The Lie algebra of an algebraic group

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Contents

1 Introduction

The aim of this talk is to give a geometric description of the Lie algebra attached to an algebraic group. We will first give a rather naive intuitive idea, which we will later make precise.

In this introduction the reader will neither find precise definitions, nor satisfactory explanations for the calculations performed. In this spirit we shall say:

Definition 1.0.1 (Naive idea of a Lie algebra) The *Lie algebra* $\mathrm{Lie}(G) = \mathfrak{g}$ attached to an algebraic group G is

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- seen as a vector space equal to the *tangent space at the unit element* $\mathcal{T}_G(1)$, i.e. the first order approximation of G, and it is
- equipped with a natural bracket reflecting a second order effect coming from a group commutator.

We shall now try to illustrate this by considering some important examples.

- *General linear group* GL_n:

  We regard GL_n as a subspace of the set M_n of all $n \times n$-matrices M_n. As GL_n is the complement of the hypersurface defined by $\det = 0$ it is an open subset of M_n. Consequently, its tangent space is all of it: $\mathfrak{gl}_n = M_n = \mathcal{T}_{\mathrm{GL}_n}(1)$.

  There seems to be just one "canonical" choice for the Lie bracket, namely

  $$[A, B] := AB - BA.$$

- *Special linear group* SL_n:

  We also consider SL_n as a subset, this time closed, of M_n. Very naively, the tangent space of SL_n at 1 consists of those matrices $A \in M_n$ such that $1 + \epsilon A \in \mathrm{SL}_n$ for ϵ "infinitesimally" small. Thus we get:

  $$\begin{aligned} \mathfrak{sl}_n = \mathcal{T}_{\mathrm{SL}_n}(1) &= \{ \; A \in M_n \mid 1 = \det(1 + \epsilon A) = 1 + \epsilon \operatorname{Tr}(A) + \epsilon^2 \dots \} \\ &= \{ \; A \in M_n \mid \; \operatorname{Tr}(A) = 0 \; \}, \end{aligned}$$

  neglecting the terms ϵ^r for $r \geq 2$. There is again one canonical way to define the Lie bracket, namely the same as for the general linear group. We only have this choice, if we want that a Lie algebra of a closed subgroup is also a sub Lie algebra. This will indeed be the case. The dimension of the Lie algebra is $n^2 - 1$.

  Now we illustrate what the Lie bracket has to do with second order effects and group commutators. Take two tangent vectors $1 + \epsilon_1 A$ and $1 + \epsilon_2 B$ and calculate their group commutator (modulo ϵ_1^2 and ϵ_2^2)

  $$\begin{aligned} &(1 + \epsilon_1 A)(1 + \epsilon_2 B)(1 + \epsilon_1 A)^{-1}(1 + \epsilon_2 B)^{-1} \\ \equiv &(1 + \epsilon_1 A)(1 + \epsilon_2 B)(1 - \epsilon_1 A)(1 - \epsilon_2 B) \\ \equiv &1 + \epsilon_1 \epsilon_2 (AB - BA) = 1 + \epsilon_1 \epsilon_2 [A, B]. \end{aligned}$$

- *Orthogonal group* O_n:

  We recall the definition of the orthogonal group (and assume that the characteristic is not 2):

  $$\mathrm{O}_n = \{ A \in \mathrm{GL}_n \mid A^T A = 1 \}$$

  In other words, the orthogonal group consists of those matrices respecting the standard bilinear form.

  In order to determine the tangent vectors, we need to calculate all matrices A such that $1 + \epsilon A \in \mathrm{O}_n$, i.e.

  $$1 = (1 + \epsilon A)^T (1 + \epsilon A) = 1 + \epsilon (A^T + A) + \epsilon^2 A^T A.$$

  Since we neglect the ϵ^2-term, we arrive at

  $$\mathfrak{o}_n = \mathcal{T}_{\mathrm{O}_n}(1) = \{ A \in M_n \mid A^T + A = 0 \},$$

  the skew-symmetric matrices. Hence the dimension is $\frac{n^2-n}{2}$.

  The Lie bracket is again the standard one, for the same reasons as above.

- *Symplectic group* SP_{2n}:

  Let M be the $2n \times 2n$-matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The symplectic group is defined as

  $$\mathrm{SP}_{2n} = \{ A \in \mathrm{GL}_{2n} \mid A^T M A = M \}.$$

  In other words, it consists of those matrices respecting the standard skew-symmetric bilinear form.

