Asymptotic behaviour of arithmetic properties of coefficients of modular forms

Gabor Wiese

Institut für Experimentelle Mathematik

Universität Duisburg-Essen

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Contents/Goals

- Common point of view on Sato-Tate and the Chebotarev density theorem.

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- Discuss ’desirable consequences’ (why should we care?).

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- Presentation of the ’orthogonal’ set-up.
- Raise the question on the ’correct analog’ in the ’orthogonal set-up’.
- Discuss ’desirable consequences’ (why should we care?).
- Calculations (some by Marcel Mohyla) leading to interesting questions.

Theory on the black board.

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Now our computations.

Computations

We will study:

- Sum of degrees $[\mathbb{F}_{p,[\overline{f}]} : \mathbb{F}_p]$ for all $[\overline{f}]$ in a given level N and weight k.

  ($\rightsquigarrow$ related to congruences and other phenomena.)

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These quantities will be compared to $\dim S_k(N)$, the dimension of the space of cusp forms for level N and weight k.

Congruences mod p et al.

We fix a (prime) level N and a weight k.

Consider the sum of residue degrees

$$\deg_k^{(p)}(N) = \sum_{[\overline{f}]} [\mathbb{F}_{p,[\overline{f}]} : \mathbb{F}_p]$$

where $[\overline{f}]$ runs through the $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-conjugacy classes of newforms of level N and weight k.

We will compare this with $\dim S_k(N)$.

Congruences mod p et al.

Theorem. $\dim S_k(N) = \deg_k^{(p)}(N) \overset{\text{def}}{=} \sum_{[\overline{f}]} [\mathbb{F}_{p,[\overline{f}]} : \mathbb{F}_p]$

(the mod p Hecke algebra is semisimple) $\Leftrightarrow$

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One could expect that strict inequality $\dim S_k(N) > \deg_k^{(p)}(N)$ is a rare phenomenon.

Is that true?

Congruences mod p et al.

Let us fix the prime p and the weight $k = 2$.

Plot $\deg_k^{(p)}(N)$ as a function of $\dim S_k(N)$ for all prime levels $N \leq 2000$.

First, let p be odd.

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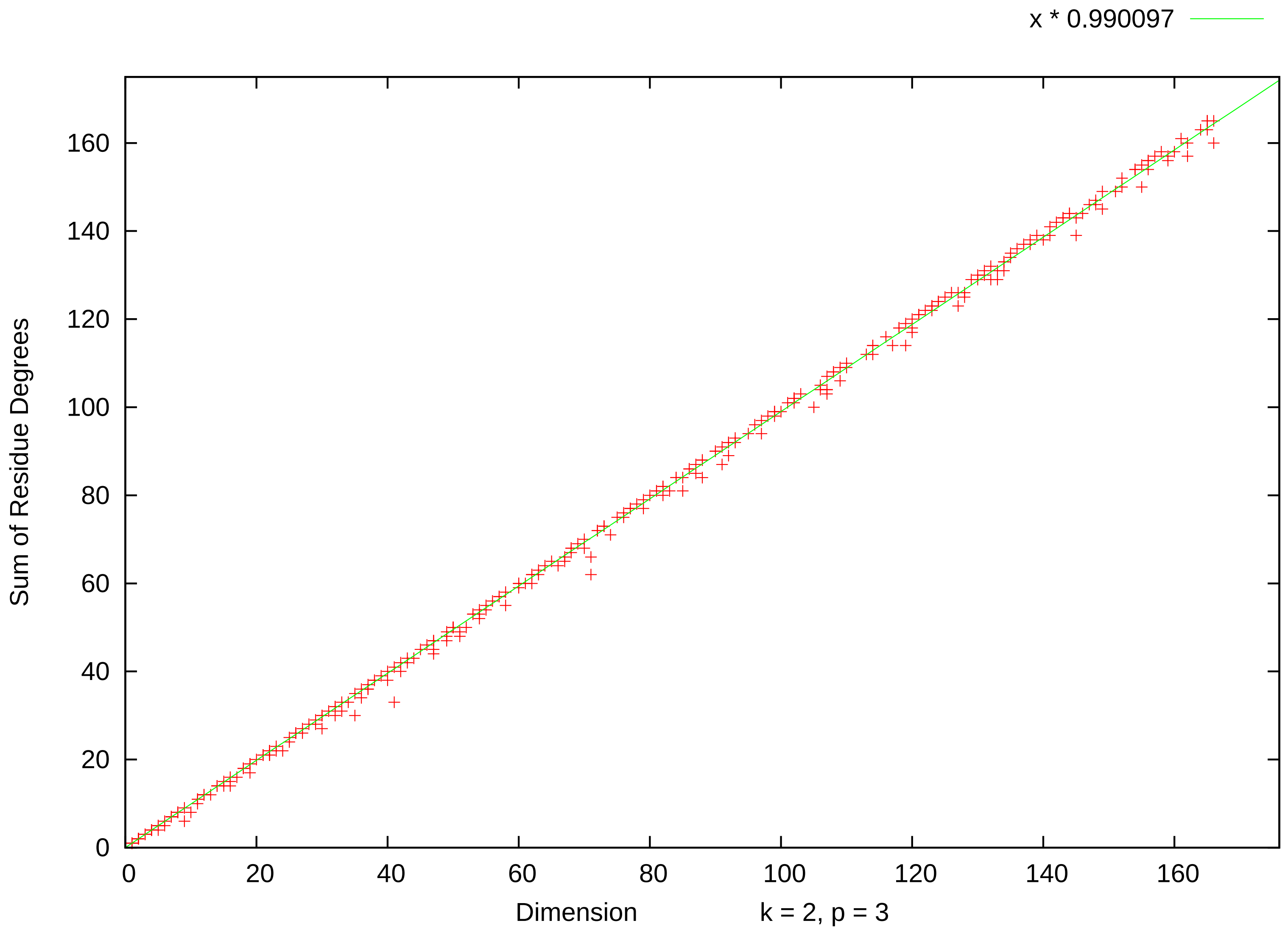
Congruences mod p et al.



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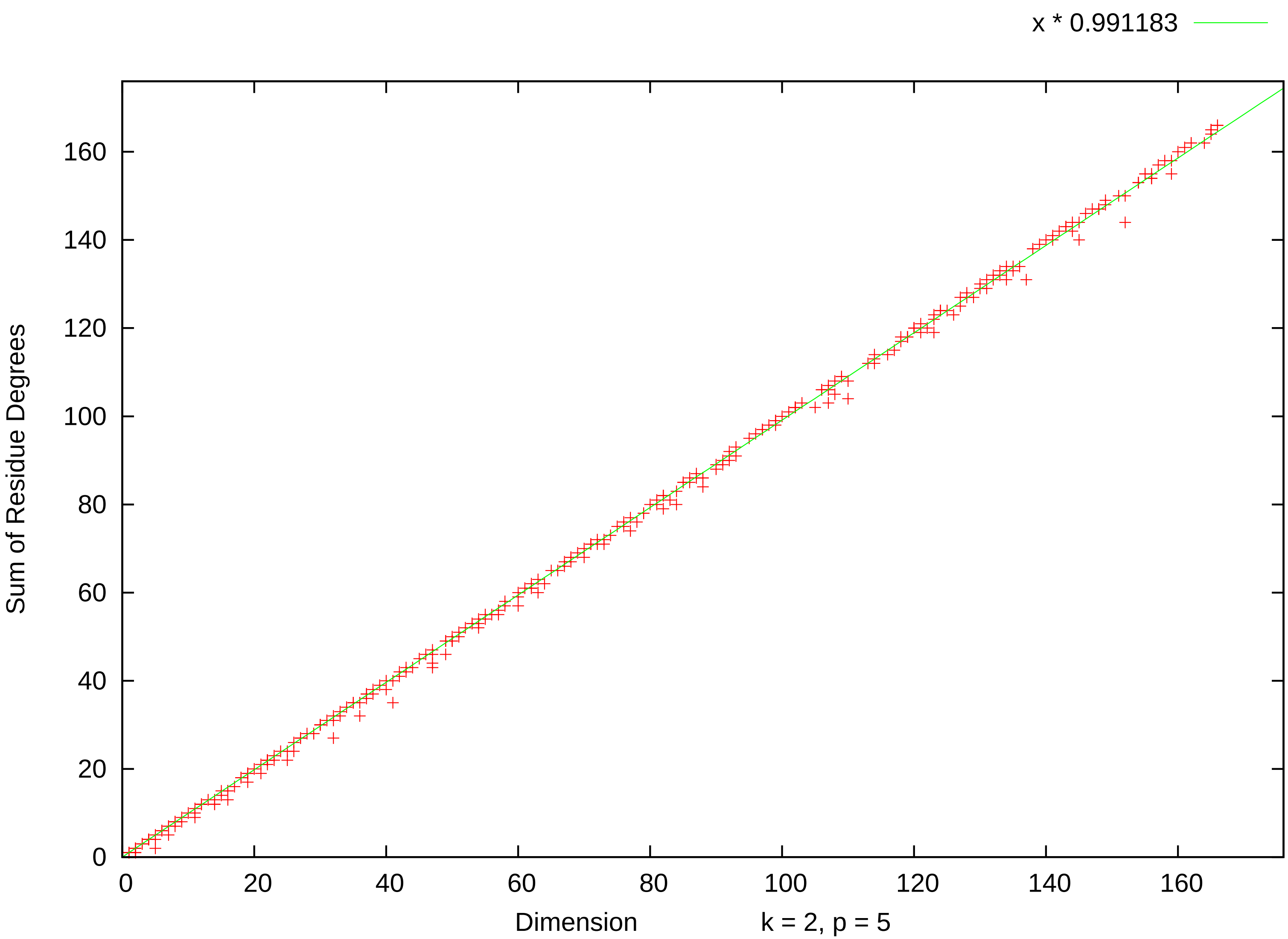

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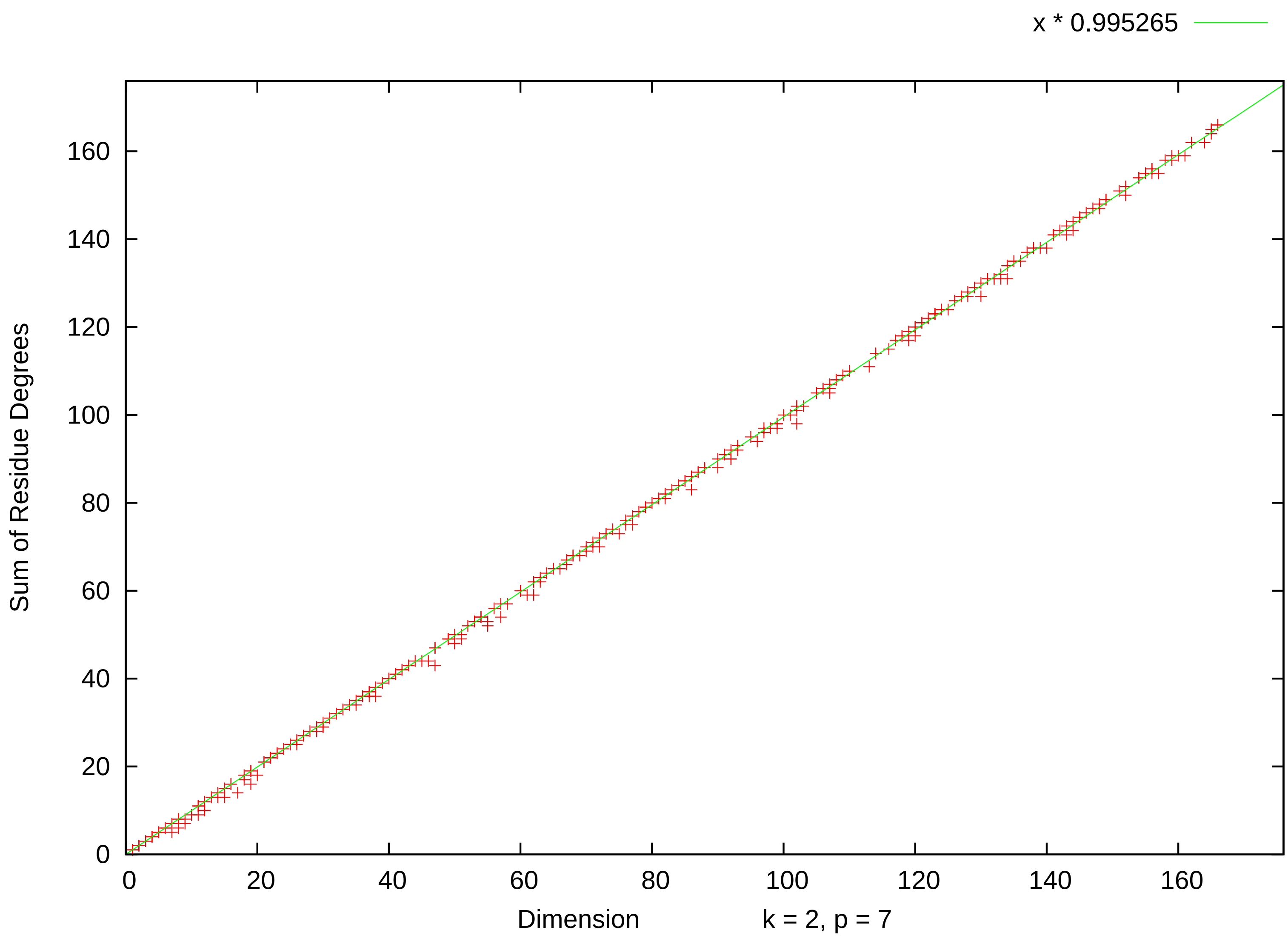

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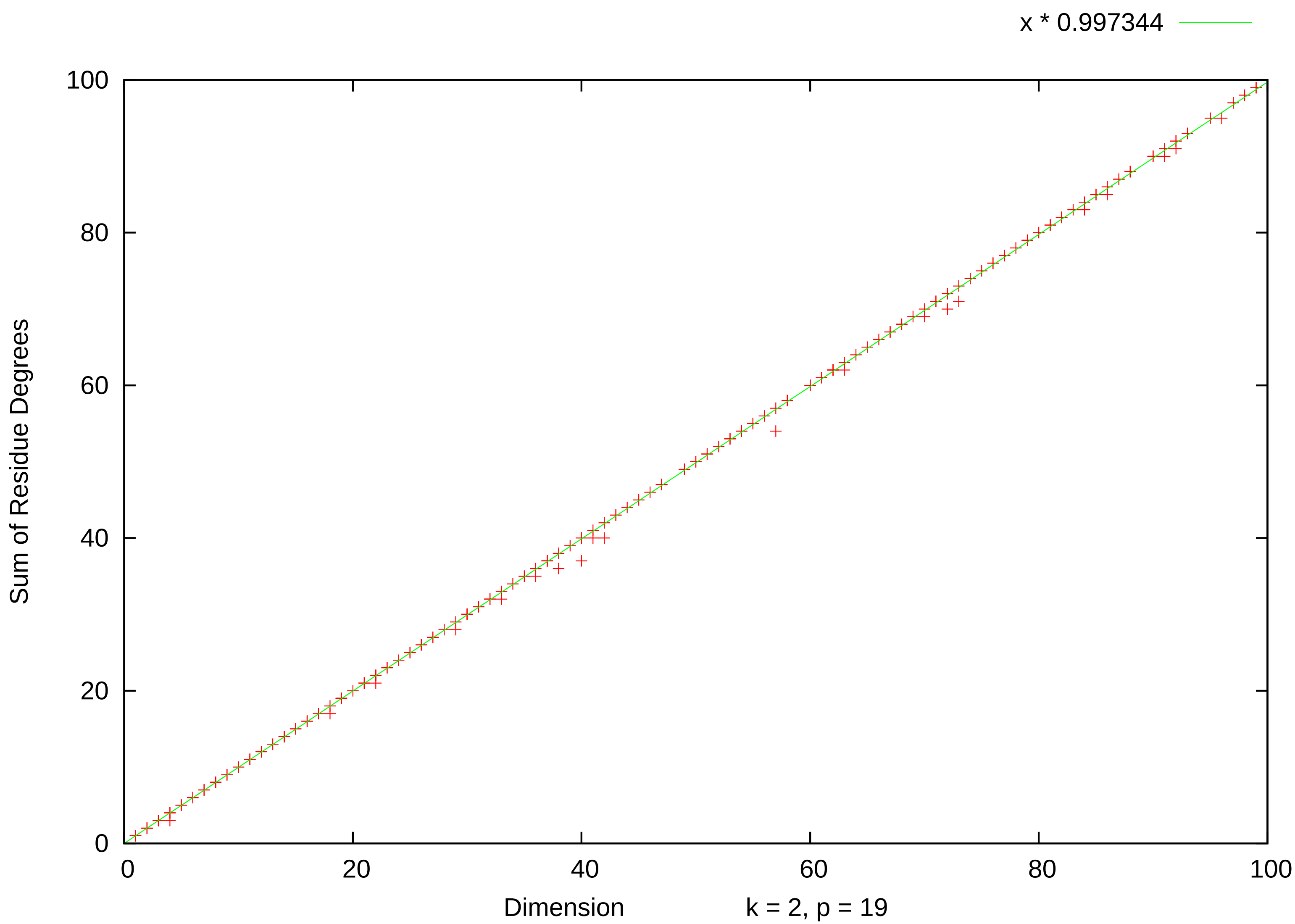

