On the arithmetic of modular forms

Gabor Wiese

28 June 2017

Number Theory

Is -1 a square?

Is -1 a square modulo a prime p?

Is -1 a square mod p?

3	squares modulo 3: $\{0, 1\} \not\ni -1$
5	
7	
11	
13	
19	
23	
29	
31	
37	

Is -1 a square mod p?

3	-1 no square mod 3
5	
7	
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Is -1 a square mod p?

3	-1 no square mod 3
5	$-1 = 2 \cdot 2 - 5$
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5	$-1 \equiv 2^2 \mod 5$
7	squares modulo 7: $\{0, 1, 2, 4\} \not\ni -1$
11	
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3	-1 no square mod 3
5	$-1 \equiv 2^2 \mod 5$
7	-1 no square mod 7
11	squares modulo 11: $\{0, 1, 3, 4, 5, 9\} \not\ni -1$
13	
19	
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13	$-1 = 5 \cdot 5 - 2 \cdot 13$
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19	squares modulo 19: $\{0, 1, 4, 5, 6, 7, 9, 11, 16, 17\} \not\ni -1$
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31	squares modulo 31: $\{0, 1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28\} \not\ni -1$
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-1 is a square modulo p

$$\Leftrightarrow$$

$p = 2$ or $p \equiv 1 \mod 4$.

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37	$-1 \equiv 31^2 \mod 37$

$$-1 \text{ is a square modulo } p \iff p = 2 \text{ or } p \equiv 1 \mod 4.$$

Proof. $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic of order $p-1$.
It contains an element of order $4 \Leftrightarrow 4 \mid p-1$.

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Examples.
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Proposition.
$X^2 + 1 \equiv (X - *) \cdot (X + *) = ()() \mod p \Leftrightarrow p \equiv 1, 2 \mod 4.$

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Generalisation:

How does $P(X) = X^6 - 6X^4 + 9X^2 + 23$ factor modulo p?

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p	factorisation
5	$(X^2+3)(X^2+X+1)(X^2+4X+1)$
13	$(X^3+10X+4)(X^3+10X+9)$
17	$(X^2+3)(X^2+2X+6)(X^2+15X+6)$
19	$(X^2+9)(X^2+X+12)(X^2+18X+12)$
31	$(X^3+28X+15)(X^3+28X+16)$
47	$(X^3+44X+20)(X^3+44X+27)$
53	$(X^2+22)(X^2+5X+25)(X^2+48X+25)$
59	$(X+9)(X+21)(X+29)(X+30)(X+38)(X+50)$
73	$(X^3+70X+14)(X^3+70X+59)$
97	$(X^2+39)(X^2+41X+42)(X^2+56X+42)$
101	$(X+4)(X+28)(X+32)(X+69)(X+73)(X+97)$

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Rule ????

The answer is given by a modular form...

What is a modular form?

Es gibt fünf Grundoperationen: Addition, Subtraktion, Multiplikation, Division und Modulformen.

Martin Eichler (1912-1992)

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J'aime bien les formes modulaires. [...] C'est un sujet sur lequel on n'a jamais de mauvaises surprises: si l'on devine un énoncé, c'est un énoncé encore plus beau qui est vrai !

Jean-Pierre Serre (*1926)

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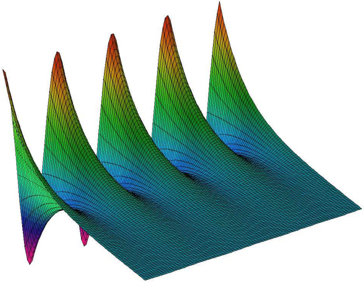
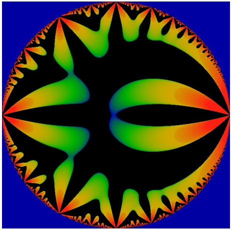
Their coefficients are

arithmetically significant.

What is a modular form?

Modular forms are highly symmetric functions.





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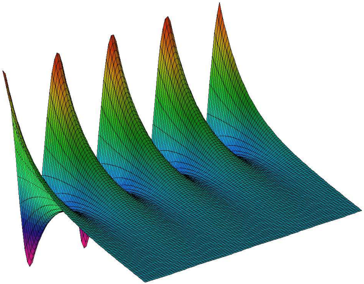

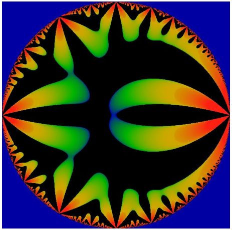

- Complex Analysis: Fourier series with certain transformation properties.