  Lets write $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$. Then the tangent condition is

  $$\begin{aligned} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= (1 + \epsilon \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix})^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (1 + \epsilon \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}) \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \epsilon (\begin{pmatrix} -A_3^T & A_1^T \\ -A_4^T & A_2^T \end{pmatrix} + \begin{pmatrix} A_3 & A_4 \\ -A_1 & -A_2 \end{pmatrix}) + \epsilon^2 \ldots \end{aligned}$$

  Consequently, we find

  $$\mathfrak{sp}_{2n} = \mathcal{T}_{\mathrm{SP}_{2n}}(1) = \{ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathrm{GL}_{2n} \mid A_1 = -A_4^T,\ A_2 = A_2^T,\ A_3 = A_3^T \}.$$

  Thus the dimension is $n^2 + n + n^2 = 2n^2 + n$.

We have seen that with our naive point of view, we have obtained interesting, and also the most important Lie algebras, without any effort.

Maybe, this is a good point to complement the picture with the Lie algebras of $\mathbb{G}_m$ and $\mathbb{G}_a$, since we have looked a lot at those groups before. Both are immediately seen to be 1-dimensional (and smooth); in fact they are a hyperbola resp. a line. Hence, their tangent spaces are 1-dimensional. But there is just one one-dimensional Lie algebra. Its bracket satisfies $[x, y] = 0$ for all x, y. So it is by definition an *abelian* Lie algebra.

More generally we can say that abelian subgroups of GL_n always give abelian Lie algebras, because group commutators are trivial.

2 Tangent spaces

We are now going define the tangent space at a k-rational point of a scheme over k in different manners.

2.1 Concept of points

Since the concept of points of (affine) schemes will be of central interest to us, we shall recall it briefly (however not in utmost generality).

Let k be a field and X a scheme over $\operatorname{Spec} k$, by which we mean that there is a morphism of schemes $X \to \operatorname{Spec} k$. Given a k-algebra R, we define the *set of R-points* to be the set $X(R) = \mathrm{Mor}_{\operatorname{Spec} k}(\operatorname{Spec} R, X)$, which by definition consists of the morphisms of schemes satisfying the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} R & \xrightarrow{P} & X \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} k & \end{array}$$

If $X = \operatorname{Spec} A$ is an affine scheme, then an R-point is equivalent to giving a commutative diagram of k-algebras

$$\begin{array}{ccc} R & \xleftarrow{P^*} & A \\ & \nwarrow \quad \nearrow & \\ & k & \end{array}$$

An important special case is that of k-rational points. They are precisely the sections

$$X \underset{P}{\overset{\longrightarrow}{\longleftarrow}} \operatorname{Spec} k.$$

By looking at the stalk at P of the structure sheaf, we obtain an exact sequence

$$0 \to \mathfrak{m}_P \to \mathcal{O}_X(P) \to k \to 0.$$

The local ring $\mathcal{O}_X(P)$ can be thought of as the ring of quotients f/g of two polynomials over k defined in a neighbourhood of P such that $g(P) \neq 0$. Then the map P^* is precisely the evaluation of this function at P and consequently the maximal ideal $\mathfrak{m}_P$ consists of those functions vanishing at P. We illustrate this by looking at the affine case $X = \operatorname{Spec} A$, where A is $k[X_1, \ldots, X_n]/I$. Then giving a k-rational point means precisely giving a point $P = (p_1, \ldots, p_n)$ in the affine n-space. This point is uniquely determined by $p_i = P^*(X_i)$, i.e. by the exact sequence

$$0 \to \mathfrak{p}_P \to k[X_1, \ldots, X_n]/I \to k \to 0.$$

The last map can now be seen as evaluation at a point $P = (p_1, \ldots, p_n)$.

The concept of a point has already frequently been used in this seminar, when defining, what we do again here, an *affine group scheme over* k as a functor

$$G : \{\ k\text{-algebras}\ \} \quad \to \quad \{\ \text{groups}\ \}$$

such that $\{\ k\text{-algebras}\ \} \xrightarrow{G} \{\ \text{groups}\ \} \xrightarrow{\text{forget}} \{\ \text{sets}\ \}$ is representable by a k-algebra A of finite type. This, however, means precisely that the set of R-points of the affine scheme $G := \operatorname{Spec} A$ for each R form a group, namely $G(R)$, and that a homomorphism $R \to S$ gives us a group homomorphism $G(R) \to G(S)$.