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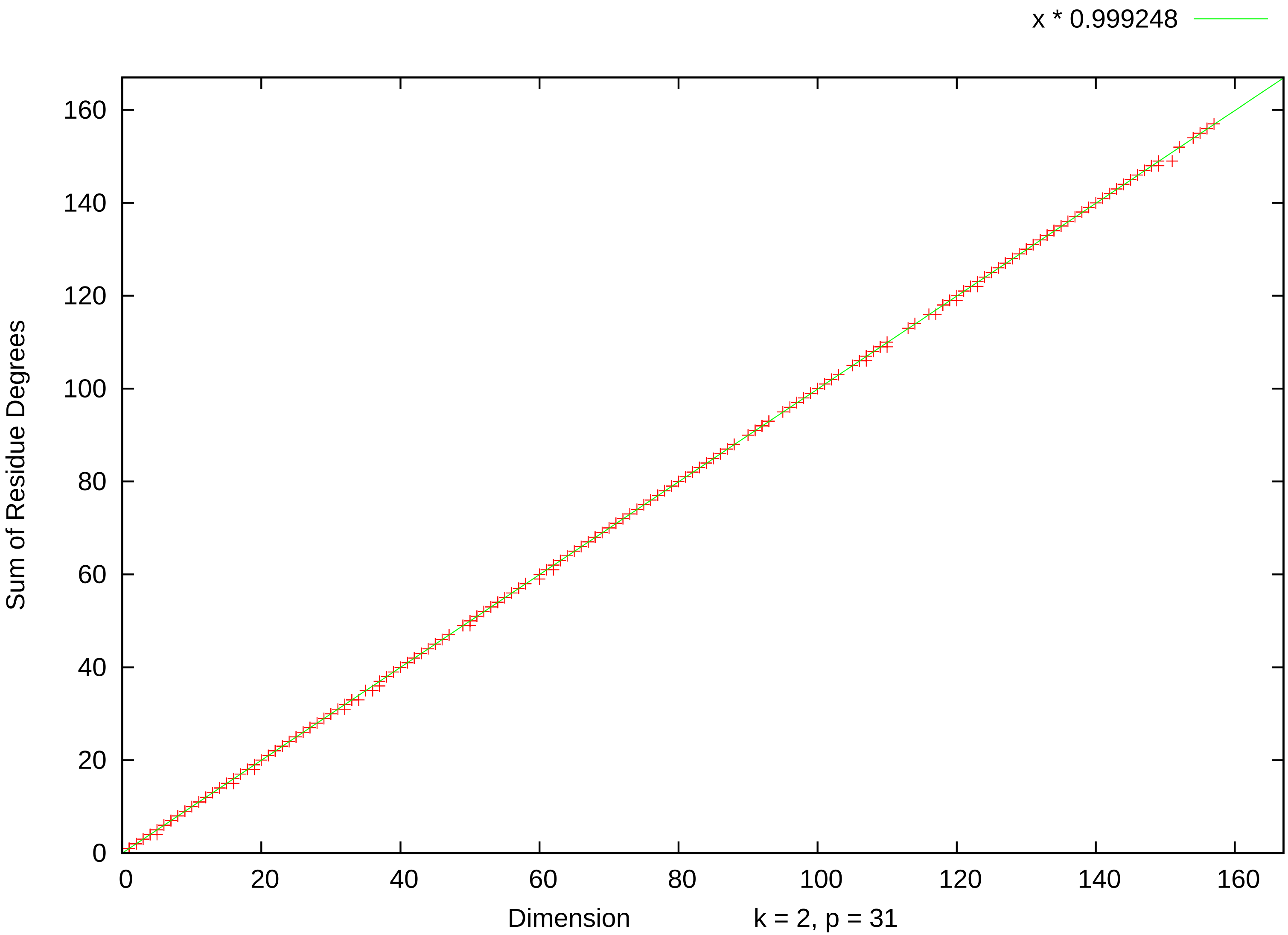

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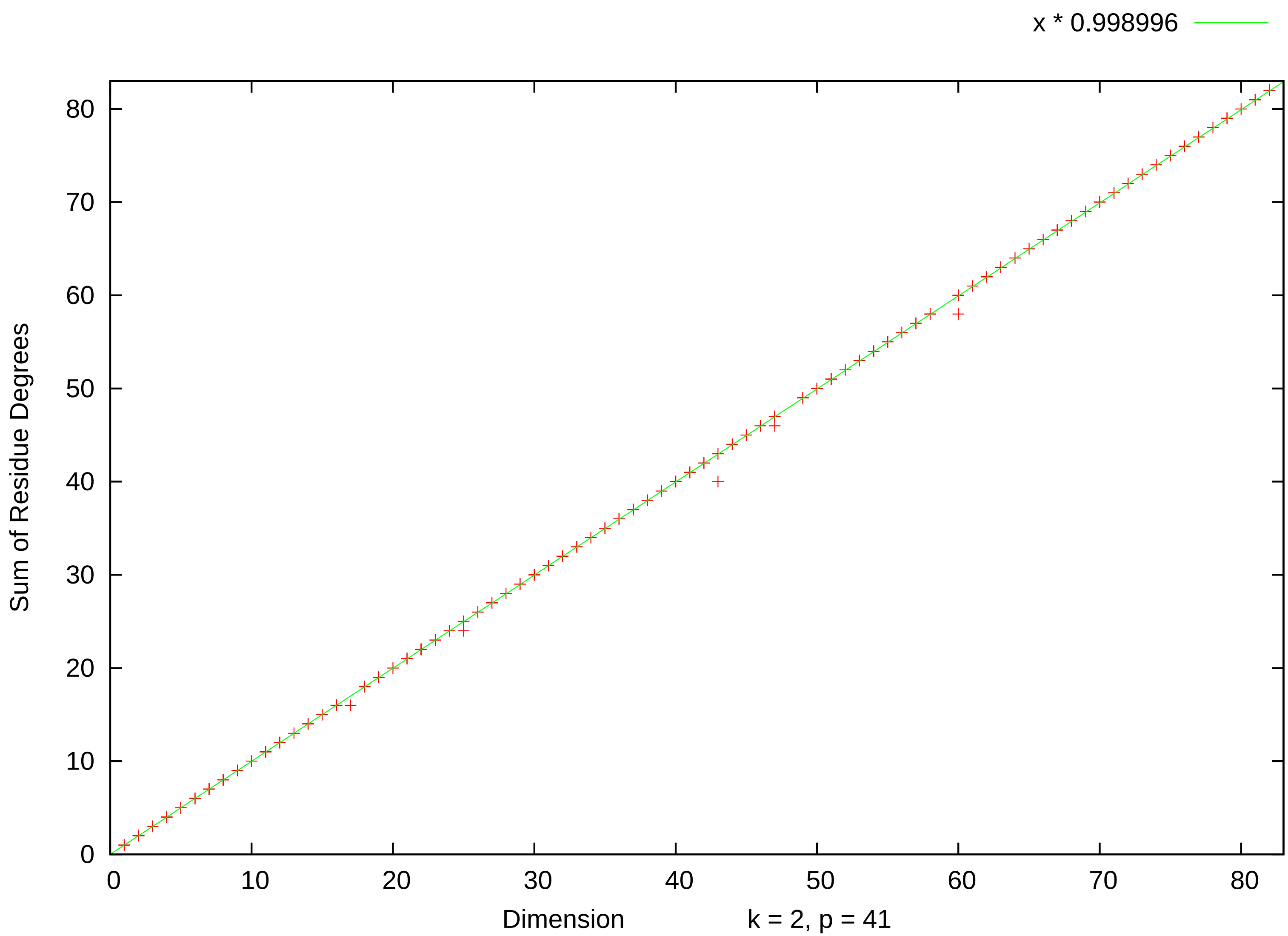

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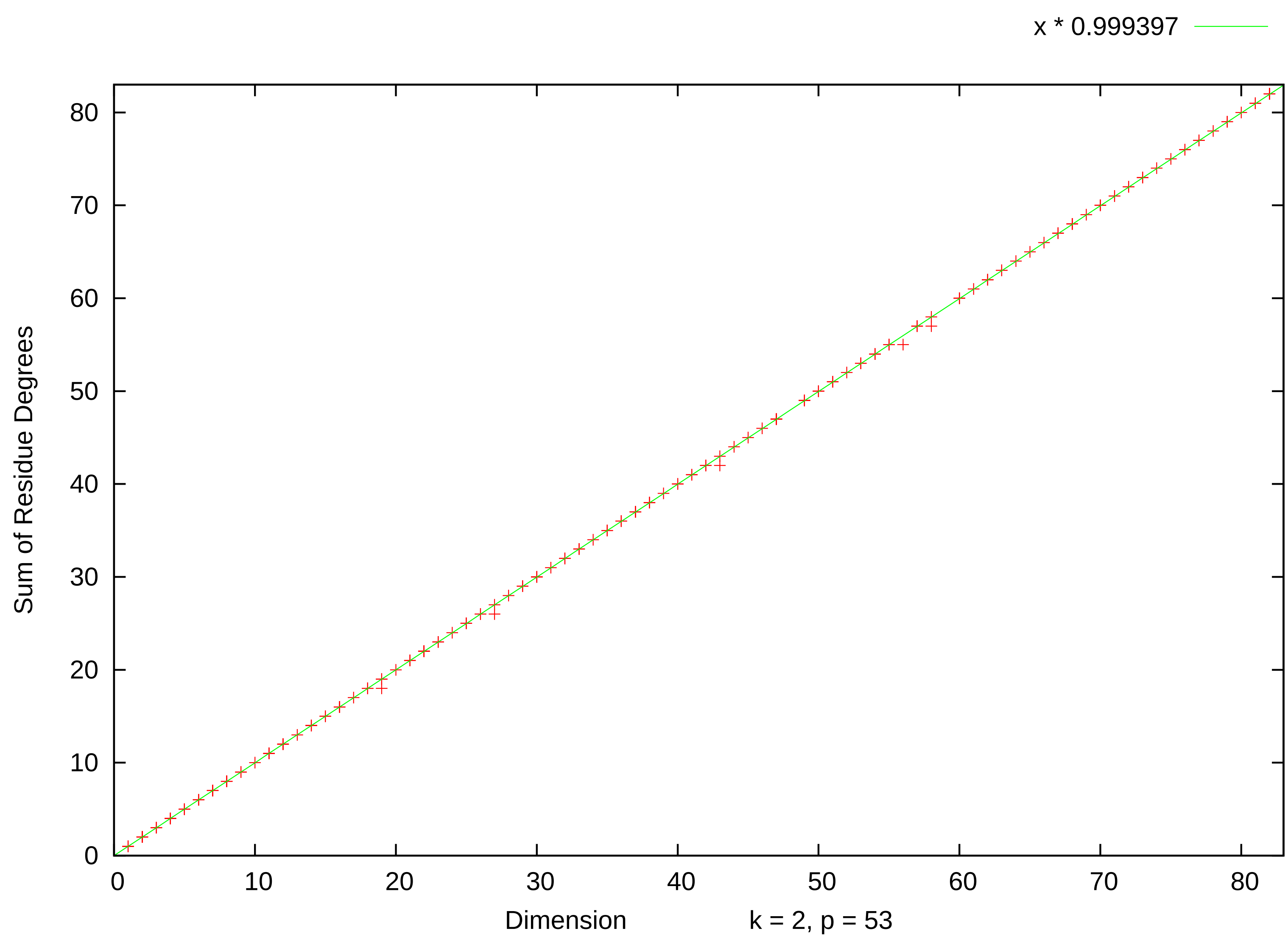

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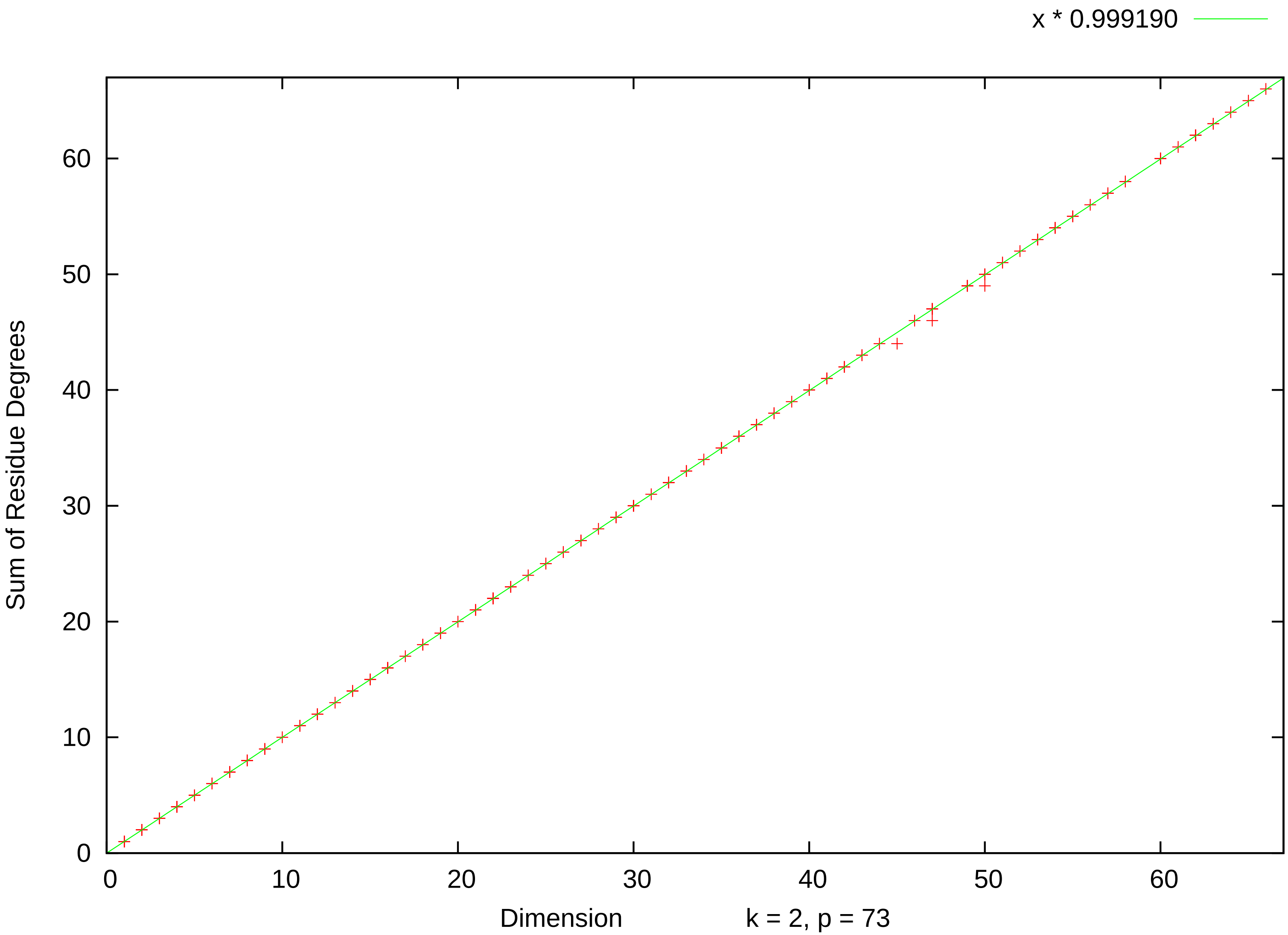

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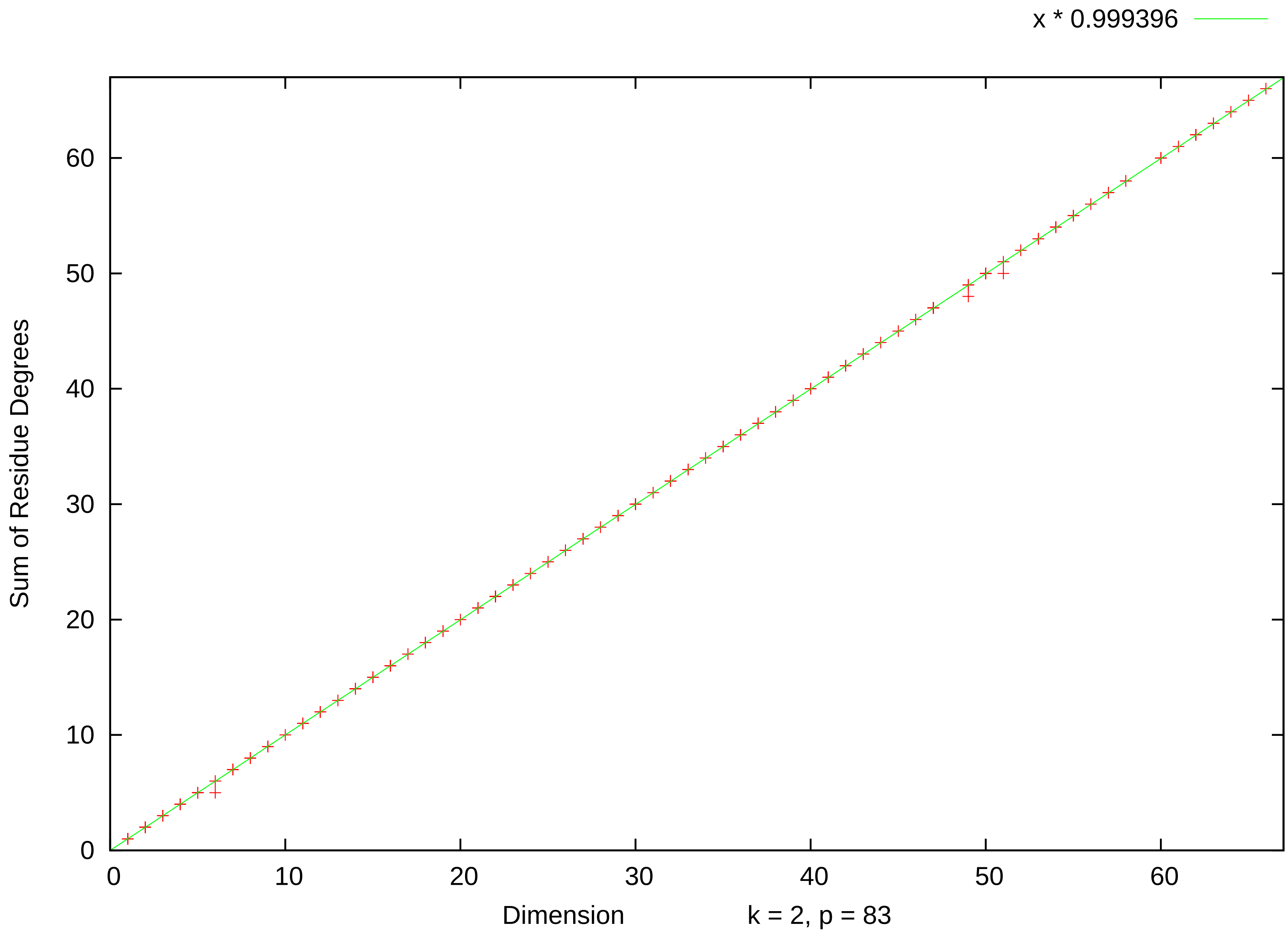