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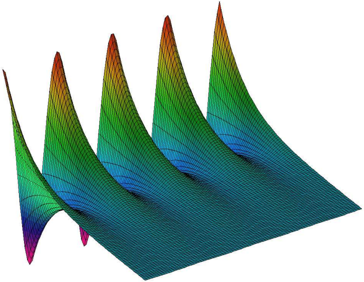

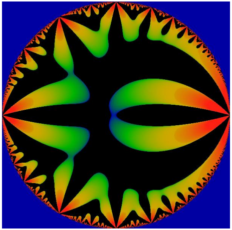

- Complex Analysis: Fourier series with certain transformation properties.
- Geometry: Differential forms on modular curves.
  Modular curves are curves parametrising elliptic curves.

What is a modular form?

A modular form is an object from geometry and/or analysis.

Definition. *A modular form of weight k is a holomorphic function*

$$f : \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\} \to \mathbb{C}$$

such that

- $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ *for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

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This is the definition for **level** 1. More generally, level $N \in \mathbb{N}$.

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The Fourier coefficients of a Hecke eigenform satisfy

$$a_n a_m = a_{nm} \qquad \text{if } \gcd(n, m) = 1.$$

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The Fourier coefficients of Hecke eigenforms a_n are algebraic integers.

Arithmetic significance of coefficients of modular forms

Examples (19th century):

Gotthold Eisenstein (1823-1852)

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$$E_k = * \sum_{(n,m)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^k}$$

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$$E_k = \frac{(k-1)!}{(2\pi i)^k} \cdot \zeta(k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) \cdot q^n, \quad q = e^{2\pi i z},$$

where $\sigma_{k-1}(n) = \sum_{0<d|n} d^{k-1}$.

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Coefficients: Special zeta-value and divisor function.

Matching Jacobi's Theta-series with Eisenstein series, one gets:

$$\#\{x \in \mathbb{Z}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = n\} = 8 \sum_{4 \nmid d | n, 1 \le d \le n} d.$$

Symmetries of equations

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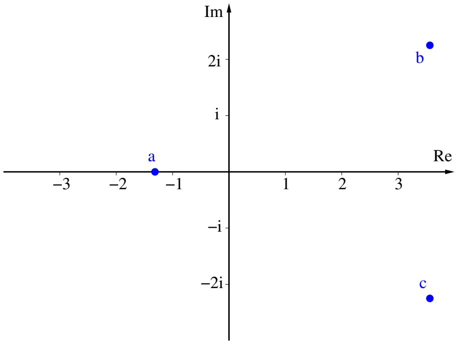
Algebraically speaking: a symmetry is a field automorphism.

Symmetries of equations

Consider $X^3 - 6X^2 + 9X + 23 = 0$. Three solutions $a, b, c \in \mathbb{C}$:



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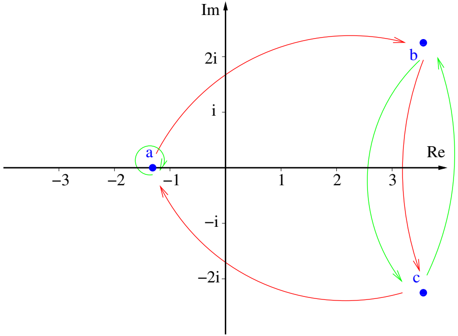

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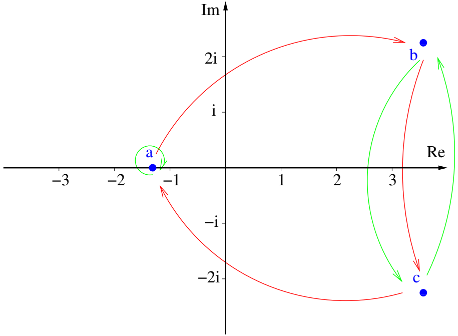

There are 6 symmetries in this example.

The symmetry group is called the Galois group.

Arithmetic significance of coefficients of modular forms

Another view on Eisenstein series.

Recall: $E_k = \frac{(k-1)!}{(2\pi i)^k} \cdot \zeta(k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) \cdot q^n$.

