)^3 = n^6$.

(b) Let f be a map from $E = \{1, 2, 3, 4\}$ to $F = \{0, 1, 3, 5, 7, 10\}$ such that $f(1) = 3$, $f(2) = 5$, $f(3) = 5$ and $f(4) = 0$.

Solution: $f(\{2, 3\}) = \{5\}$, $\mathrm{im}(f) = \{0, 3, 5\}$.
$f^{-1}(\{5\}) = \{2, 3\}$, $f^{-1}(\{0, 1, 3\}) = \{3, 4\}$, $f^{-1}(\{1, 10\}) = \emptyset$.
The map f is not injective because $f(2) = f(3)$ and it is not surjective (e.g. 10 is not in the image), consequently it is not bijective either.

9. *More on injectivity, surjectivity, bijectivity.*

(a) Find an injective but not bijective map from $\mathbb{N}$ to $\mathbb{N}$.
Solution. For example: $f : \mathbb{N} \to \mathbb{N}$, $n \mapsto 2n$.

(b) Find a surjective but not bijective map from $\mathbb{N}$ to $\mathbb{N}$.
Solution. For example: $f : \mathbb{N} \to \mathbb{N}$, $n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$

(c) Find a bijection between $\mathbb{Z}$ and $\mathbb{N}$.
Solution. For example: $f : \mathbb{N} \to \mathbb{Z}$, $n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$

10. *Some proofs concerning maps.*
Let A, B, C be sets and $f : A \to B$ and $g : B \to C$ maps. Prove:

(a) If f and g are both injective (resp. surjective, resp. bijective), then $g \circ f$ is injective (resp. surjective, resp. bijective).
Solution. Suppose f, g are injective. Let $a_1, a_2 \in A$ such that $g \circ f(a_1) = g \circ f(a_2)$. This means $g(f(a_1)) = g(f(a_2))$. As g is injective, we infer $f(a_1) = f(a_2)$. Further, as f is injective we arrive at $a_1 = a_2$. Thus we have shown that $g \circ f$ is injective.
Suppose f, g are surjective. Let $c \in C$. As g is surjective, there is $b \in B$ such that $g(b) = c$. Further, as f is surjective, there is $a \in A$ such that $f(a) = b$. Putting things together we have $g \circ f(a) = g(f(a)) = g(b) = c$, showing that $g \circ f$ is surjective.
The statement about bijectivity follows immediately from the statements about injectivity and surjectivity.

(b) If $g \circ f$ is injective, then f is injective.
Solution. Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. We conclude $g(f(a_1)) = g(f(a_2))$, that is, $g \circ f(a_1) = g \circ f(a_2)$. Using now that $g \circ f$ is injective, we infer $a_1 = a_2$. This now means that f is injective.

(c) If $g \circ f$ is surjective, then g is surjective.
Solution. Let $c \in C$ be given. As $g \circ f$ is surjective, there is $a \in A$ such that $g \circ f(a) = c$. Putting $b := f(a) \in B$ we see $g(b) = c$, showing the surjectivity of g.

(d) Suppose that both f and g are bijective with inverses f^{-1} and g^{-1}, respectively.
Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
Solution. We use the fact that we can put brackets as we like and compute:

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ \mathrm{id} \circ g^{-1} = g \circ g^{-1} = \mathrm{id}$$

and

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ \mathrm{id} \circ f = f^{-1} \circ f = \mathrm{id}.$$

11. *Invertibility*

Consider the function

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} |x+1| \text{ if } x < 0 \\ |x-1| \text{ if } x \geq 0. \end{cases}$$

Make a sketch of f.

(a) Is the function f a bijection?

Solution. No, for instance $f(-1) = f(1) = 0$.

(b) Find the biggest closed interval $[a, 10] \subseteq \mathbb{R}$ such that f restricted to $[a, 10]$ is injective.

Solution. $[1, 10]$

(c) Write $g : [a, 10] \to f([a, 10])$ for the restriction of f to $[a, 10]$ with a from (b). Now g is bijective. Describe the inverse of g explicitly.