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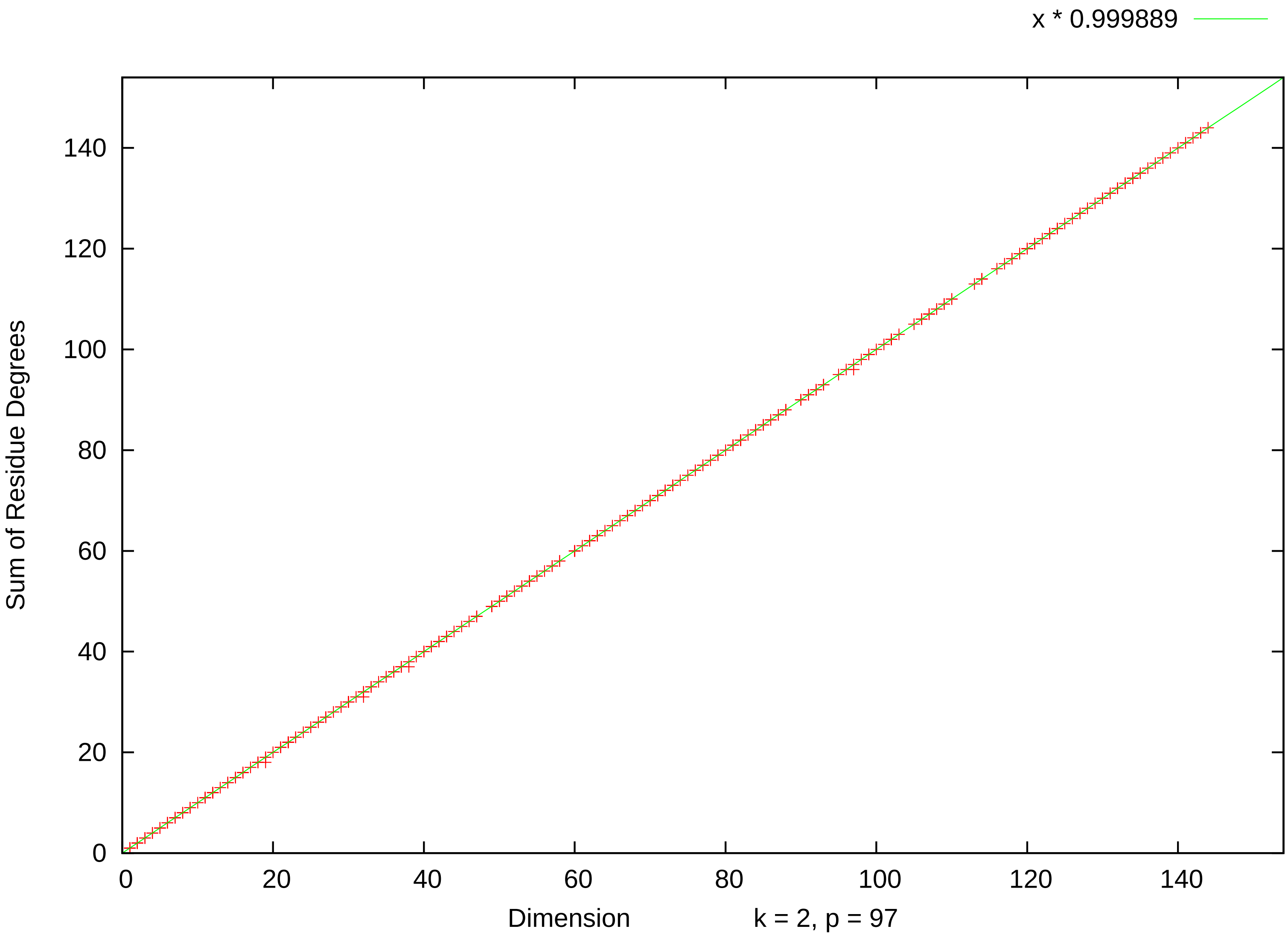
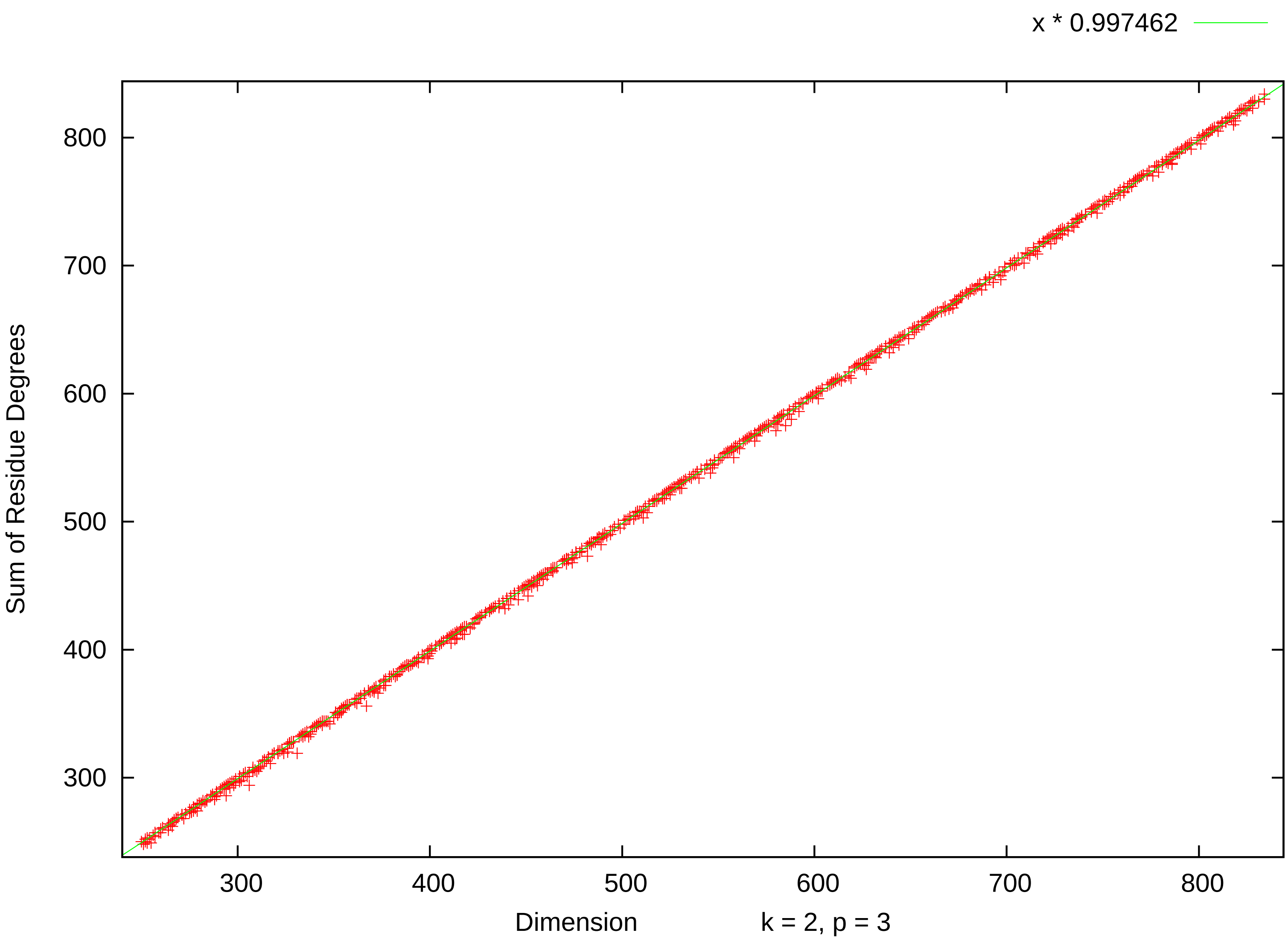


Congruences mod p et al.

Consider $p = 3$ and weight $k = 2$ in a bigger range.

Plot $\deg_k^{(p)}(N)$ as a function of $\dim S_k(N)$.

Congruences mod p et al.



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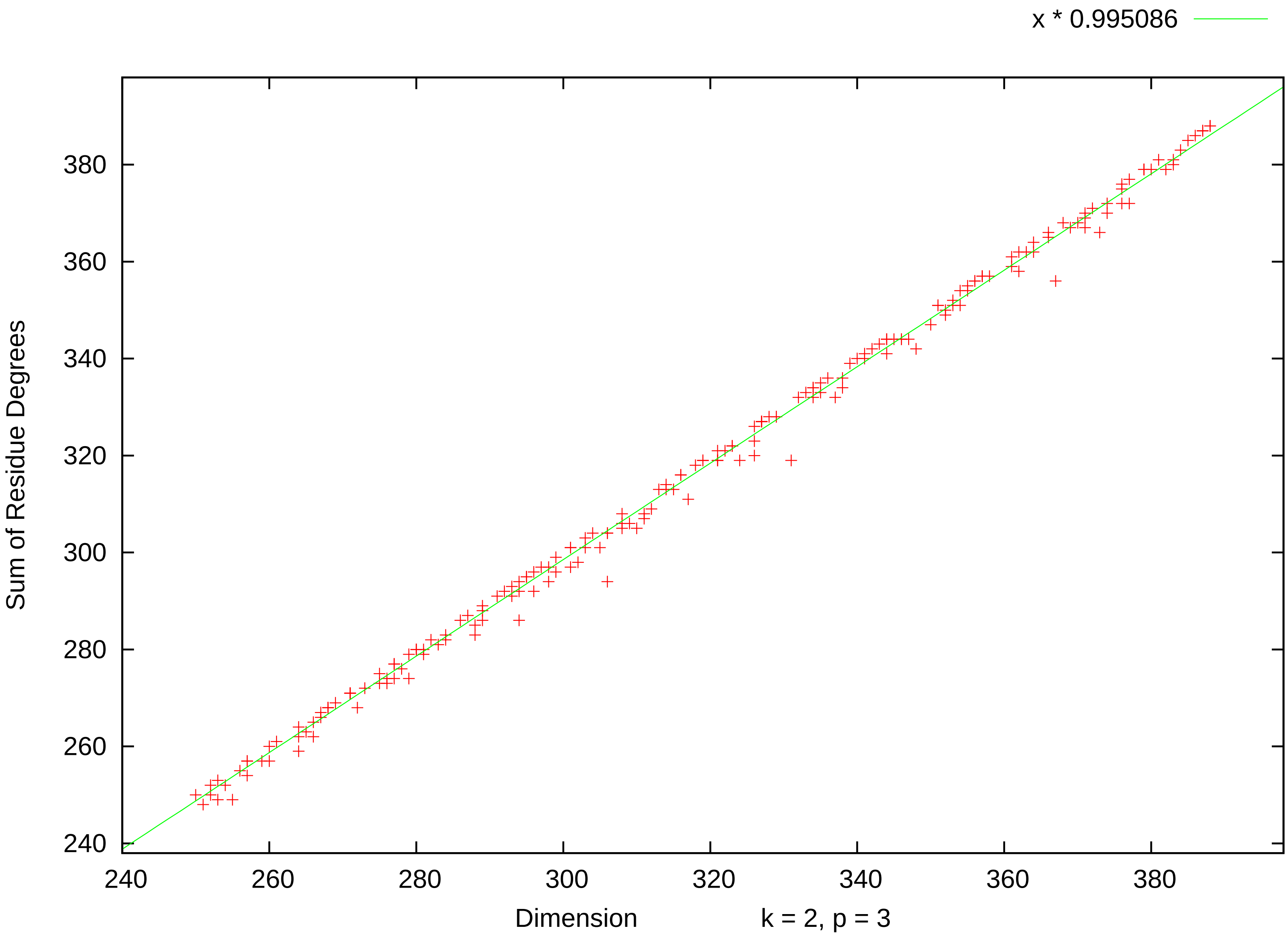

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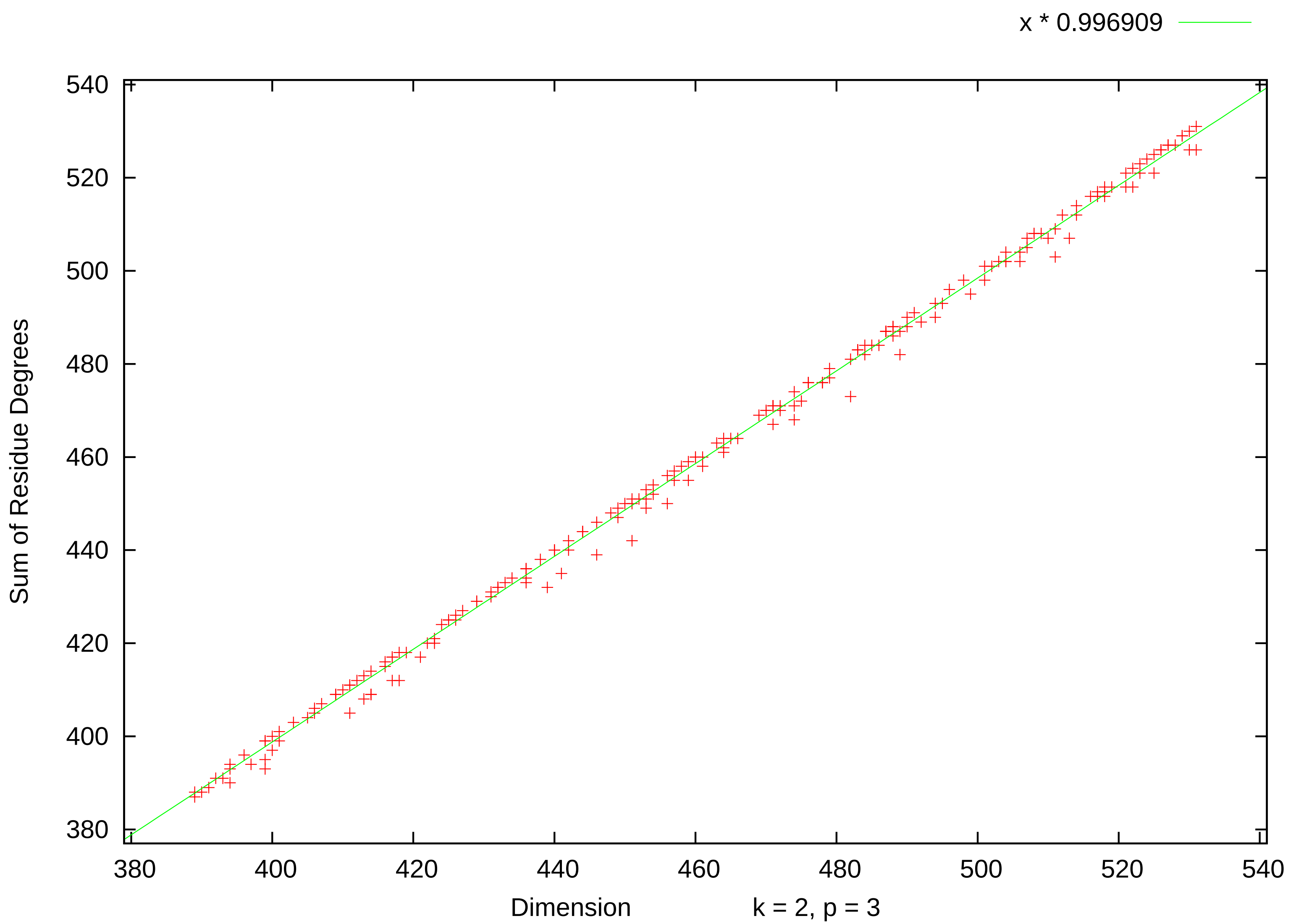

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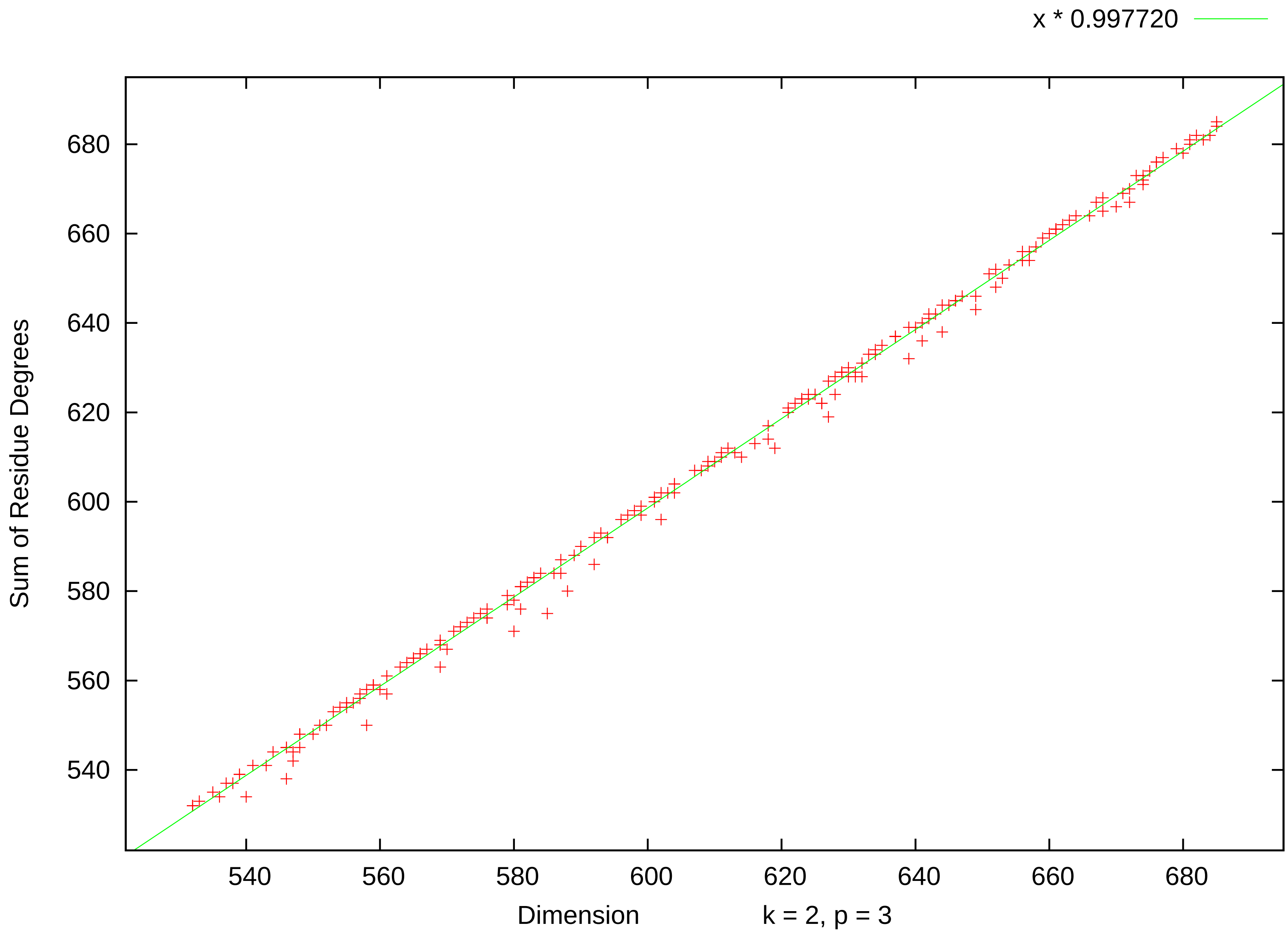