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$$\chi : G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$$

given by the action on the ℓ-power roots of unity:

$$\sigma(\zeta_{\ell^n}) = \zeta_{\ell^n}^{\chi(\sigma)}.$$

Particularly, $\mathrm{Frob}_p(\zeta_{\ell^n}) = \zeta_{\ell^n}^p = \zeta_{\ell^n}^{\chi(\mathrm{Frob}_p)}$, whence $\chi(\mathrm{Frob}_p) = p$.

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Consider the reducible semi-simple Galois representation

$$\rho := 1 \oplus \chi^{k-1} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_\ell), \quad \rho(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \chi^{k-1}(\sigma) \end{pmatrix}.$$

In particular,

$$\rho(\mathrm{Frob}_p) = \begin{pmatrix} 1 & 0 \\ 0 & \chi^{k-1}(\mathrm{Frob}_p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^{k-1} \end{pmatrix}.$$

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$\Rightarrow$ $\mathrm{Tr}(\rho(\mathrm{Frob}_p)) = 1 + p^{k-1} = \sigma_{k-1}(p)$.

This is the p-th coefficient of the Eisenstein series of weight k.

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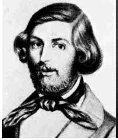
In a sense, the modular form is the character of the Galois representation.

One says that ρ is attached to the Eisenstein series.

From Geometry to Number Theory

Geometry/Analysis	**Number Theory**
Modular Forms	**Galois Representations**

From Geometry to Number Theory

Geometry/Analysis **Modular Forms**	**Number Theory** **Galois Representations**
 	

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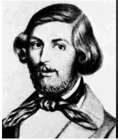
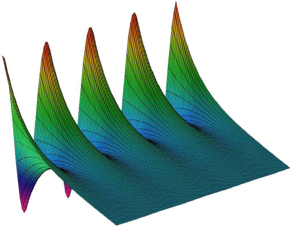
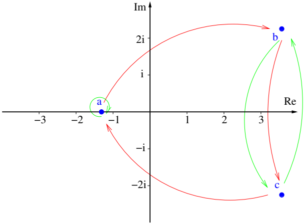
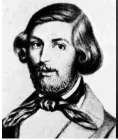






From Geometry to Number Theory

Geometry/Analysis
Modular Forms




Hecke eigenforms
$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$
with $a_1 = 1$

Number Theory
Galois Representations

Galois repres.
$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Z}}_\ell)$
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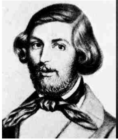 		
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f level N	$\mapsto$	ρ_f unramified outside $N\ell$ $\mathrm{Tr}(\rho_f(\mathrm{Frob}_p)) = a_p$

Shimura, Deligne, Serre.

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Hilbert

Given a finite group G.
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Under the assumption of Maeda's Conjecture, the density is 1.

How does $P(X) = X^6 - 6X^4 + 9X^2 + 23$ factor modulo p?

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p	factorisation
5	$(X^2+3)(X^2+X+1)(X^2+4X+1)$
13	$(X^3+10X+4)(X^3+10X+9)$
17	$(X^2+3)(X^2+2X+6)(X^2+15X+6)$
19	$(X^2+9)(X^2+X+12)(X^2+18X+12)$
31	$(X^3+28X+15)(X^3+28X+16)$
47	$(X^3+44X+20)(X^3+44X+27)$
53	$(X^2+22)(X^2+5X+25)(X^2+48X+25)$
59	$(X+9)(X+21)(X+29)(X+30)(X+38)(X+50)$
73	$(X^3+70X+14)(X^3+70X+59)$
97	$(X^2+39)(X^2+41X+42)(X^2+56X+42)$
101	$(X+4)(X+28)(X+32)(X+69)(X+73)(X+97)$

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p	factorisation	a_p
5	()()()	0
13	()()	-1
17	()()()	0
19	()()()	0
31	()()	-1
47	()()	-1
53	()()()	0
59	()()()()()()	2
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Proposition.

There is a modular form (of weight 1 and level 23) $f = \sum_{n=1}^{\infty} a_n q^n$ s.t.
3 factors $\Leftrightarrow a_p = 0$,
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P mod p	Frob_p	$\rho(\mathrm{Frob}_p)$	trace
()()()()()()	identity	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2
()()	2 3-cycles	$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}, \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta \end{pmatrix}, \zeta = e^{2\pi i/3}$	-1
()()()	3 2-cycles	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{pmatrix}$	0

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What can one compute explicitly?

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$\Rightarrow$ Compute modular forms to learn about number theory.

Arithmetic significance of coefficients of modular forms

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(III) In how far are Galois representations governed by modular forms?

