Modular Forms and the Inverse Galois Problem

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Inverse Galois Problem

Question of Hilbert:

Given a finite group G.
Is there a Galois extension $K/\mathbb{Q}$ such that

$$\mathrm{Gal}(K/\mathbb{Q}) \cong G?$$

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In this talk focus on two cases:

- The GL_2-case: $G = \mathrm{PSL}_2(\mathbb{F}_{\ell^d})$.
- The GSp_{2n}-case: $G = \mathrm{PSp}_{2n}(\mathbb{F}_{\ell^d})$.

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Introduction: GL_2-case

Consider a cuspidal modular form

$$f = \sum_{n=1}^{\infty} a_n q^n \qquad (q = e^{2\pi i z})$$

s.t. $a_1 = 1$ (normalised), Hecke eigenform, no CM, any weight, on $\Gamma_1(N)$, nebentype $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

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Theorem (Deligne, Shimura, Eichler, Igusa, Serre).

For each prime ℓ, $\exists$ Galois representation

$$\overline{\rho}_{f,\ell}^{\mathrm{proj}} : G_{\mathbb{Q}} \xrightarrow{\overline{\rho}_{f,\ell}} \mathrm{GL}_2(\overline{\mathbb{F}}_\ell) \xrightarrow{\text{nat. proj.}} \mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$$

unramified outside $N\ell$ such that for all $p \nmid N\ell$

$$\mathrm{Tr}(\overline{\rho}_{f,\ell}(\mathrm{Frob}_p)) \equiv a_p \text{ ‘mod } \ell\text{'}.$$

One speaks of a compatible system.

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(II) Image of $\overline{\rho}_{f,\ell}^{\mathrm{proj}}$?

Note: $\mathrm{Gal}(\overline{\mathbb{Q}}^{\ker(\overline{\rho}_{f,\ell}^{\mathrm{proj}})}/\mathbb{Q}) \cong \overline{\rho}_{f,\ell}^{\mathrm{proj}}(G_{\mathbb{Q}})$.

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(III) Prove the existence of f such that for fixed ℓ, d:

$$\overline{\rho}_{f,\ell}^{\mathrm{proj}}(G_{\mathbb{Q}}) \cong \mathrm{PSL}_2(\mathbb{F}_{\ell^d}),$$

i.e. realise $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ as Galois group over $\mathbb{Q}$.

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Answer:

If $\overline{\rho}_{f,\ell}$ is irreducible, then $\overline{\rho}_{f,\ell}^{\mathrm{proj}}$ can be defined over residue field (above ℓ) of the <u>global</u> field $\mathbb{Q}(\frac{a_p^2}{\psi(p)} \mid p \nmid N)$

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Answer:

From (I): $\overline{\rho}_{f,\ell}^{\mathrm{proj}}$ definable over $\mathbb{F}_{\ell^d}$. By **Dickson ($\sim$ 1900)**:

$\overline{\rho}_{f,\ell}^{\mathrm{proj}}(G_{\mathbb{Q}})$ is

- $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$, $\mathrm{PGL}_2(\mathbb{F}_{\ell^d})$
- dihedral
- $\subseteq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$
- A_4, S_4, A_5

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Ribet: For almost all ℓ: huge image.

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(III) Prove the existence of f such that for fixed ℓ, d:

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i.e. realise $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ as Galois group over $\mathbb{Q}$.

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Partial Answers:

Theorem A (W. 2008). Given ℓ, $\exists$ infinitely many d s.t. $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ occurs as $\overline{\rho}_{f,\ell}^{\mathrm{proj}}(G_{\mathbb{Q}})$ (for some f depending on d) with only ℓ and one other prime (dep. on d) ramifying.

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Theorem B (Dieulefait, W. 2011). Given d, $\exists$ positive density set of primes $\mathcal{L}$ s.t. $\forall\, \ell \in \mathcal{L}$: $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ occurs as $\overline{\rho}^{\mathrm{proj}}_{f,\ell}(G_{\mathbb{Q}})$ with only ℓ and at most three other primes (not dep. on ℓ) ramifying.