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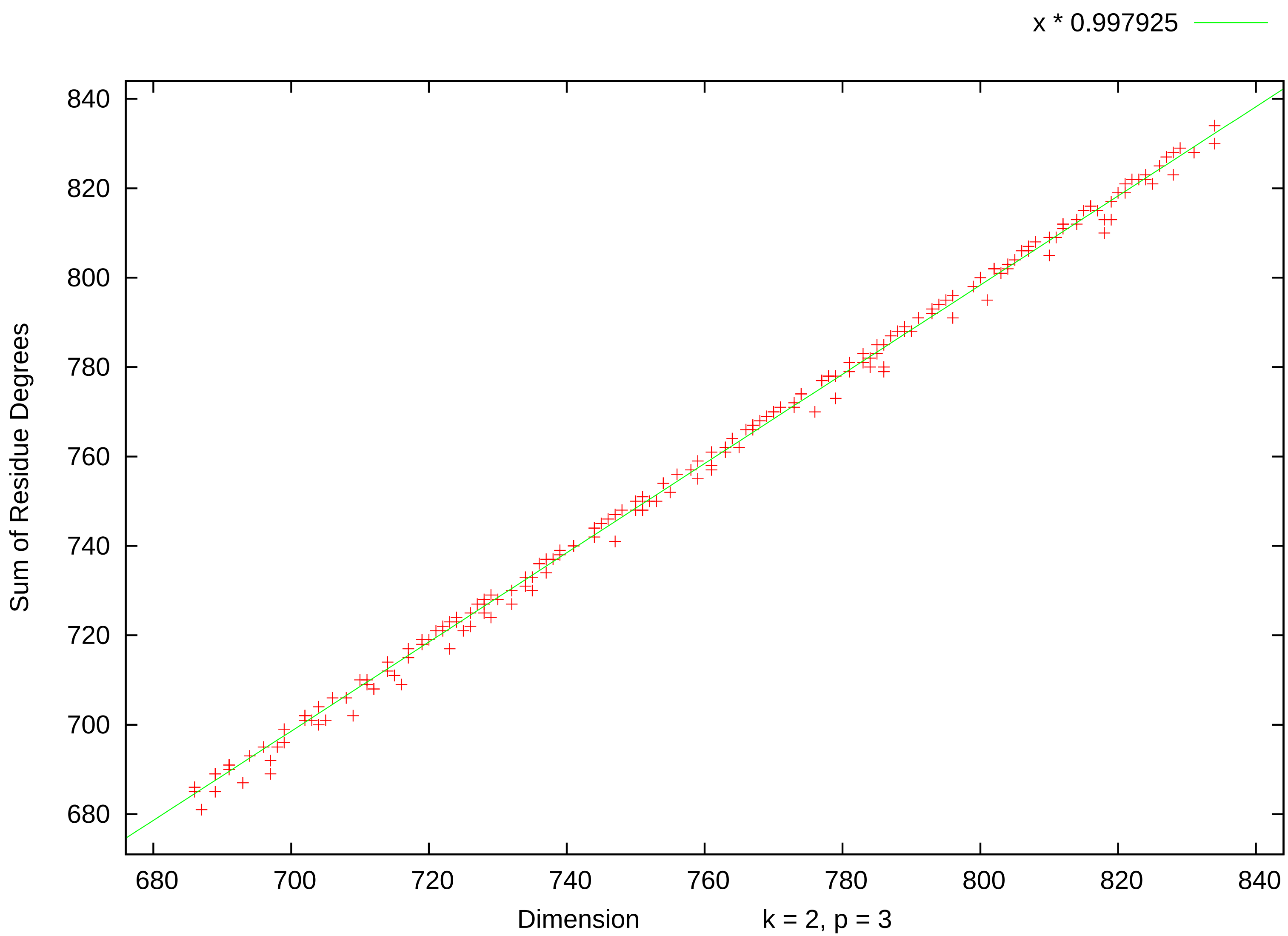

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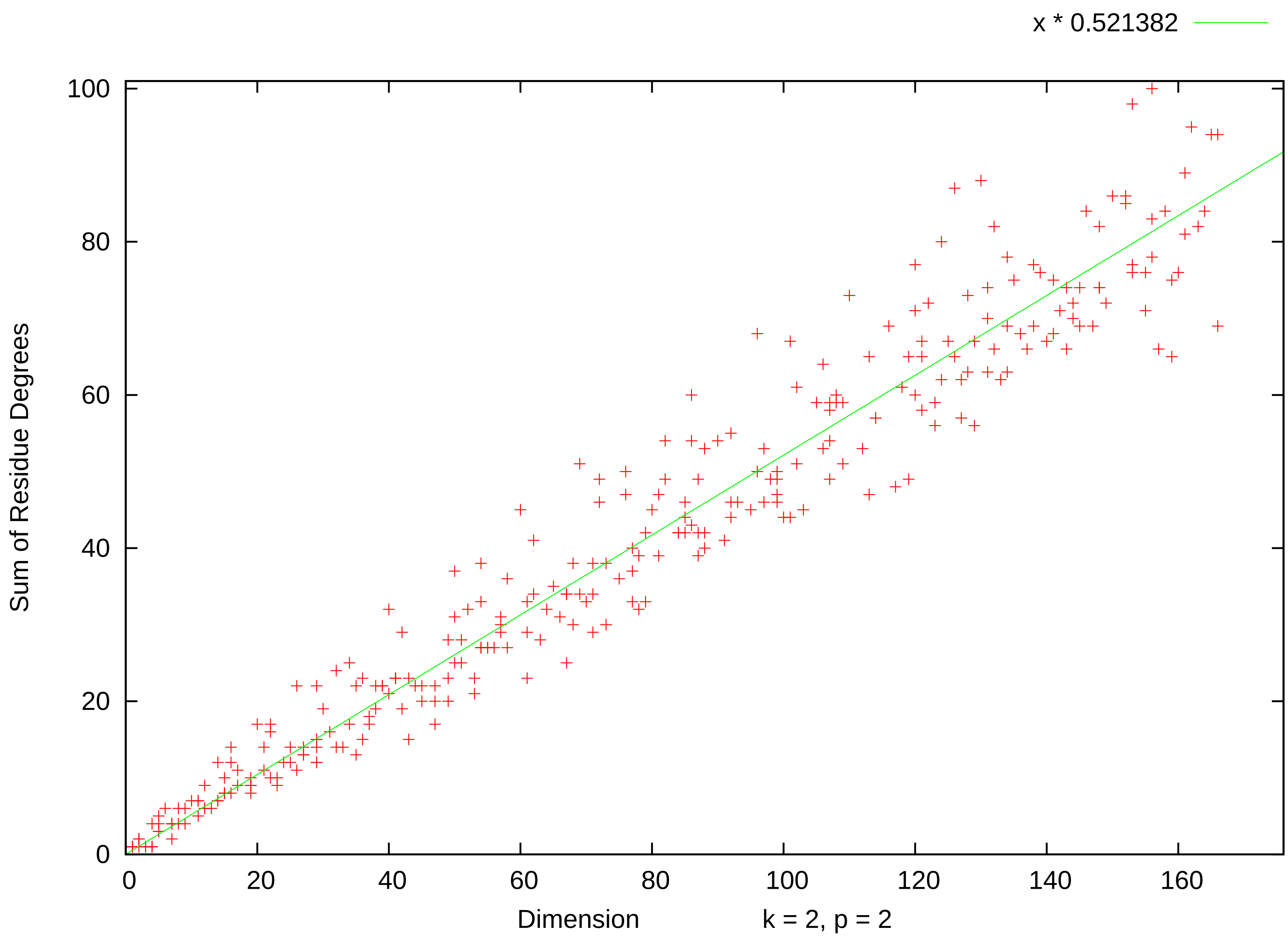
Congruences mod p et al.

Now $p = 2$.

Congruences mod p et al.



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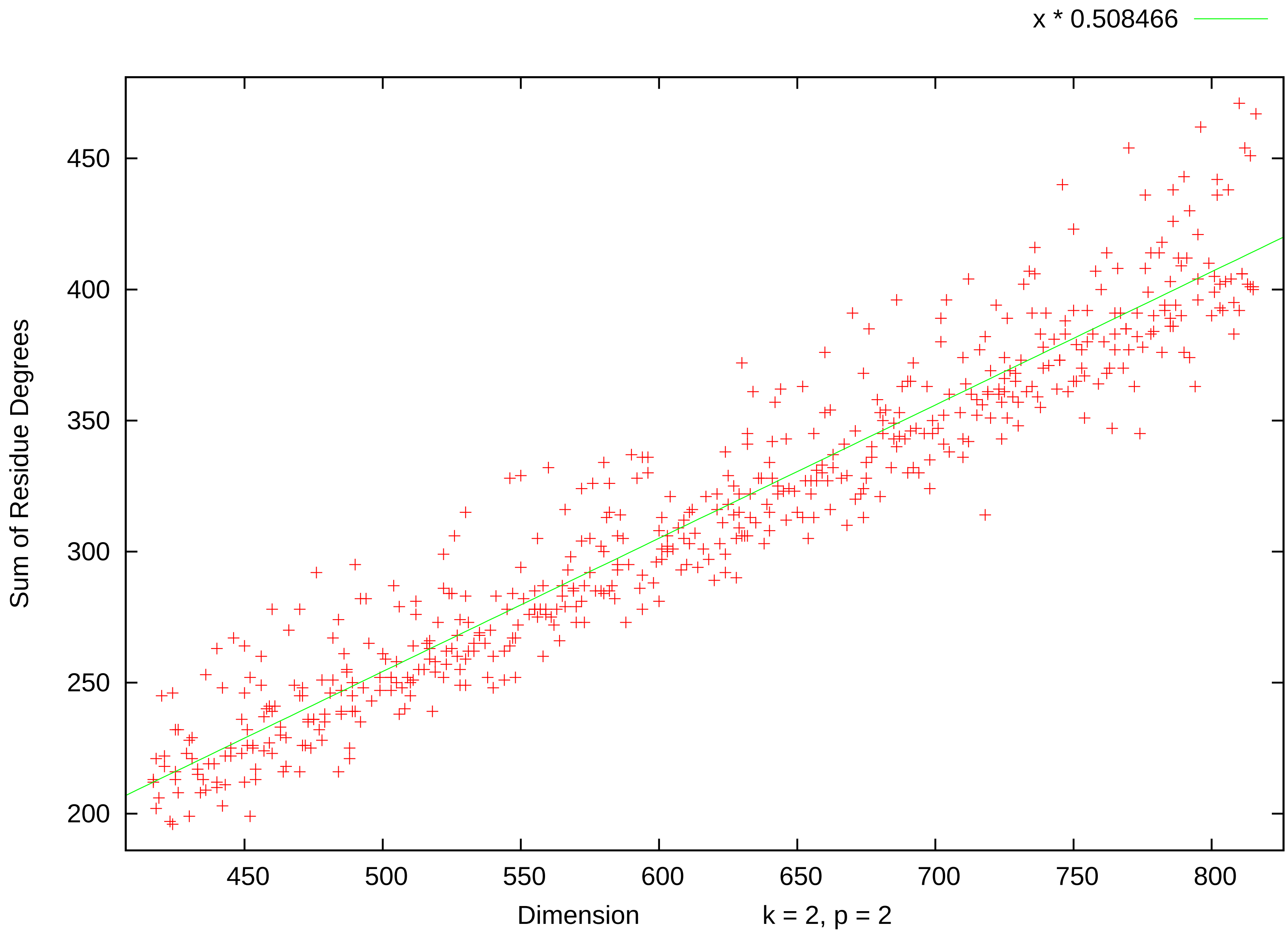

Congruences mod p et al.

How can this different behaviour at $p = 2$ be explained?

Is the Atkin-Lehner operator responsible?

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Congruences mod p et al.

Question: *Fix a prime $p > 2$ and a weight $k \geq 2$.*

Are there $0 < \alpha \leq 1$ and $C > 0$ s.t.

$$\alpha \dim S_k(N) - C \leq \deg_k^{(p)}(N) \text{ ?}$$

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Question: *Fix a weight $k \geq 2$.*

Are there $0 < \alpha \leq \beta < 1$ and $C, D > 0$ s.t.

$$\alpha \dim S_k(N) - C \leq \deg_k^{(2)}(N) \leq \beta \dim S_k(N) + D \quad ?$$

Degrees of coefficient fields

Define:

- $\max_k^{(p)}(N) := \max_{[\overline{f}]} [\mathbb{F}_{p,[\overline{f}]} : \mathbb{F}_p]$

  maximum degree of the coefficient fields mod p,

- $\mathrm{average}_k^{(p)}(N) := \frac{\sum_{[\overline{f}]} [\mathbb{F}_{p,[\overline{f}]} : \mathbb{F}_p]}{\sum_{[\overline{f}]} 1}$

  average degree of the coefficient fields mod p.

Here, $[\overline{f}]$ runs through the $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-conjugacy classes of newforms in level N and weight k.

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Here, $[\overline{f}]$ runs through the $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-conjugacy classes of newforms in level N and weight k.

Can $\text{average}_k^{(p)}(N)$ and $\max_k^{(p)}(N)$ be related to $\dim S_k(N)$?

Maximum degree mod p

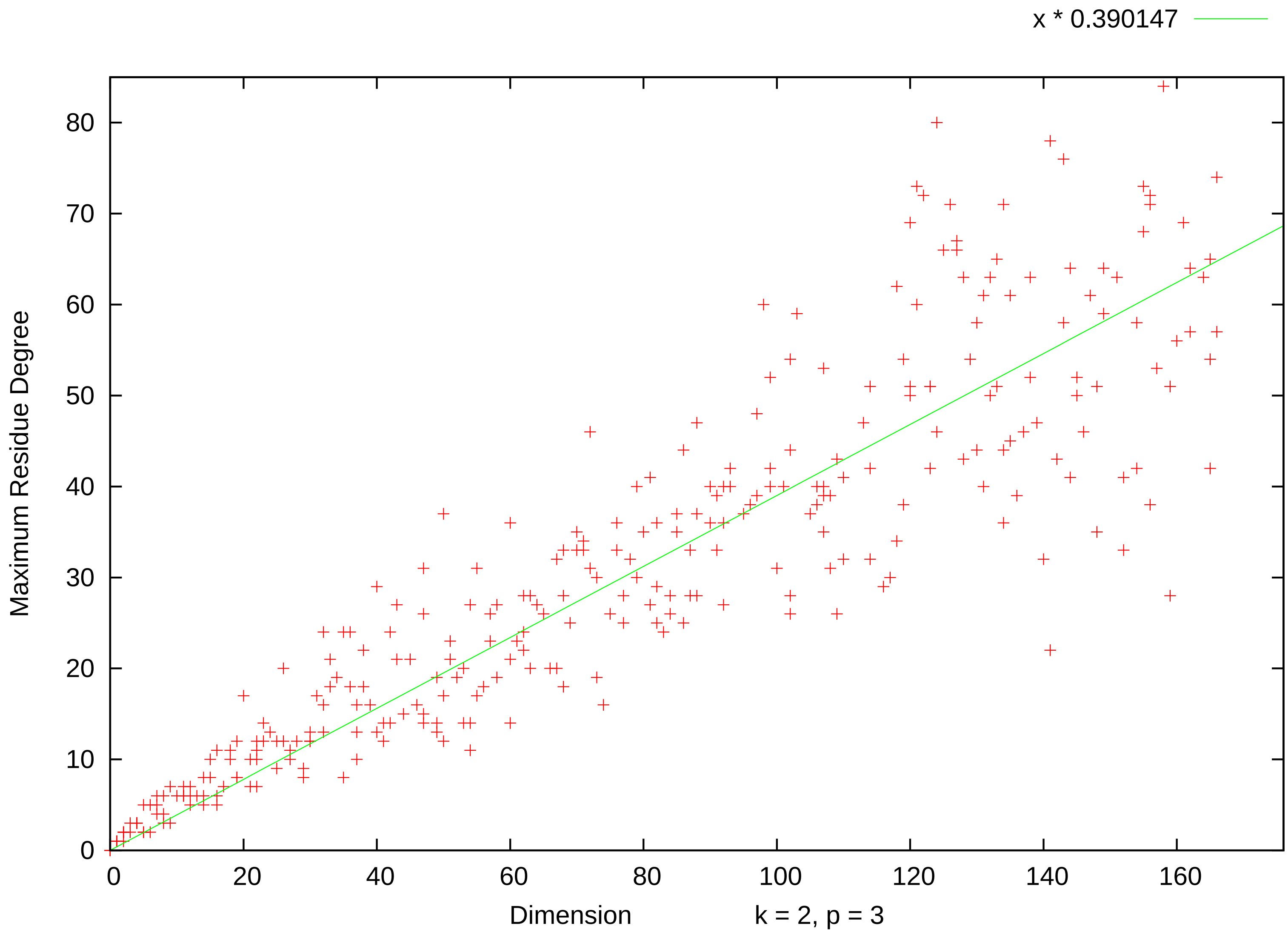
Fix $p > 2$ and $k = 2$.

Guess a dependence of the form

$$\max_k^{(p)}(N) \sim \alpha \dim S_k(N).$$

Plot $\max_k^{(p)}(N)$ as a function of $\dim S_k(N)$ for the primes $N \leq 2000$.