2.2 Tangent space in affine setting

The tangent space should become a local object, so we can define it in an affine setting. However, we wish to illustrate the definition by beginning with the special case $X = \operatorname{Spec} k[X]$, where $k[X] = k[X_1, \ldots, X_n]/I$. In other words, we consider X to be embedded into affine n-space over k. Let us fix a k-rational point $P = (p_1, \ldots, p_n) \in X$.

A tangent through P should be a line "touching" X at P, i.e. intersecting it with multiplicity greater equal 2.

Let a tangent be given by the points $P + t\underline{v}$ for $t \in k$, where $\underline{v} = (v_1, \ldots, v_n) \in k^n$, the *slope* of the tangent. Now we intersect it with X, i.e. we want to know the t, where $f(P + t\underline{v})$ is zero. For this we expand the expression as a polynomial in t (now considered as a formal parameter)

$$f(P + t\underline{v}) = f(P) + t < J(f), \underline{v} > + t^2(\cdot) + t^3(\cdot) + \ldots,$$

where $J(f)$ denotes the column vector consisting of $\frac{\partial f}{\partial X_i}(P)$ and $< a, b >= \sum_i a_i b_i$. Obviously, $t = 0$ is a zero.

We now say that $\underline{v}$ is the *slope of a tangent at P* if $t = 0$ is at least a double zero for all $f \in I$, by which we mean that

$$< J(f), \underline{v} >= 0 \quad \text{for all } f \in I.$$

We define the *tangent space at* P, denoted $\mathcal{T}_X(P)$, (in the currently considered setting) as the set of all the slopes v.

Remark 2.2.1 If $I = (f_1, \ldots, f_m)$, then it is enough to check $< J(f_i), v >= 0$ for the generators f_i. Let us define a matrix consisting of the columns $J(f_1), J(f_2), \ldots, J(f_m)$. Denote by J the transpose of this matrix. It is called *the Jacobian matrix*. Then by definition the tangent space at P is the kernel of J. Thus, its dimension is n minus the rank of J.

We notice that the function

$$k[X] \to k, \quad f \mapsto < J(f), v >$$

makes sense for all $v \in \mathcal{T}_X(P)$. An easy calculation shows that it defines a k-derivation $k[X] \to k$, where k is a $k[X]$-module by $f.x = f(P)x$. We briefly recall what that means.

Definition 2.2.2 Let R be an k-algebra and M an R-module. Then the set $\mathrm{Der}_k(R, M)$ of *k-derivations* $R \to M$ consists of the maps $D : R \to M$ satisfying

(i) $D(r_1 + r_2) = D(r_1) + D(r_2)$ for all $r_1, r_2 \in R$,

(ii) $D(x) = 0$ for all $x \in k$ and

(iii) $D(r_1 r_2) = r_1.D(r_2) + r_2.D(r_1)$ for all $r_1, r_2 \in R$.

2.3 Tangent space at P as the set of $k[\epsilon]$-points over P

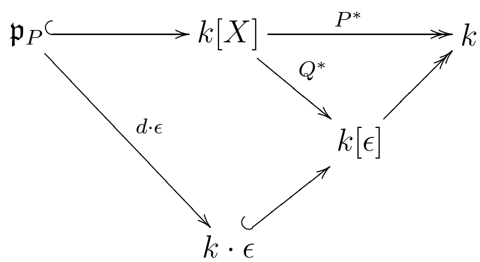
Let us for a moment stay in the setting from the previous section. We shall now give a very elegant description of the tangent space as the set of certain points lying over P.

Recall that we chose $P \in X(k)$ to be a k-rational point. We define the k-algebra $k[\epsilon] = k[X]/(X^2)$ and a k-algebra homomorphism $k[\epsilon] \to k$ by mapping ϵ to 0. The latter gives us a map $X(k[\epsilon]) \to X(k)$.

We want to prove the equality

$$\boxed{\mathcal{T}_X(P) = \{ \, Q \in X(k[\epsilon]) \mid Q \mapsto P \, \}.}$$

For that we consider the following commutative diagram.