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Partial Answers:

Theorem C (W. 2012). Given d even. Assume *Maeda's conjecture*. Then the density of the set of primes ℓ such that $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ occurs as $\overline{\rho}_{f,\ell}^{\mathrm{proj}}(G_{\mathbb{Q}})$ with only ℓ ramifying is 1.

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Maeda's conjecture. Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(1)$ be a level 1 newform of any weight k. Let $\mathbb{Q}_f := \mathbb{Q}(a_2, a_3, a_4, \dots)$. Then

- $[\mathbb{Q}_f : \mathbb{Q}] = \dim_{\mathbb{C}} S_k(1) =: d_k$ and
- the Galois group of splitting field of $\mathbb{Q}_f/\mathbb{Q}$ is Sym_{d_k}.

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The conjecture has been veried numerically for $k \le 12000$ (work of Ghitza and student).

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Reasons behind the proof:

- $K/\mathbb{Q}$, deg $n > d$, Galois gp Sym_n has subfields of deg d.
- If K and L two such (with Sym_m, Sym_n, $m > n \geq 5$), then $K \cap L$ at most quadratic (A_n simple!).
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Generalisation to GSp_{2n} any n:

Theorem A (Khare, Larsen, Savin, 2008).
Given ℓ, $\exists$ infinitely many d s.t. $\mathrm{PSp}_{2n}(\mathbb{F}_{\ell^d})$ or $\mathrm{PGSp}_{2n}(\mathbb{F}_{\ell^d})$ occurs as image of the residual Galois representation attached to a suitable automorphic form on GL_{2n} over $\mathbb{Q}$.

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Joint work with Sara Arias-de-Reyna and Luis Dieulefait:

(I) Determine projective field of definition of compatible system of symplectic Galois representations. (DONE. Explain now.)

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(III) Generalise Theorem B.
(ALMOST DONE, subject to a 'promised theorem' by others).

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Inner twists

Let K be a field, $\overline{K}$ separable closure. Consider:

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- Let $\epsilon : G_{\mathbb{Q}} \to \overline{K}^{\times}$ char. $\Rightarrow (\rho \otimes \epsilon)^{\mathrm{proj}} = \rho^{\mathrm{proj}}$.

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- Suppose $\rho_1^{\text{proj}} \sim \rho_2^{\text{proj}}$.

  Put $\epsilon(g) := M^{-1}\rho_1(g) M \rho_2(g)^{-1} \in \overline{K}^{\times}$.

  $\Rightarrow \rho_1 \sim \rho_2 \otimes \epsilon$.

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Galois action on coefficients: for $\sigma \in G_K$ consider
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ρ has complex multiplication (CM) if $\sigma = \mathrm{id}$, $\epsilon \neq 1$.

Suppose ρ is irreducible and has no CM. Then:

${}^\sigma\rho \sim \rho \otimes \epsilon \Leftrightarrow$
$\sigma(\mathrm{Tr}(\rho(\mathrm{Frob}_p))) = \mathrm{Tr}(\rho(\mathrm{Frob}_p))\epsilon(\mathrm{Frob}_p)$ $\forall$ unramified p.

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Def.: $H_\rho := \bigcap_\epsilon \ker(\epsilon) \lhd G_\mathbb{Q}$ for ϵ occuring in an inner twists.
$\Gamma_\rho := \{\sigma \in G_K \mid \sigma$ occurs in an inner twist$\}$.
$K_\rho := \overline{K}^{\Gamma_\rho}$, called projective field of definition of ρ.

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$K_\rho := \overline{K}^{\Gamma_\rho}$, called projective field of definition of ρ.

Theorem (Arias-de-Reyna, Dieulefait, W., 2012).
Suppose $\rho|_{H_\rho}$ is irreducible. Then:

(1) $\exists\, \rho'$ such that $\rho'^{\mathrm{proj}} \sim \rho^{\mathrm{proj}}$ and ρ'^{proj} factors through K_ρ.

(2) K_ρ is the smallest subfield of $\overline{K}$ with this property.

Morale: The inner twists determine the smallest field over which ρ^{proj} can be defined.