Maximum degree mod p



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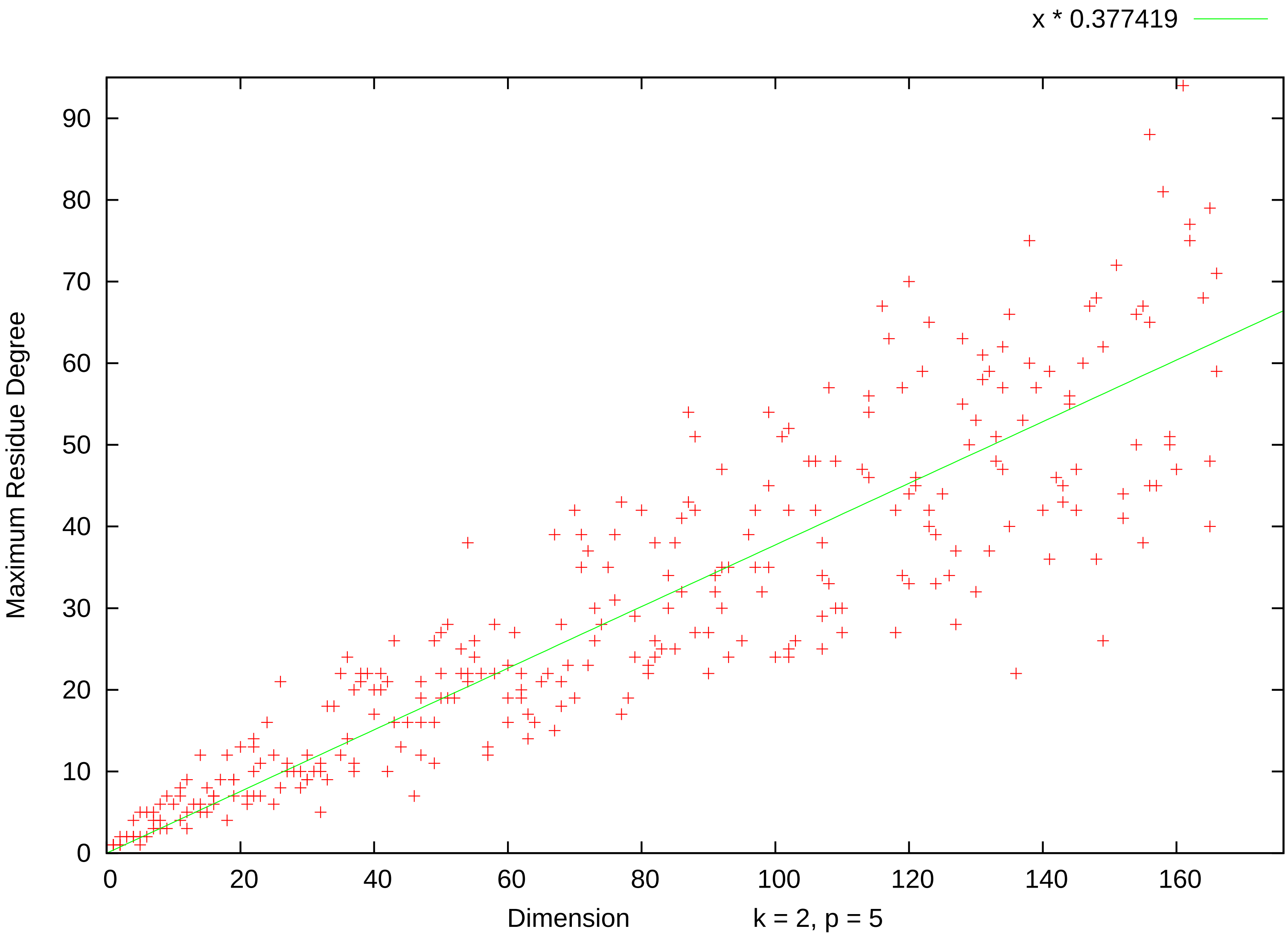

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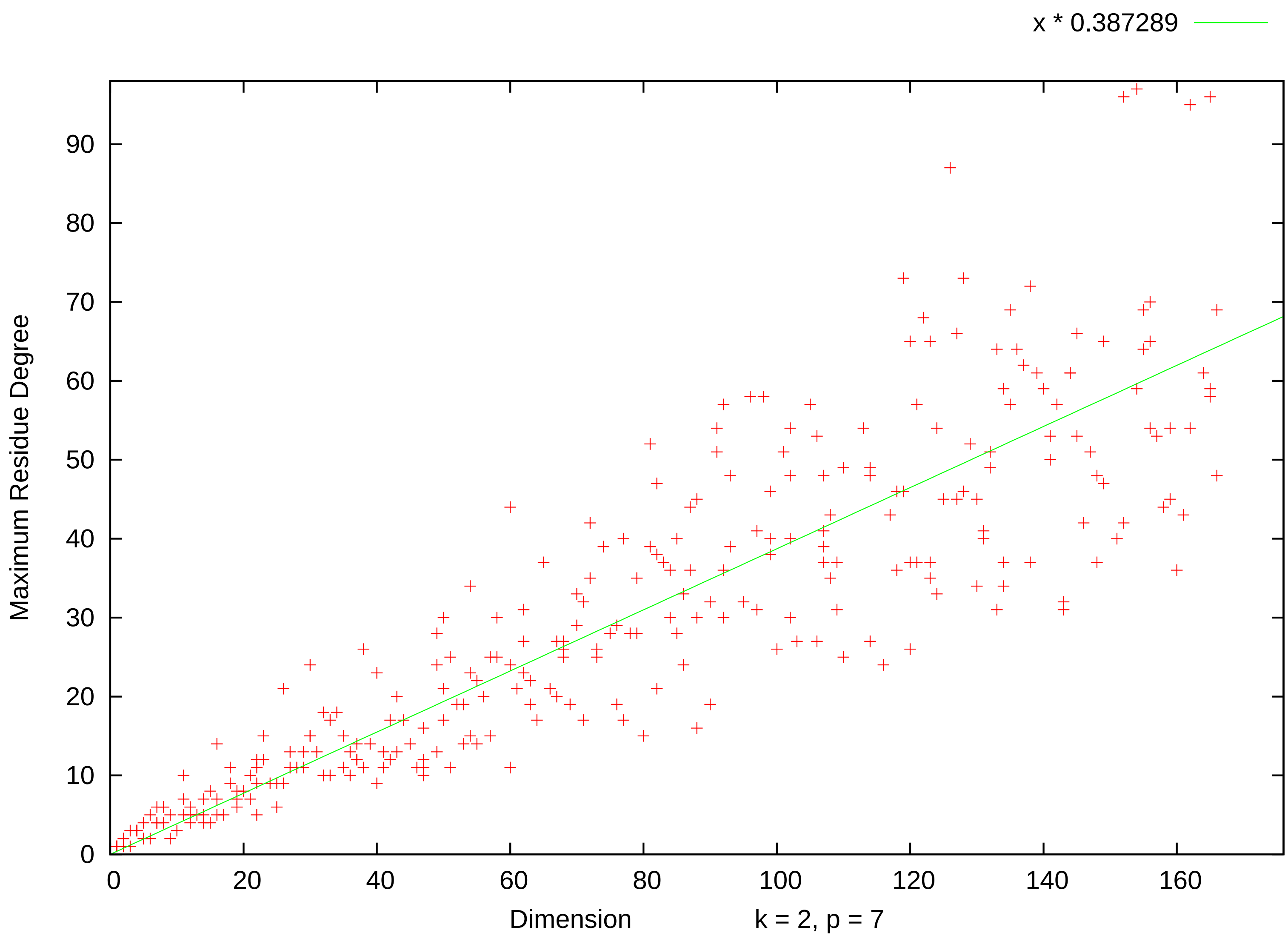

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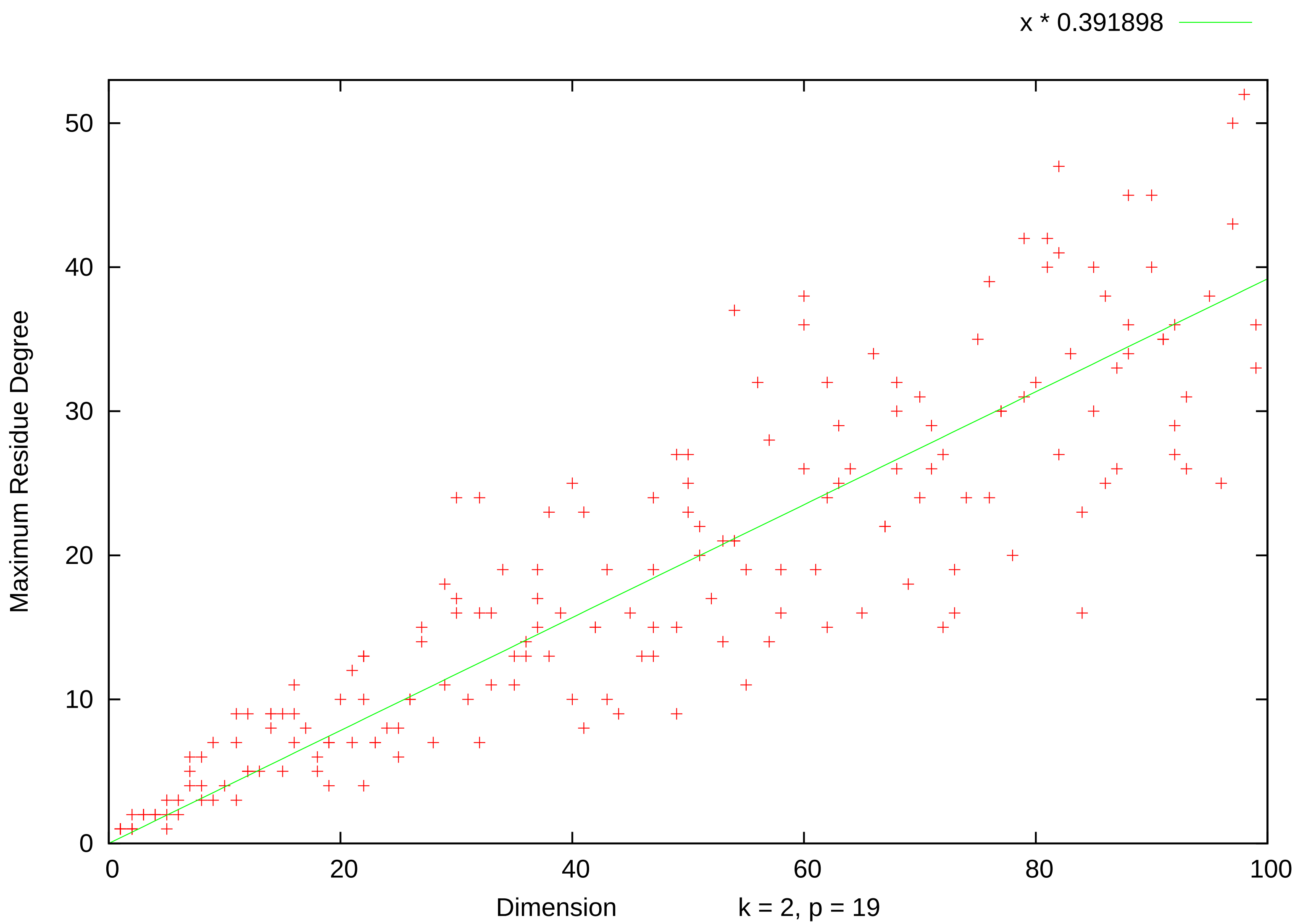

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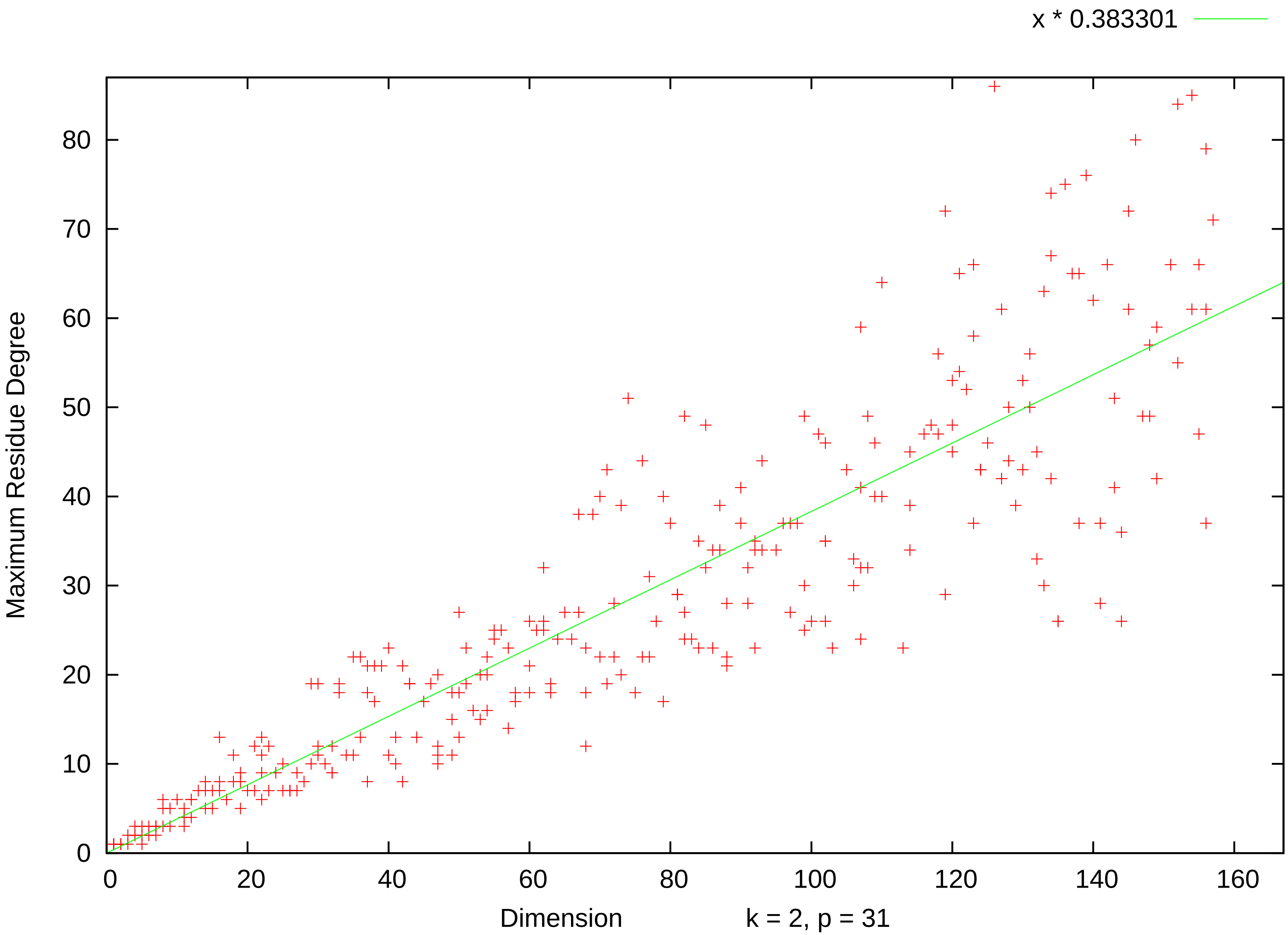

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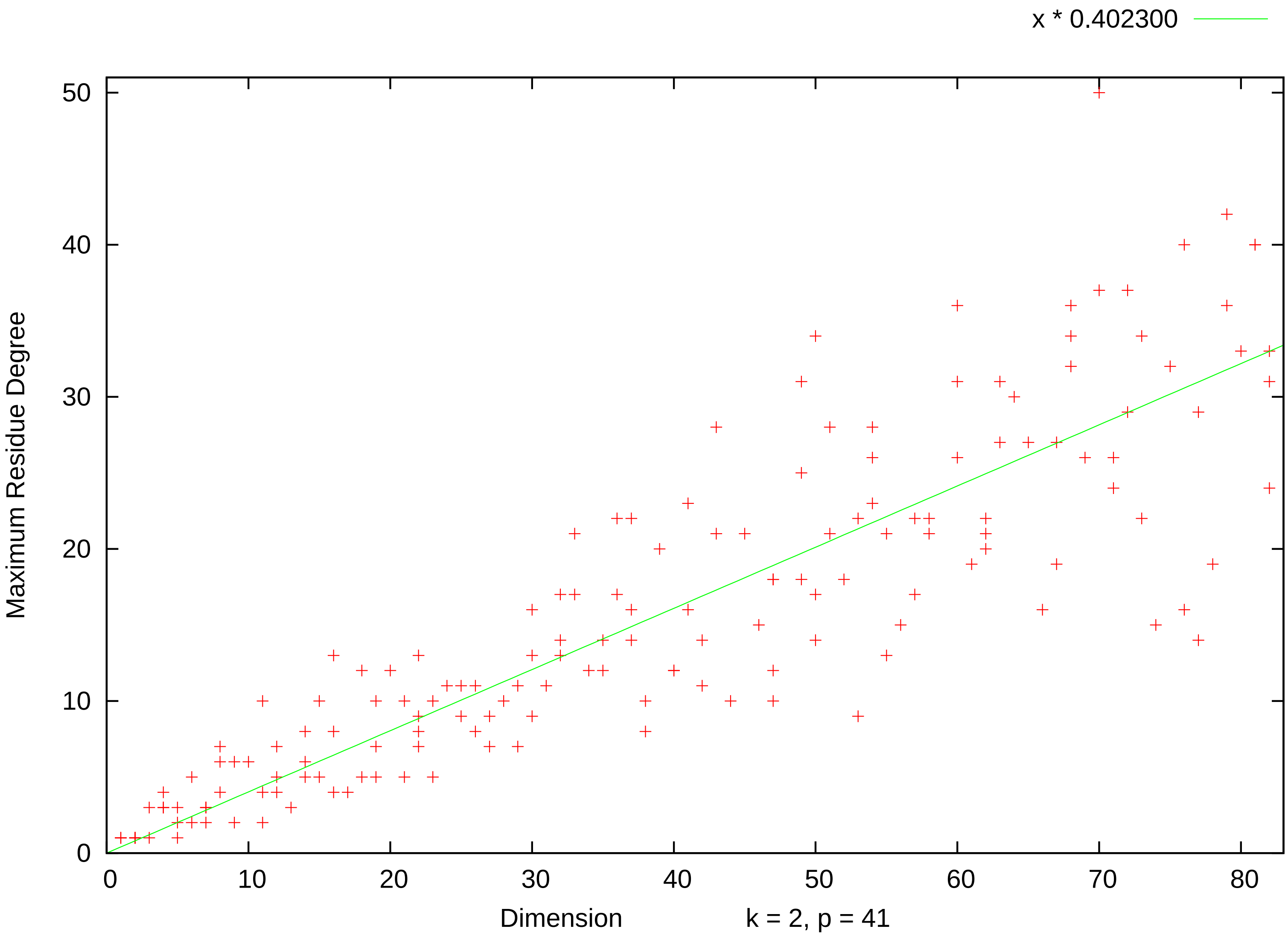