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(2) 'Real distribution'.

Normalise the coefficients $b_p = \frac{a_p}{p^{(k-1)/2}} \in [-2, 2]$.

How are the b_p distributed over $[-2, 2]$?

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The density of the set $\{p \mid a_p \equiv b \mod 7\}$ equals

b	0	1	2	3	4	5	6
density	$\frac{1}{2}$	0	$\frac{1}{6}$	0	0	0	$\frac{1}{3}$

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Nice illustration by Andrew Sutherland.

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Computations carried out with Marcel Mohyla suggest that the maximum residue degree in level q with q.

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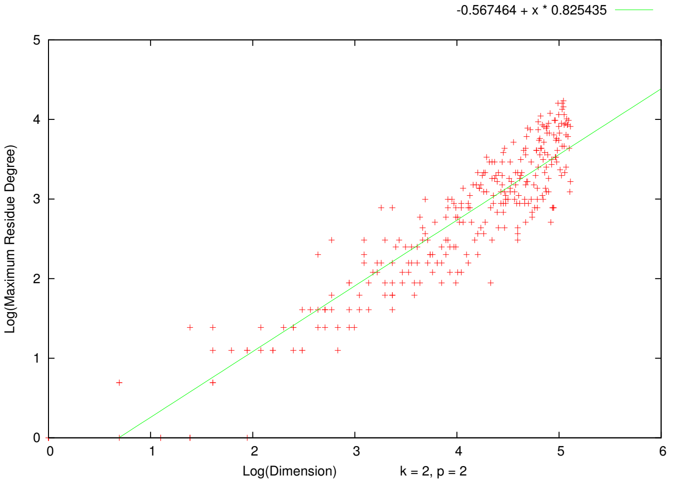
Degrees of residual coefficient fields mod ℓ for $k = 2$ in prime levels.



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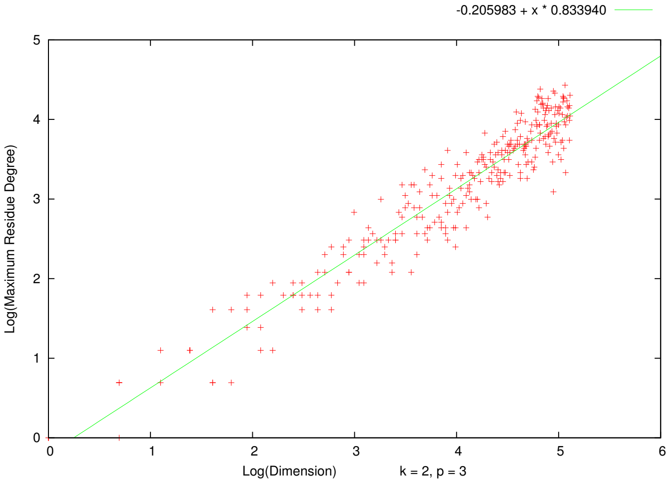

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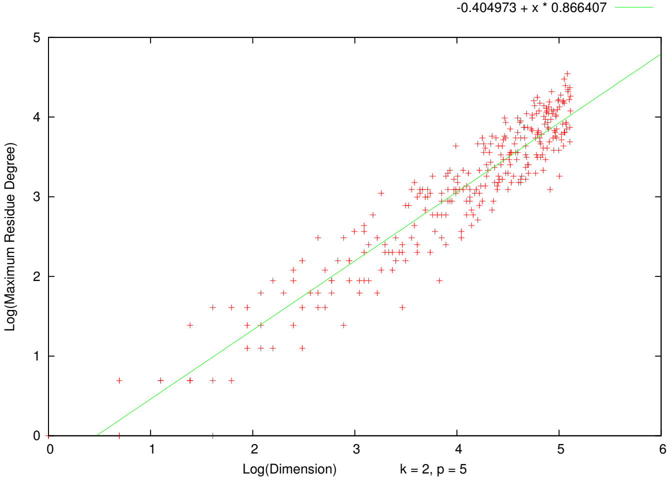