We can always write $Q^*(f) = f(P) + \epsilon d(f)$ and take this equality as the definition of d. A straightforward calculation shows that d is a derivation, more precisely $d \in \mathrm{Der}_k(k[X], k)$, where again k is considered as a $k[X]$-module via evaluation at P. Conversely, if we are given such a derivation d, then we can define a k-algebra homomorphism $Q^* : k[X] \to k[\epsilon]$ by setting $Q^*(f) = f(P) + \epsilon d(f)$.

Hence, we receive

$$\boxed{\mathrm{Der}_k(k[X], k) = \{\ Q \in X(k[\epsilon]) \mid Q \mapsto P\ \}.}$$

Yet another description of these objects is easily obtained. First, we note that Q^* is uniquely determined by its values on $\mathfrak{p}_P$, because $Q^*(f) = f(P) + \epsilon d(f) = f(P) + \epsilon d(f - f(P))$ as R^* is a derivation. Next, it is straightforward that Q^* factors through $\mathfrak{p}_P^2$. On the other hand, any k-linear map $\phi : \mathfrak{p}_P/\mathfrak{p}_P^2 \to k$ defines a valid Q^* by setting $Q^*(f) = f(P) + \epsilon\phi(f)$. This yields the equality

$$\boxed{(\mathfrak{p}_P/\mathfrak{p}_P^2)^\vee = \mathrm{Hom}_{k-v.s.}(\mathfrak{p}_P/\mathfrak{p}_P^2, k) = \{\ Q \in X(k[\epsilon]) \mid Q \mapsto P\ \}.}$$

We associated to an element $\underline{v} \in \mathcal{T}_X(P)$ a derivation $f \mapsto < J(f), \underline{v} >$. Following the maps described above, we find an injection $\mathcal{T}_X(P) \to (\mathfrak{p}_P/\mathfrak{p}_P^2)^\vee$ sending a tangent slope $\underline{v}$ to $f \mapsto < J(f), \underline{v} >$, which by duality of vector spaces corresponds to the surjective homomorphism

$$\mathfrak{p}_P/\mathfrak{p}_P^2 \to (\mathcal{T}_X(P))^\vee, \ \ f \mapsto (\underline{v} \mapsto < J(f), \underline{v} >).$$

However, it is easy to see that it is in fact an isomorphism. This is immediate from the fact that $\mathfrak{p}_P^2 = \{\ f \in \mathfrak{p}_P \mid J(f) = 0\ \}$.

Hence we have finished proving all the boxed equalities.

In order to get rid of the assumption that X is embedded in affine n-space we note that via localization at the maximal ideal $\mathfrak{p}_P$ we receive an isomorphism $\mathfrak{p}_P/\mathfrak{p}_P^2 \cong \mathfrak{m}_P/\mathfrak{m}_P^2$, where $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_X(P)$.

Let X now be any scheme over k and $P \in X(k)$ a k-rational point. Then we define the *tangent space of X at P* as

$$\boxed{\mathcal{T}_X(P) = (\mathfrak{m}_P/\mathfrak{m}_P^2)^\vee = \mathrm{Der}_k(\mathcal{O}_X(P), k) = \{\ Q \in X(k[\epsilon]) \mid Q \mapsto P\ \}.}$$

Here again, k is considered as a $\mathcal{O}_X(P)$-module via P^*. The isomorphisms are seen by exactly the same arguments as above.

We can state various facts arising directly from the various equivalent definitions.

(i) The tangent space is a local object and, in particular, does not depend on an embedding into affine space.

(ii) The tangent space is a k-vector space.

(iii) A morphism of schemes $\phi : X \to Y$ induces a homomorphism of tangent spaces: $\mathcal{T}_X(P) \to \mathcal{T}_Y(\phi(P))$ (e.g. consider the definition by $k[\epsilon]$-points).

(iv) A closed immersion of k-schemes $X \to Y$ induces an injective homomorphism $\mathcal{T}_X(P) \to \mathcal{T}_Y(\phi(P))$. Closed immersion means that the map on the stalks of the structure sheaves is surjective implying that the map between the spaces of derivations is an injection.

(v) For products we have $\mathcal{T}_{X\times_k Y}((P,Q)) = \mathcal{T}_X(P) \oplus \mathcal{T}_Y(Q)$.