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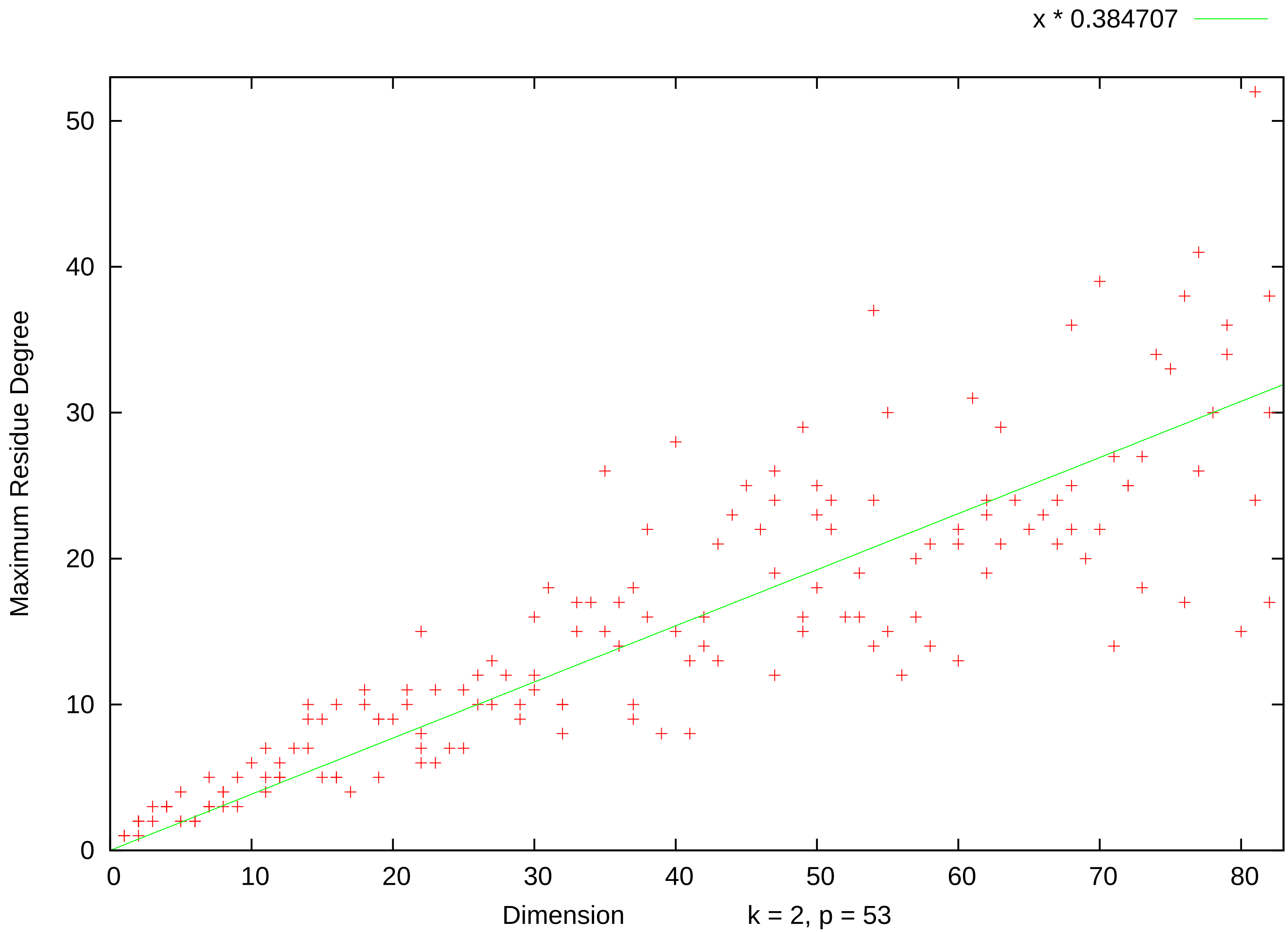

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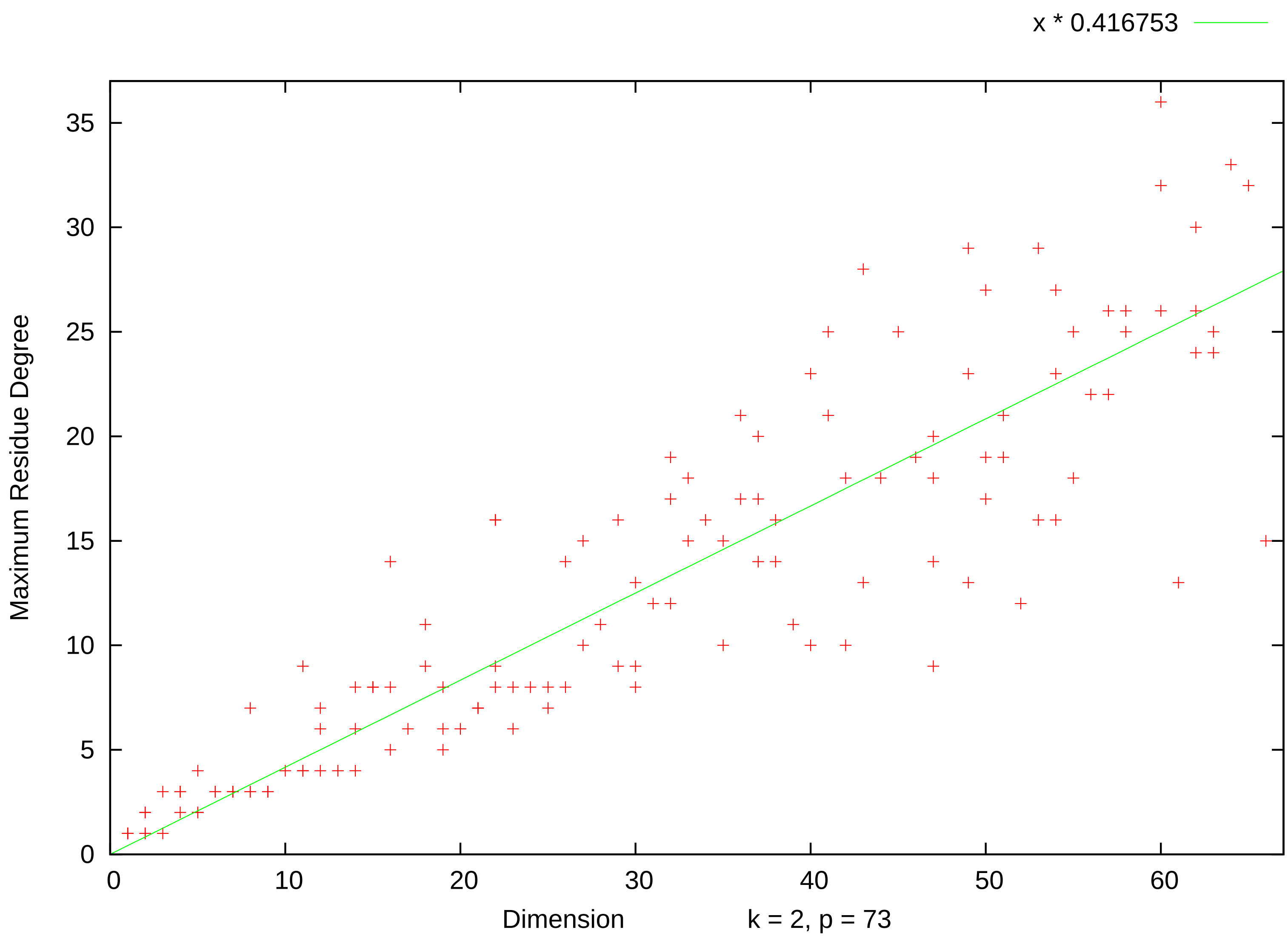

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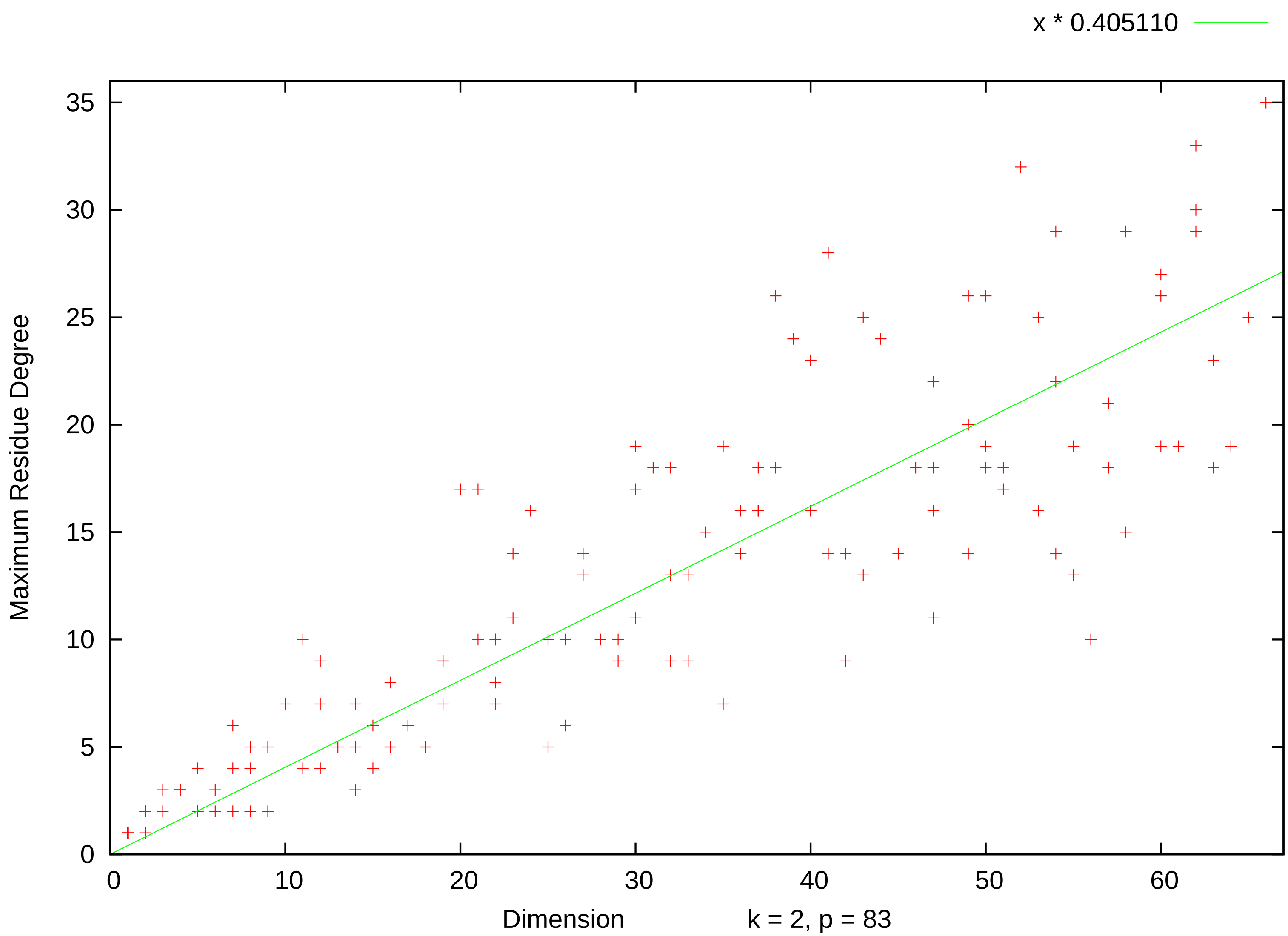

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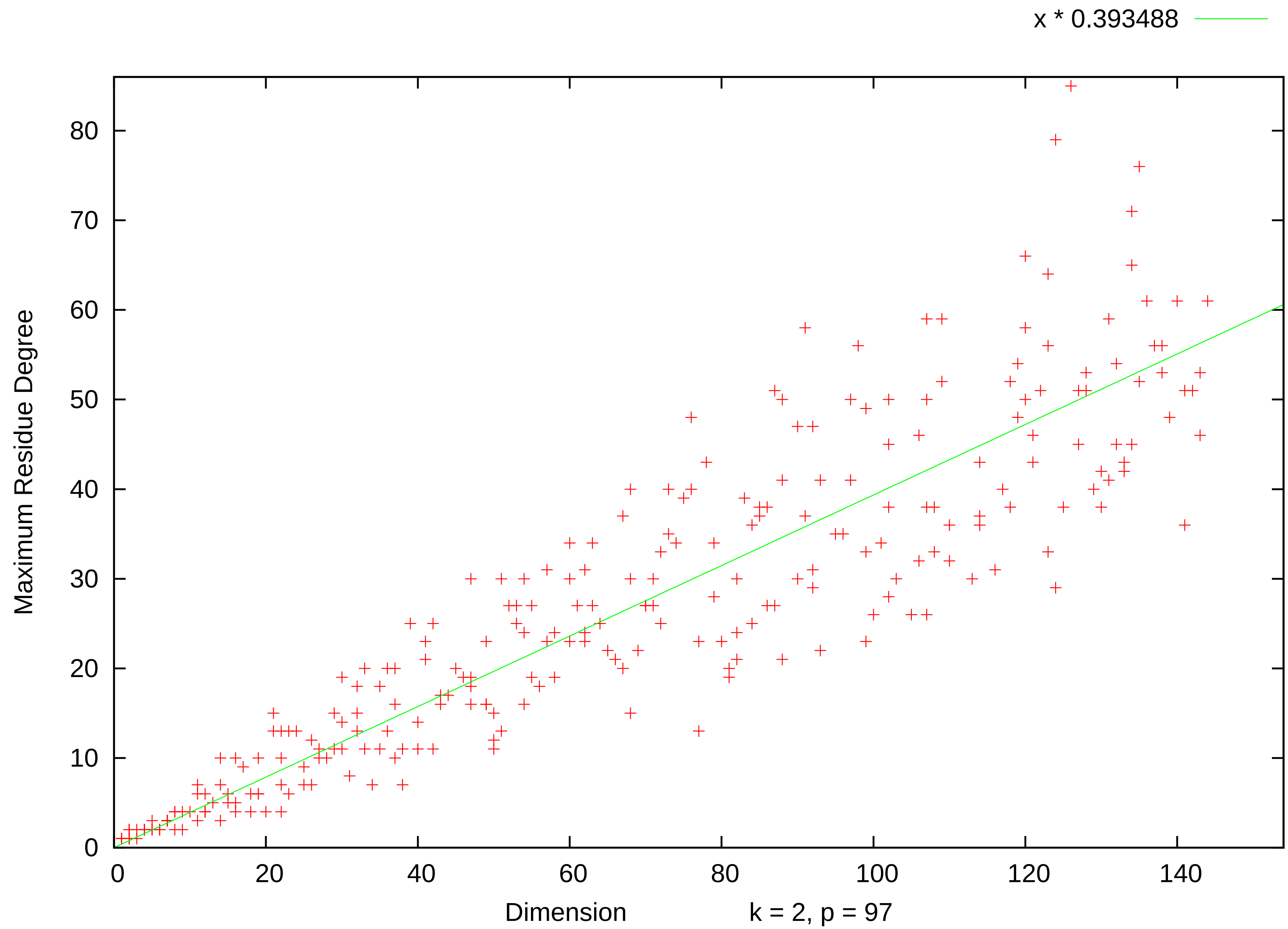

Maximum degree mod p

Consider $p = 3$ and weight $k = 2$ in a bigger range.

Plot $\max_k^{(p)}(N)$ as a function of $\dim S_k(N)$.

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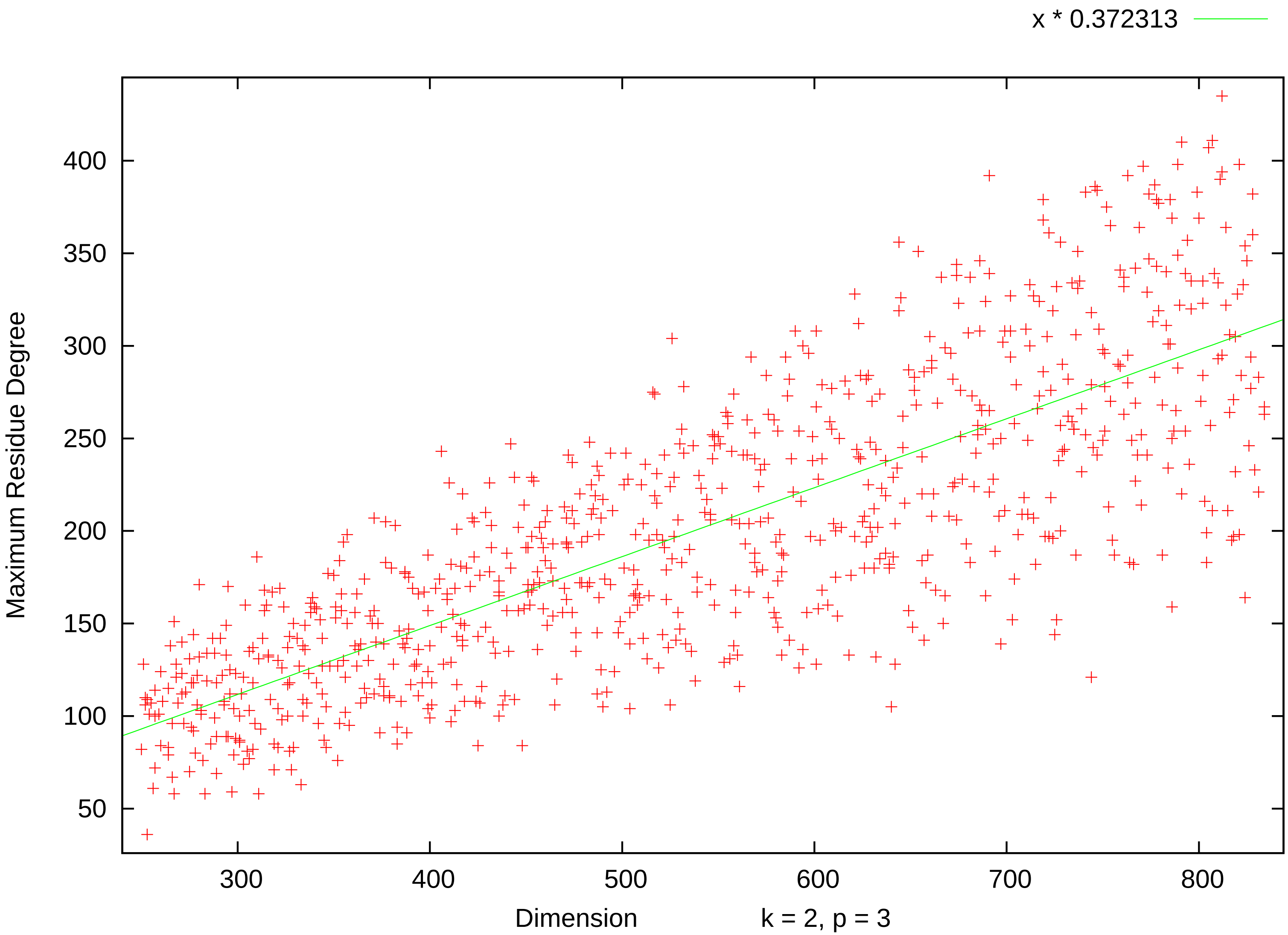
Maximum degree mod p