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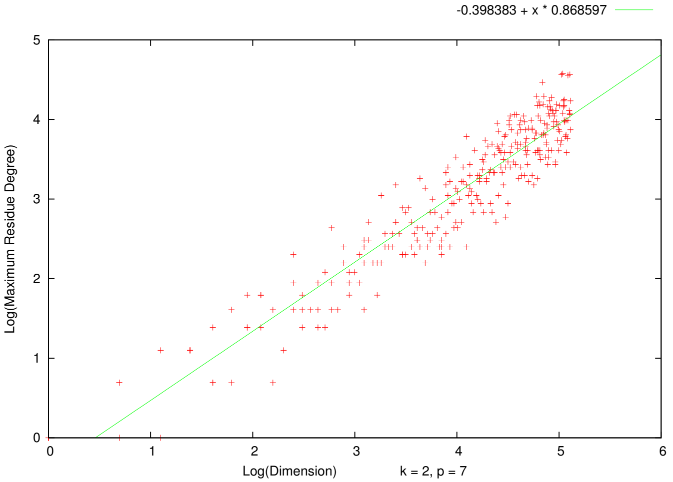

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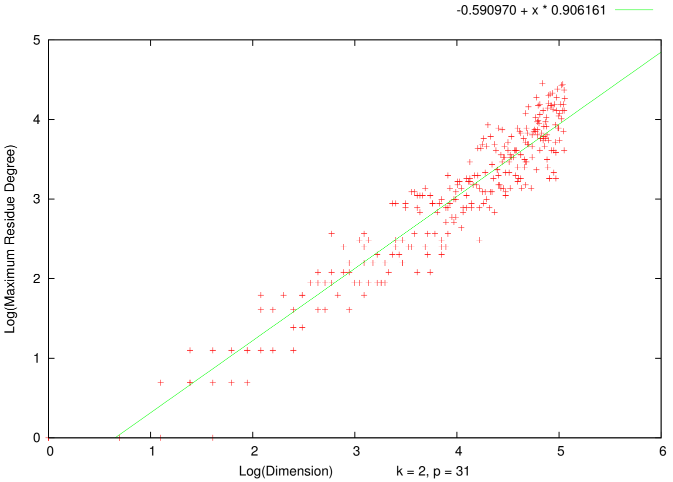

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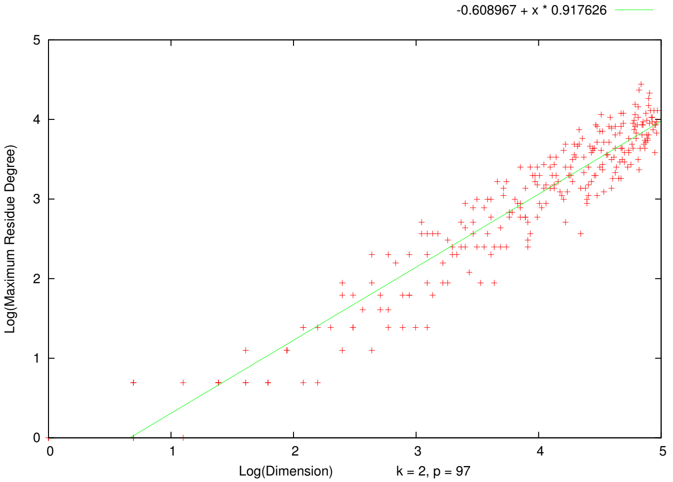

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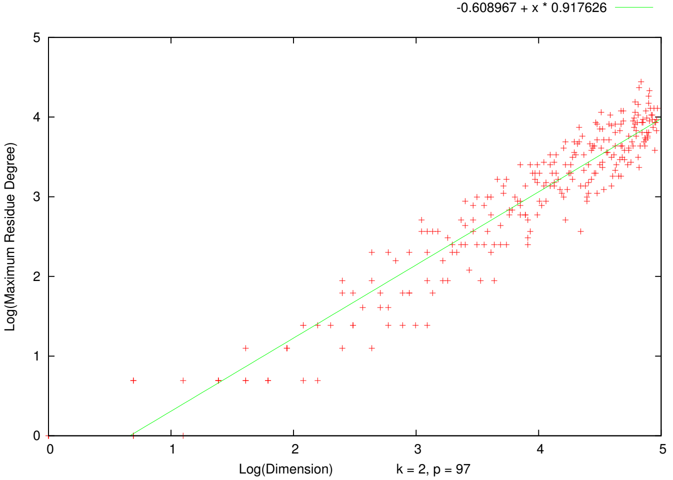

Any idea?

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Theorem (Gross, Coleman-Voloch, W.) *If f is of weight one, prime-to-ℓ level and geometrically defined over $\overline{\mathbb{F}}_\ell$, then the attached Galois representation $\overline{\rho}_f$ is unramified at ℓ.*

Moreover, this characterises weight one among all weights (at least if $\ell > 2$).