Solution. On $[1, 10]$, the function f simply is $f(x) = x - 1$. Its inverse is $h : [0, 9] \to [1, 10]$ such that $x \mapsto x + 1$.

12. *Involution*

Let E be a set and $f : E \to E$ a map such that: $f \circ f = \mathrm{id}_E$.
Prove that f is bijective.
What is its inverse?

Solution. Since $f \circ f$ is the identity, it is bijective. From exercises 4(b) and 4(c), we thus obtain that f is injective and surjective, hence bijective. The inverse is simply f itself.

13. *Sine function*

Let $\sin : \mathbb{R} \to [-1, 1]$ be the sine function (known from school):

(a) Is $\sin$ bijective?

Solution. No, it is not injective. For instance, $\sin(0) = \sin(\pi) = 0$.

(b) $\sin^{-1}(\{0\}) = \{n\pi \mid n \in \mathbb{Z}\}$.

(c) $\sin^{-1}(\{1\}) = \{\frac{\pi}{2} + 2\pi n \mid n \in \mathbb{Z}\}$.

14. *Maps and power sets.*

Let E be a non-empty set, $\mathcal{P}(E)$ its power set, and $A, B \in \mathcal{P}(E)$. One defines

$$f : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto (A \cap X) \cup (B \cap \overline{X}^E),$$

where $\overline{X}^E = E \setminus X$ is the complement of X in E.
Analyse the equality $f(X) = \emptyset$.
Deduce a necessary condition for f to be bijective.

Solution. We have $f(X) = \emptyset$ if and only if $A \cap X = \emptyset$ and $B \cap \overline{X}^E = \emptyset$. The second condition is equivalent to $B \subseteq X$. We thus have proved the equivalence:

$$f(X) = \emptyset \Leftrightarrow A \cap X = \emptyset \wedge B \subseteq X.$$

If f is bijective, then, in particular, $\emptyset$ is in the image of f. This, however, means that a subset $X \subseteq E$ exists such that $X \cap A = \emptyset$ and $B \subseteq X$. This implies that $A \cap B = \emptyset$. That A and B are disjoint is hence a necessary condition for f to be surjective.

15. *Increasing maps.*
Let $I \subseteq \mathbb{R}$ and $J \subseteq \mathbb{R}$ be two intervals in $\mathbb{R}$. Let $f : I \longrightarrow J$ be a strictly increasing function.

(a) Show that f is injective.

Solution. Strictly increasing means that for any $x_1 < x_2$ one has $f(x_1) < f(x_2)$. This immediately implies injectivity.

(b) Determine the unique subset $K \subseteq J$ such that $f : I \longrightarrow K$ is bijective.

Solution. The sought for set K is just $\operatorname{im}(f)$.

16. *Maps from* $\mathbb{N}$ *to* $\mathbb{N}$
Consider a map $u : \mathbb{N} \to \mathbb{N}$ and assume that

$$\forall k \in \mathbb{N} : u(k+1) > u(k).$$

(a) Show rigorously (justifying each step of your argument) that for any $k, l \in \mathbb{N}$ with $k < l$, one has $u(k) < u(l)$.

Solution. This can be proved by induction. More precisely, for any $k \in \mathbb{N}, k \geq 1$ we perform an induction in order to prove the assertion: $\forall\, n \geq 1 : u(k) < u(k+n)$.

Initialisation: The assertion for $n = 1$ is precisely the assumption.

Induction step: Suppose the assertion is proved for n, we now prove it for $n+1$:

$$u(k) < u(k+n) < u((k+n)+1),$$

where the second inequality is the assumption applied to $k+n$.

(b) Is u necessarily injective? Justify your answer by a precise argument or by a counterexample.

Solution. Let $k, l \in \mathbb{N}$ be distinct. By possibly renaming them, we may suppose $k < l$. By (a) we have $u(k) < u(l)$, in particular, $u(k) \neq u(l)$. This proves the injectivity.

(c) Is u necessarily surjective? Justify your answer by a precise argument or by a counterexample.

Solution. No. A counterexample is provided by $u : \mathbb{N} \to \mathbb{N}$ given by $u(n) = 2n$.