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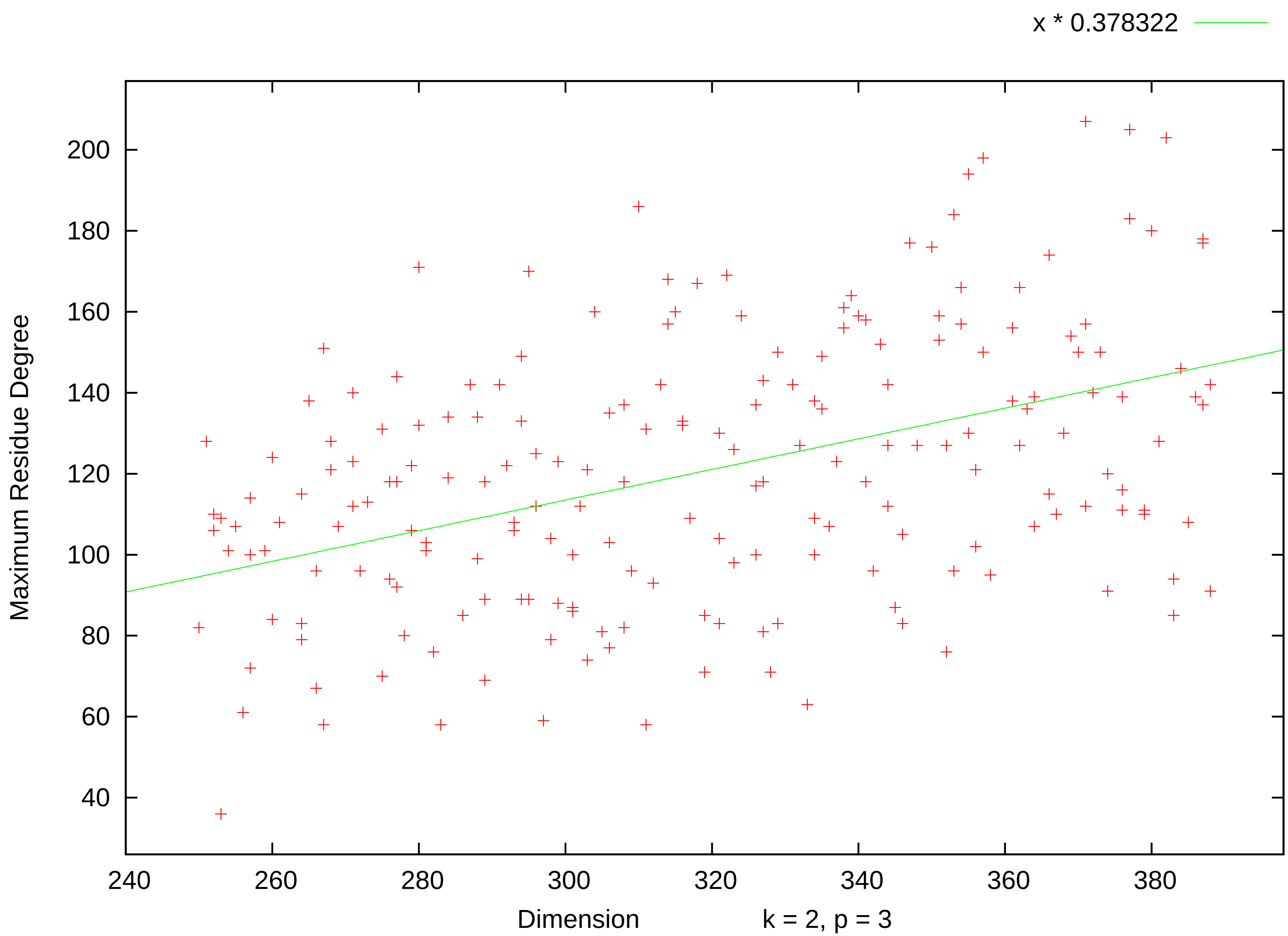

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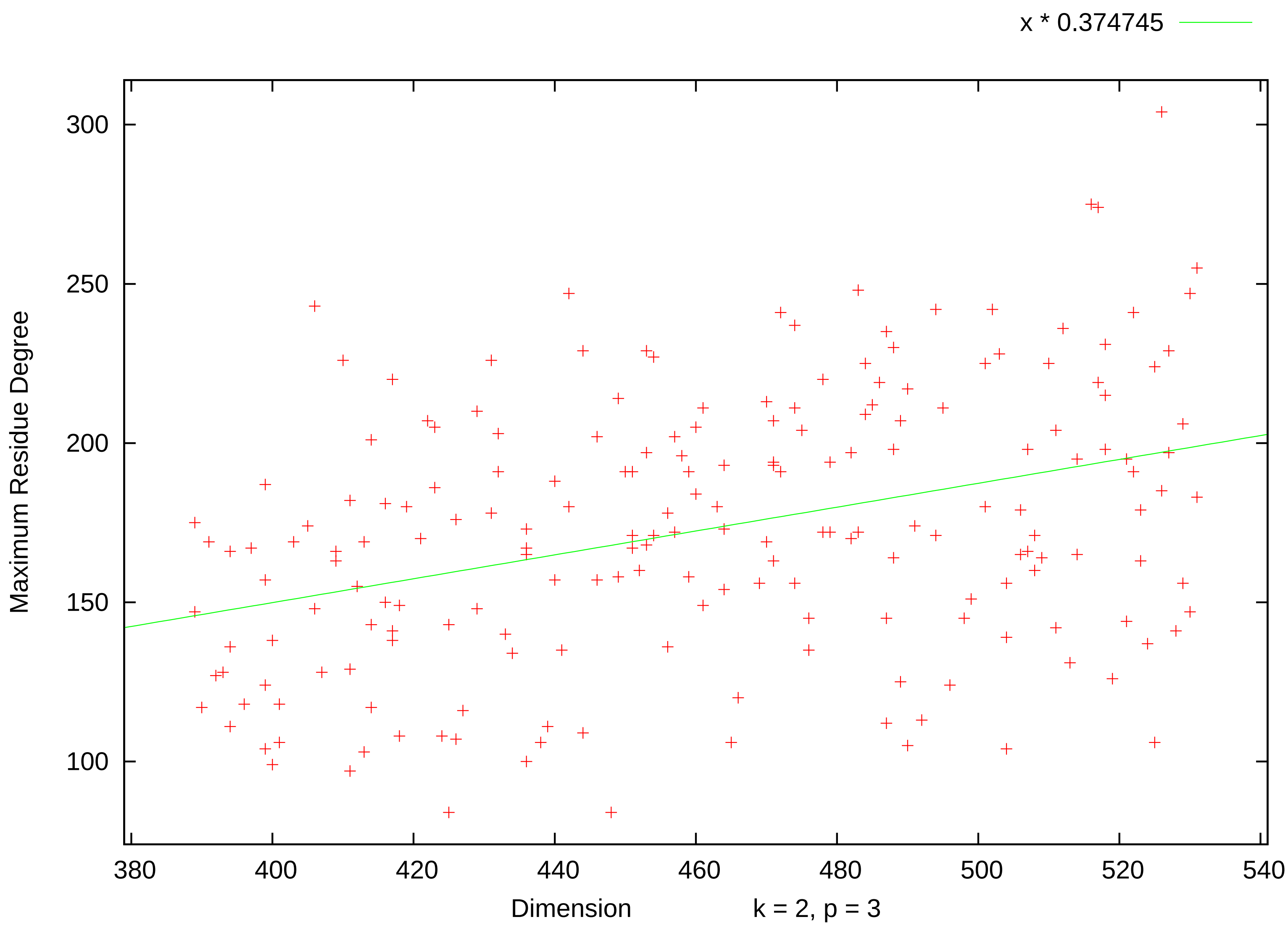

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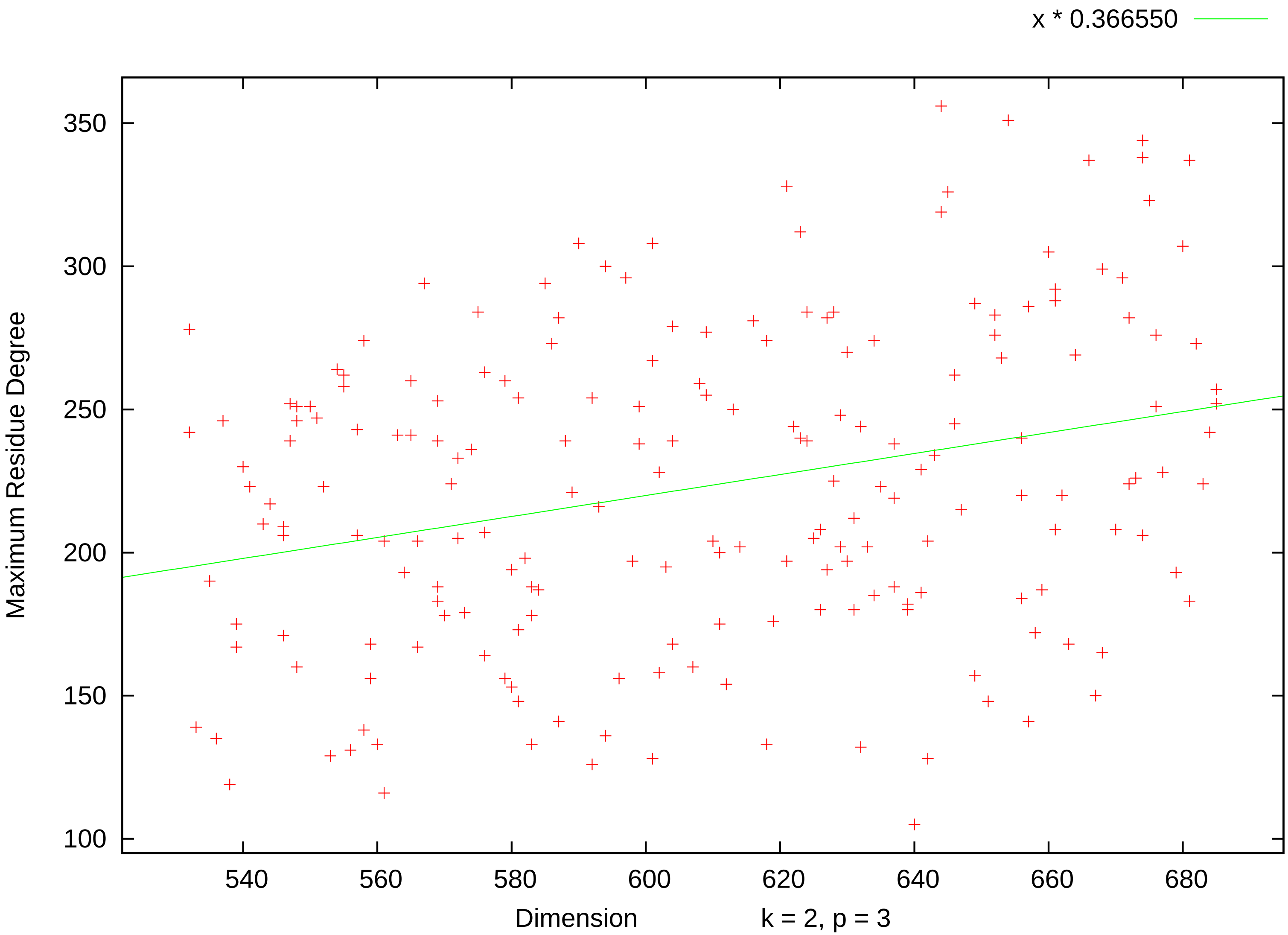

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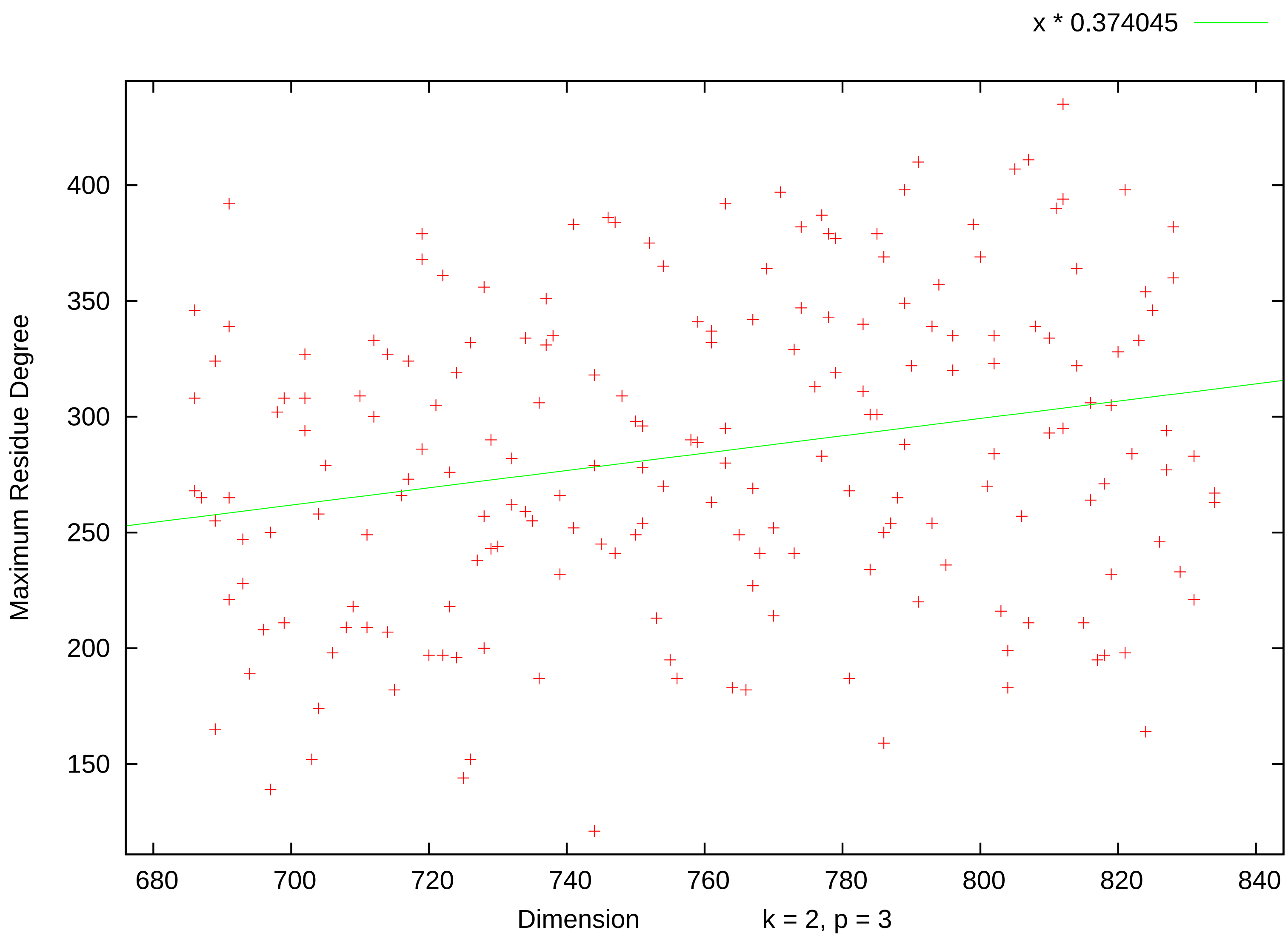
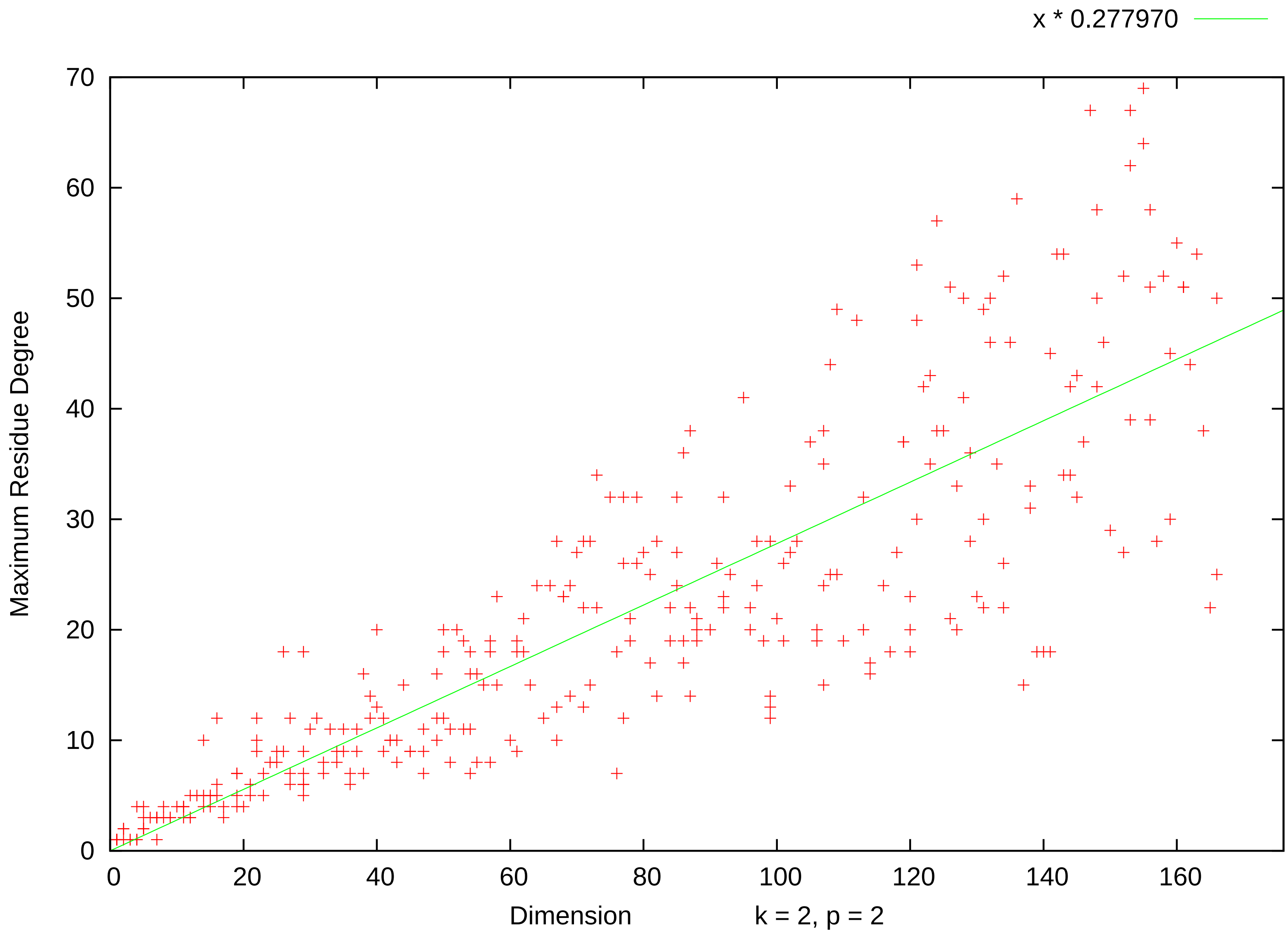


Maximum degree

Now $p = 2$.