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It is believed and partially proved that this characterises parallel weight one forms among all Hilbert Hecke eigenforms.

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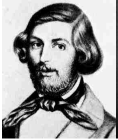

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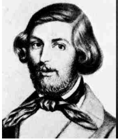


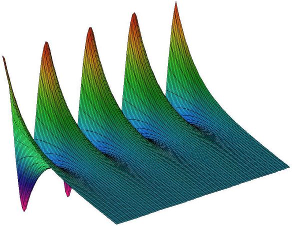

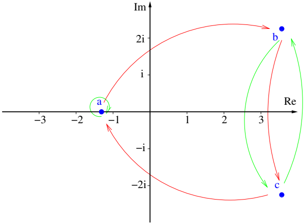

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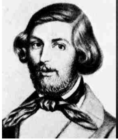


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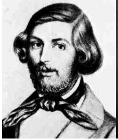
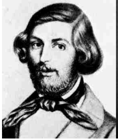



Hecke eigenforms	$\longrightarrow$	Galois repres.
$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$		$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$
with $a_1 = 1$		s.t. $\det(\overline{\rho}(\text{compl. conj.})) = -1$
f	$\mapsto$	$\overline{\rho}_f$
level N		unramified outside $N\ell$
		$\mathrm{Tr}(\overline{\rho}_f(\mathrm{Frob}_p)) = a_p$

Shimura, Deligne, Serre.

From Geometry to Number Theory and Back

Geometry/Analysis		**Number Theory**
Modular Forms		**Galois Representations**



Hecke eigenforms $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ with $a_1 = 1$	$\longrightarrow$ $\longleftarrow$	Galois repres. $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ s.t. $\det(\overline{\rho}(\text{compl. conj.})) = -1$
f level N	$\mapsto$	$\overline{\rho}_f$ unramified outside $N\ell$ $\mathrm{Tr}(\overline{\rho}_f(\mathrm{Frob}_p)) = a_p$

Shimura, Deligne, Serre.

Serre's Conjecture (1987).

Khare, Wintenberger, Kisin (2009).

The fabulous world of modular forms

From the front page of the
New York Times of 24 June 1993

At Last, Shout of 'Eureka!' In Age-Old Math Mystery

By GINA KOLATA

More than 350 years ago, a French mathematician wrote a deceptively simple theorem in the margins of a book, adding that he had discovered a marvelous proof of it but lacked space to include it in the margin. He died without ever offering his proof, and mathematicians have been trying ever since to supply it.

Now, after thousands of claims of success that proved untrue, mathematicians say the daunting challenge, perhaps the most famous of unsolved mathematical problems, has at last been surmounted.

The problem is Fermat's last theorem, and its apparent conqueror is Dr. Andrew Wiles, a 40-year-old English mathematician who works at Princeton University. Dr. Wiles announced the result yesterday at the last of three lectures given over three days at Cambridge University in England.

Within a few minutes of the conclusion of his final lecture, computer mail messages were winging around the world as mathematicians alerted each other to the startling and almost wholly unexpected result.

Dr. Leonard Adelman of the University of Southern California said he received a message about an hour after Dr. Wiles's announcement. The frenzy is justified, he said. "It's the most exciting thing that's happened in — geez — maybe ever, in mathematics."

Impossible Is Possible

Mathematicians present at the lecture said they felt "an elation," said Dr. Kenneth Ribet of the University of California at Berkeley, in a telephone interview from Cambridge.

The theorem, an overarching statement about what solutions are possible for certain simple equations, was stated in 1637 by Pierre de Fermat, a 17th-century French mathematician and physicist. Many of the brightest minds in mathematics have struggled to find the proof ever since, and many have concluded that Fermat, contrary to his tantalizing claim, had probably failed to develop one despite his considerable

Continued on Page D22, Column 1

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Pierre de Fermat, whose theorem may have been proved.

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Dr. Andrew Wiles explaining his proof of Fermat's last theorem yesterday at Cambridge University in England.

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At Last, a 'Eureka!' in an Age-Old Math Mystery

Continued From Page A1

Dr. Andrew Wiles explaining his proof of Fermat's last theorem yesterday at Cambridge University in England.

Fermat's Last Theorem:

$$a^n + b^n = c^n$$

for $n \geq 3$,

$$a, b, c \in \mathbb{Z}_{>0}$$

impossible!!!

Thank you for your attention!