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Compatible systems

Let $n \in \mathbb{N}$, $L/\mathbb{Q}$ Galois number field, $N, k \in \mathbb{N}$, $\psi : G_{\mathbb{Q}} \to L^{\times}$, for all $p \nmid N$: $P_p(X) = X^{2n} - a_p X^{2n-1} + \cdots \in L[X]$.

A compatible system $\rho_{\bullet}$ is:

Compatible systems

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A compatible system $\rho_\bullet$ is:
for each λ place of L a Galois representation

$$\rho_\lambda : G_{\mathbb{Q}} \to \mathrm{GSp}_{2n}(L_\lambda) \text{ such that}$$

- abs. irred., unramified outside $N\ell$ (for $\Lambda \mid \ell$),
- $\forall\, p \nmid N\ell : \mathrm{charpoly}(\rho_\lambda(\mathrm{Frob}_p)) = P_p$,
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We consider: $\overline{\rho}_\lambda$ (residual representation), $\rho_\lambda^{\mathrm{proj}}$, and $\overline{\rho}_\lambda^{\mathrm{proj}}$.

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Def.: (σ, ϵ) (with $\sigma \in \mathrm{Gal}(L/K)$ and $\epsilon : G_{\mathbb{Q}} \to L^\times$) inner twist of $\rho_\bullet$ if

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Def.: $\Gamma_{\rho_\bullet} := \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma$ occurs in an inner twist of $\rho_\bullet\}$.
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Theorem 1 (Arias-de-Reyna, Dieulefait, W., 2012).
Assume moreover: $\rho_\bullet$ is *strictly compatible with regular Hodge-Tate weights* and $\overline{\rho}_\lambda$ is absolutely irreducible for almost all λ.
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Morale: The global field $K_{\rho_\bullet}$ (depending only on the inner twists) determines the projective field of definition of $\overline{\rho}_\lambda^{\mathrm{proj}}$.

This field is the GSp_{2n}-replacement of $\mathbb{Q}(\frac{a_p^2}{\psi(p)} \mid p \nmid N)$.

Classification result

Theorem 2 (Arias-de-Reyna, Dieulefait, W., 2012).
Let $\ell \geq 5$ and $\overline{\rho} : G \to \mathrm{GSp}_{2n}(\overline{\mathbb{F}}_\ell)$ be irreducible.
Assume: $\overline{\rho}(G)$ contains a non-trivial transvection.

Then either $\overline{\rho}(G) \supseteq \mathrm{PSp}_{2n}(\mathbb{F}_\ell)$ (huge image)
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Morale: Our replacement of Dickson's theorem for GL_2:

Recall: $\overline{\rho}^{\mathrm{proj}}_{f,\ell}(G_\mathbb{Q})$ is

• $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$, $\mathrm{PGL}_2(\mathbb{F}_{\ell^d})$	huge image
• dihedral	induced
• $\subseteq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$	reducible
• A_4, S_4, A_5	exceptional

Inverse Galois Problem

Theorem 3 (Arias-de-Reyna, Dieulefait, W., 2012).
Let $\rho_\bullet$ be as in Theorem 1. Assume moreover:

- $\overline{\rho}_\lambda(G_\mathbb{Q})$ contains a transvection for almost all λ.
- *'Good dihedral prime' (Khare, Wintenberger, Larsen, Savin)*:
  $\exists$ prime q, $\exists$ suitable character $\delta : G_{\mathbb{Q}_{q^{2n}}} \to L^\times$ of order $2t$ (t prime), $2n \mid (t-1)$ such that $\overline{\rho}_\lambda|_{G_{\mathbb{Q}_q}} \sim \mathrm{Ind}_{G_{\mathbb{Q}_{q^{2n}}}}^{G_{\mathbb{Q}_q}}(\delta)$.

Then for all $d \mid \frac{t-1}{2n}$, the set of places λ of L such that $\mathrm{PSp}_{2n}(\mathbb{F}_{\ell^d})$ or $\mathrm{PGSp}_{2n}(\mathbb{F}_{\ell^d})$ equals $\overline{\rho}_\lambda^{\mathrm{proj}}(G_\mathbb{Q})$ has a positive density.

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Morale: If such a $\rho_\bullet$ exists, then we obtain the desired application to the inverse Galois problem.

Thank you for your attention.