Math Upgrade Week: Sets and Functions

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University of Luxembourg
Prof. Dr. Gabor Wiese

Exercise sheet 2

1. *Injectivity, surjectivity, bijectivity.*
   Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{A, B, C, D\}$.
   (a) Describe a surjective map from A to B.
   (b) Describe a map from A to B which is neither surjective nor injective.
   (c) Does there exist an injective map from A to B? Why?
   (d) Describe an injective map from B to A.
   (e) Describe a map from B to A which is neither surjective nor injective.
   (f) Does there exist a surjective map from B to A? Why?

2. *Domain, image, preimage*
   (a) Let f be a map from $\mathbb{N}$ to $\mathbb{Z}$ defined by $f(n) = n^3$ and g a map from $\mathbb{Z}$ to $\mathbb{N}$ defined by $g(n) = n^2$. Calculate the image of 2 under f and determine $f \circ g$.
   (b) Let f be a map from $E = \{1, 2, 3, 4\}$ to $F = \{0, 1, 3, 5, 7, 10\}$ such that $f(1) = 3$, $f(2) = 5$, $f(3) = 5$ and $f(4) = 0$.
   Determine $f(\{2, 3\})$, $\mathrm{im}(f)$.
   Determine $f^{-1}(\{5\})$, $f^{-1}(\{0, 1, 3\})$ and $f^{-1}(\{1, 10\})$.
   Is f injective, surjective, bijective?

3. *More on injectivity, surjectivity, bijectivity.*
   (a) Find an injective but not bijective map from $\mathbb{N}$ to $\mathbb{N}$.
   (b) Find a surjective but not bijective map from $\mathbb{N}$ to $\mathbb{N}$.
   (c) Find a bijection between $\mathbb{Z}$ and $\mathbb{N}$.

4. *Some proofs concerning maps.*
   Let A, B, C be sets and $f : A \to B$ and $g : B \to C$ maps. Prove:
   (a) If f and g are both injective (resp. surjective, resp. bijective), then $g \circ f$ is injective (resp. surjective, resp. bijective).
   (b) If $g \circ f$ is injective, then f is injective.
   (c) If $g \circ f$ is surjective, then g is surjective.
   (d) Suppose that both f and g are bijective with inverses f^{-1} and g^{-1}, respectively.
   Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

5. *Invertibility*
Consider the function
$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} |x+1| \text{ if } x < 0 \\ |x-1| \text{ if } x \geq 0. \end{cases}$$
Make a sketch of f.
   (a) Is the function f a bijection?
   (b) Find the biggest closed interval $[a, 10] \subseteq \mathbb{R}$ such that f restricted to $[a, 10]$ is injective.
   (c) Write $g : [a, 10] \to f([a, 10])$ for the restriction of f to $[a, 10]$ with a from (b). Now g is bijective. Describe the inverse of g explicitly.

6. *Involution*
Let E be a set and $f : E \to E$ a map such that: $f \circ f = \mathrm{id}_E$.
Prove that f is bijective.
What is its inverse?

7. *Sine function*
Let $\sin : \mathbb{R} \to [-1, 1]$ be the sine function (known from school):
   (a) Is $\sin$ bijective?
   (b) Describe the preimage $\sin^{-1}(\{0\})$.
   (c) Describe the preimage $\sin^{-1}(\{1\})$.

8. *Maps and power sets.*
Let E be a non-empty set, $\mathcal{P}(E)$ its power set, and $A, B \in \mathcal{P}(E)$. One defines
$$f : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto (A \cap X) \cup (B \cap \overline{X}^E),$$
where $\overline{X}^E = E \setminus X$ is the complement of X in E.
Analyse the equality $f(X) = \emptyset$.
Deduce a necessary condition for f to be bijective.

9. *Increasing maps*.
Let $I \subseteq \mathbb{R}$ and $J \subseteq \mathbb{R}$ be two intervals in $\mathbb{R}$. Let $f : I \longrightarrow J$ be a strictly increasing function.
   (a) Show that f is injective.
   (b) Determine the unique subset $K \subseteq J$ such that $f : I \longrightarrow K$ is bijective.

10. *Maps from $\mathbb{N}$ to $\mathbb{N}$*
Consider a map $u : \mathbb{N} \to \mathbb{N}$ and assume that
$$\forall k \in \mathbb{N} : u(k+1) > u(k).$$
   (a) Show rigorously (justifying each step of your argument) that for any $k, l \in \mathbb{N}$ with $k < l$, one has $u(k) < u(l)$.
   (b) Is u necessarily injective? Justify your answer by a precise argument or by a counter-example.
   (c) Is u necessarily surjective? Justify your answer by a precise argument or by a counter-example.