3 The Lie algebra of a group scheme

We illustrate the use of $k[\epsilon]$-points in the case of a closed subgroup G of some GL_n. It can hence be described by equations defining an ideal $I = (f_1, \dots, f_m)$. We have

$$A \in \mathcal{T}_G(1) \Leftrightarrow < J(f_i), A >= 0\ \forall i \Leftrightarrow f_i(1+\epsilon A) = 0\ \forall i \Leftrightarrow 1 + \epsilon A \in G(k[\epsilon]),$$

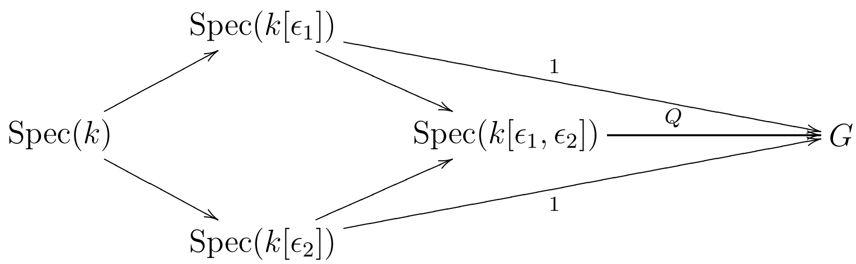
where the f_i are now polynomials in $k[\epsilon][X]$. We know how to multiply two such points $1+\epsilon A$ and $1+\epsilon B$, namely by multiplying out. We had also seen that the group commutator of those gives us the Lie bracket as the term of $\epsilon_1\epsilon_2$.

Let G now be a group scheme over a field k. We recall that as a k-vector space we define the Lie algebra as

$$\mathfrak{g} = \mathrm{Lie}(G) = \mathcal{T}_G(1) = \mathrm{Ker}(G(k[\epsilon]) \to G(k)) = \{\ Q \in G(k[\epsilon]) \mid Q \mapsto P\ \}.$$

Consequently, it carries a group structure, which can be shown to be the vector space addition.

Now we define the Lie bracket. Let $P_1, P_2 \in T_G(1)$ be given. Consider them as elements $P_1 \in G(k[\epsilon_1])$ resp. $P_2 \in G(k[\epsilon_2])$. For $i = 1, 2$ we consider the natural k-algebra homomorphisms $k[\epsilon_i] \to k[\epsilon_1, \epsilon_2]$. They allow us to consider P_1 and P_2 as elements $P_1', P_2' \in G(k[\epsilon_1, \epsilon_2])$. Denote by Q the group commutator $P_1 P_2 P_1^{-1} P_2^{-1} \in G(k[\epsilon_1, \epsilon_2])$. Under the natural maps $G(k[\epsilon_1, \epsilon_2]) \to G(k[\epsilon_i])$ the point Q is mapped to 1. We can illustrate the points by the following commutative diagram.



This shows that the point Q factors as

$$\mathrm{Spec}(k[\epsilon_1, \epsilon_2]) \xrightarrow{\text{natural}} \mathrm{Spec}(k[\epsilon_1 \cdot \epsilon_2]) \xrightarrow{R} G.$$

We define the Lie bracket $[P_1, P_2]$ to be this point R. From the diagram above it also follows that it lies over 1. Hence $[P_1, P_2]$ is again an element of the tangent space $\mathcal{T}_G(1)$.

We had seen that closed immersions give rise to injections on the tangent spaces. The above construction shows moreover that in the case of a closed subgroup scheme $H \leq G$ the Lie brackets agree, yielding $\mathrm{Lie}(H) \leq \mathrm{Lie}(G)$. More generally, a morphism of group schemes $H \to G$ gives us for the same reasons a Lie algebra homomorphsim on the Lie algebras.

We still have to check that the Lie bracket satisfies the Jacobi identity and so on. I did not see a direct way to do that from the definition. As we are concerned with affine group schemes, which can be embedded into some GL_n, the properties needed result from those on GL_n, which are trivially true.

4 Relation with left invariant derivations for affine group schemes

Intuitively the situation is as follows. A tangent vector at 1 can be carried to a tangent vector at g by group translation. This gives us a global derivation, which is clearly left invariant. On the other hand given such a left invariant derivation, it uniquely determines a tangent vector at 1. I have not worked that out in detail. A proof based on calculations in the Hopf algebra can be found in Waterhouse's book.

One should also prove that the Lie bracket of two global derivations A, B corresponds to $A \circ B - B \circ A$. This is also done in the afore mentioned book.