Maximum degree mod p



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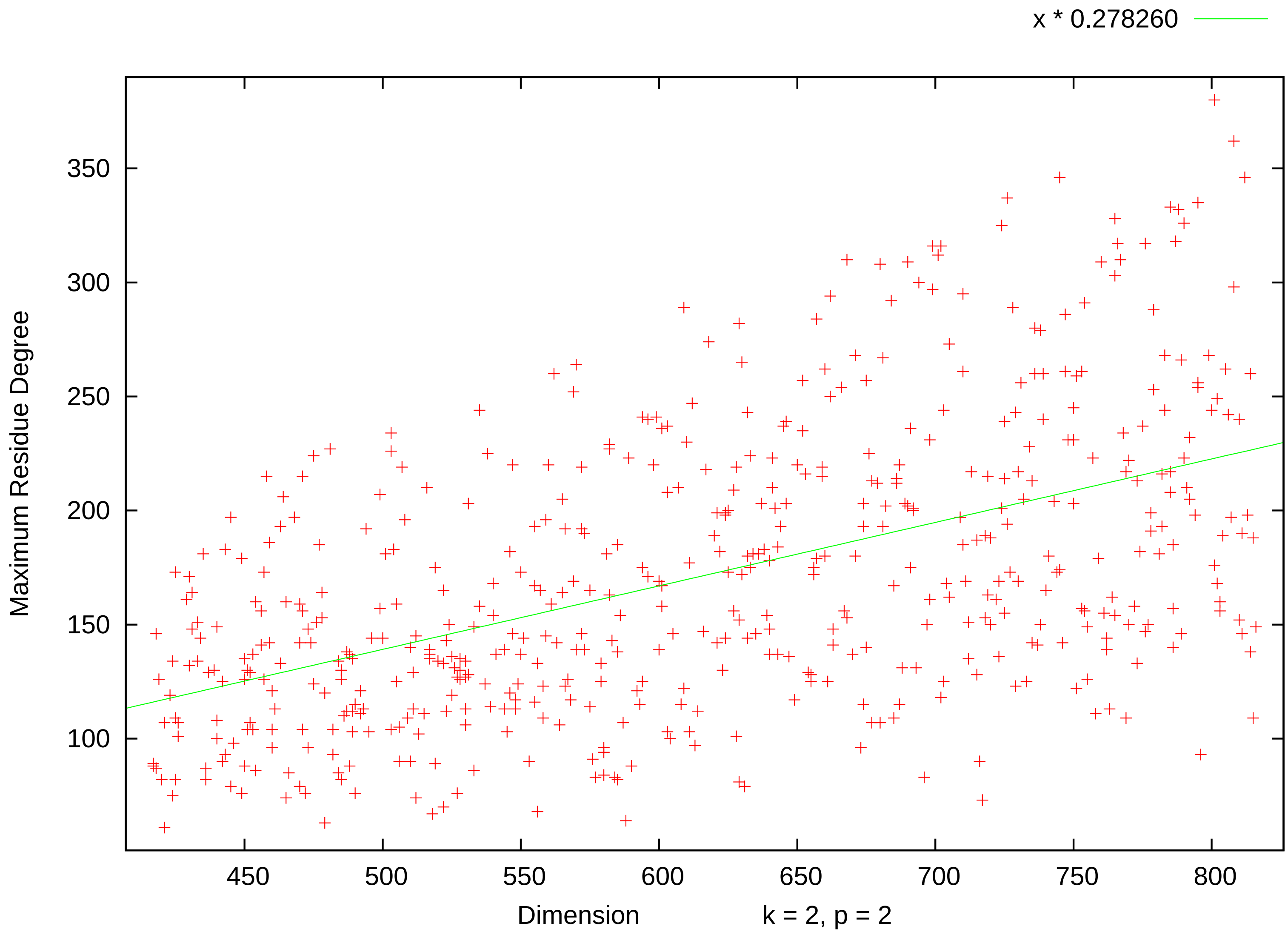

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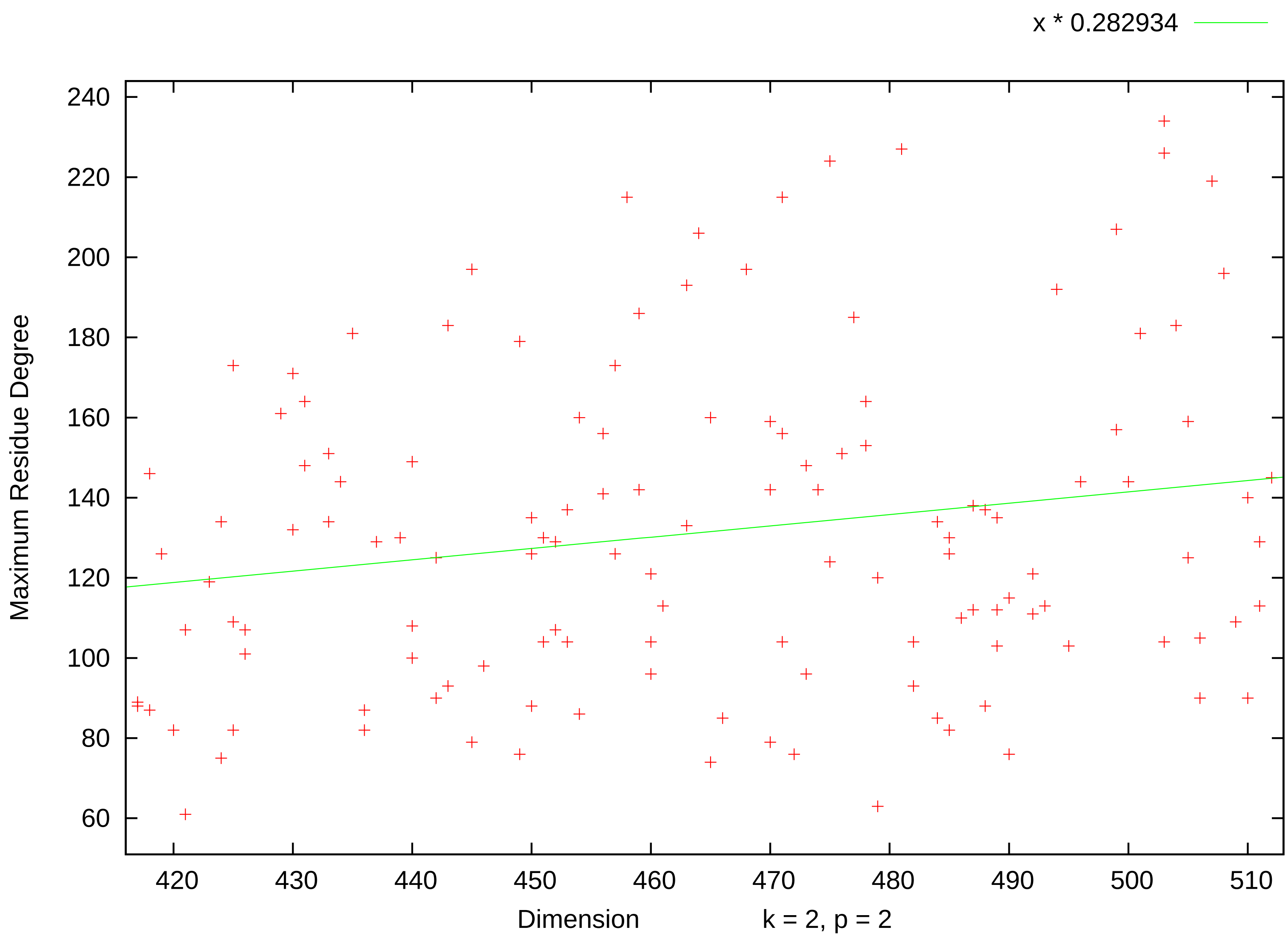

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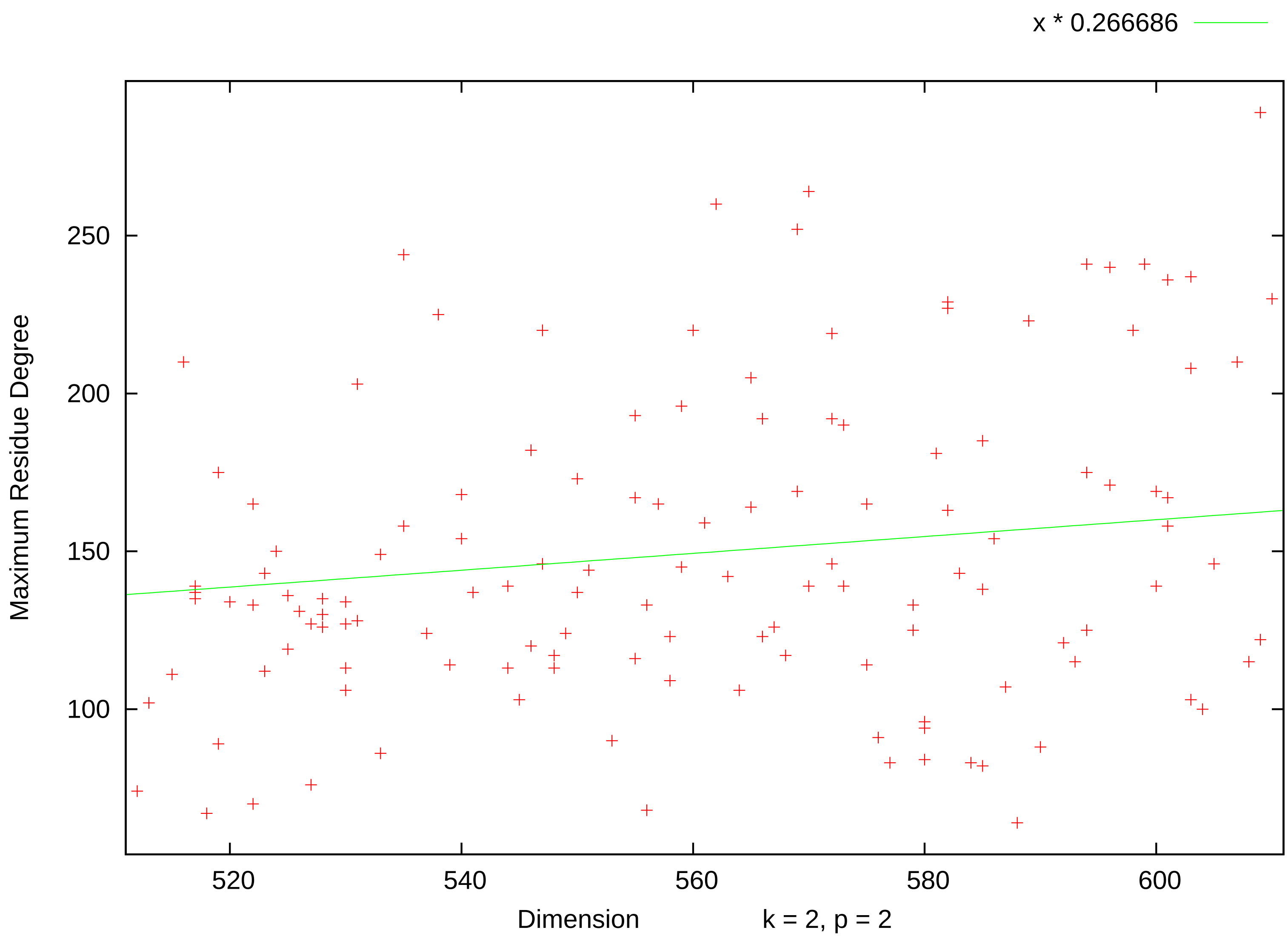

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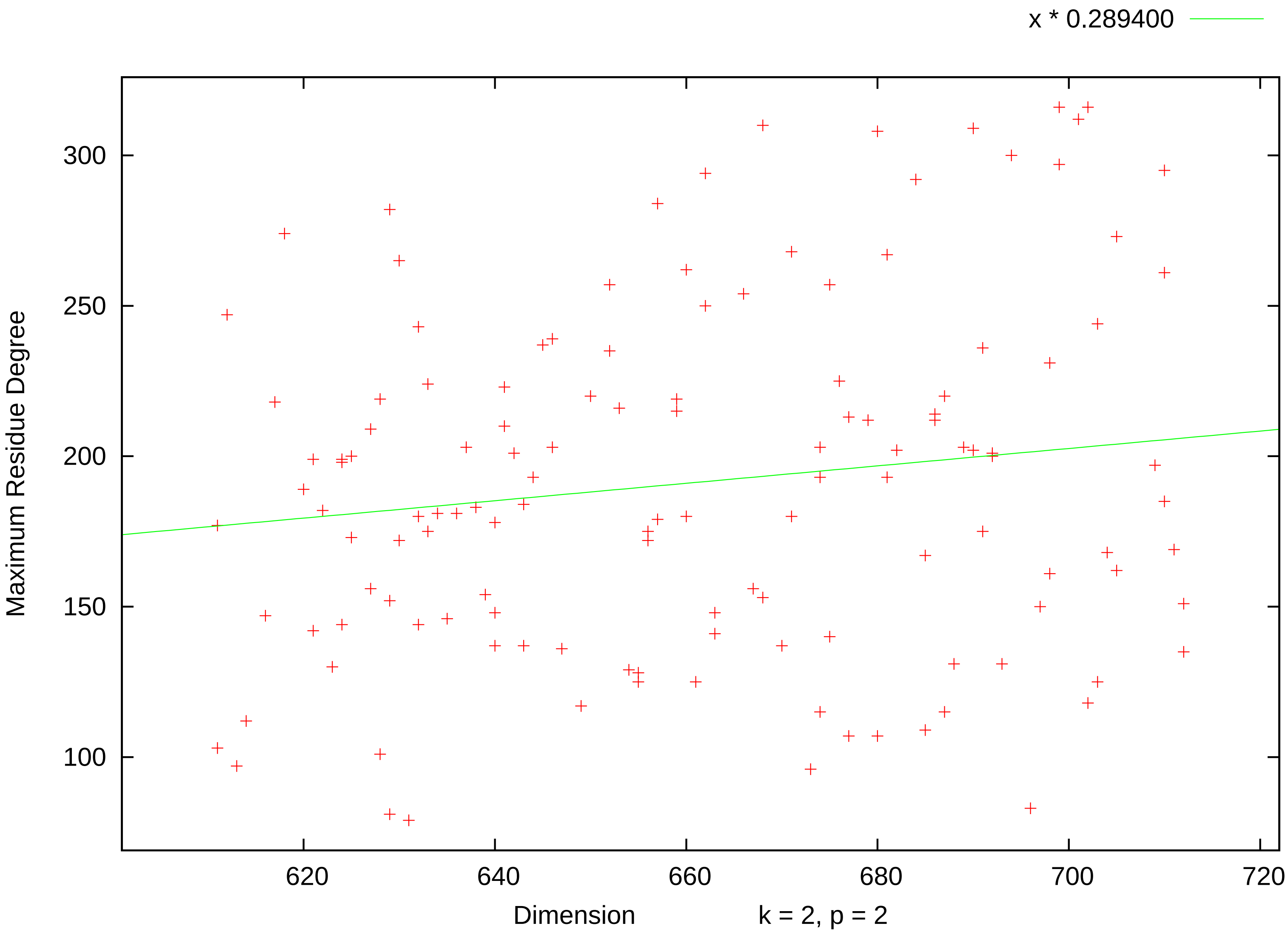

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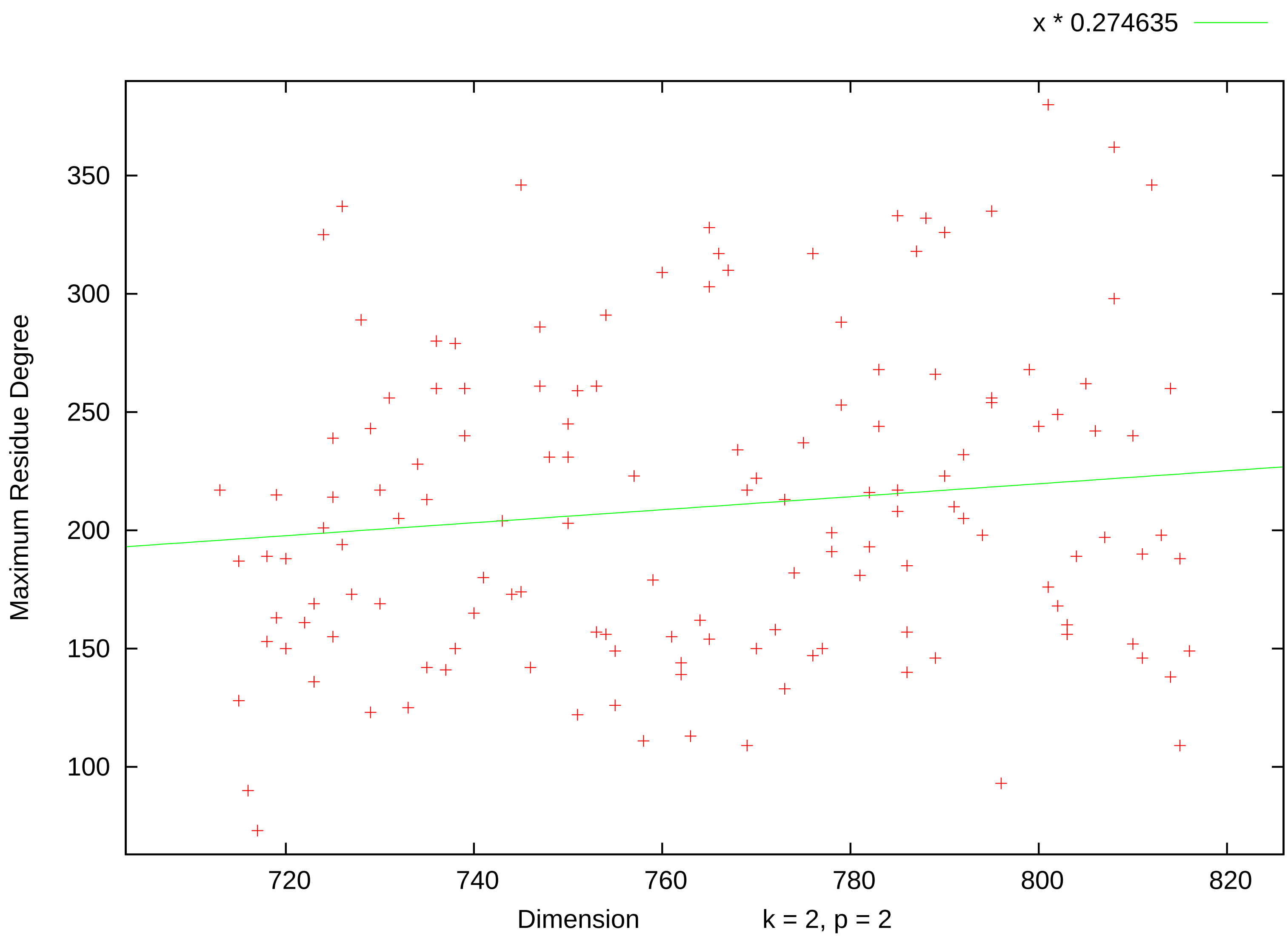

Maximum degree mod p

Question: *Fix p and the weight $k \geq 2$.*

Are there $0 < \alpha \leq \beta < 1$ and $C, D > 0$ s.t.

$$\alpha \dim S_k(N) - C \leq \max_k^{(p)}(N) \leq \beta \dim S_k(N) + D \quad ?$$

Average degree mod p

Fix p and $k = 2$.

Guess a dependence of the form

$$\text{average}_k^{(p)}(N) \sim \alpha \dim S_k(N).$$

Plot $\text{average}_k^{(p)}(N)$ as a function of $\dim S_k(N)$ for the primes $N \leq 2000$.

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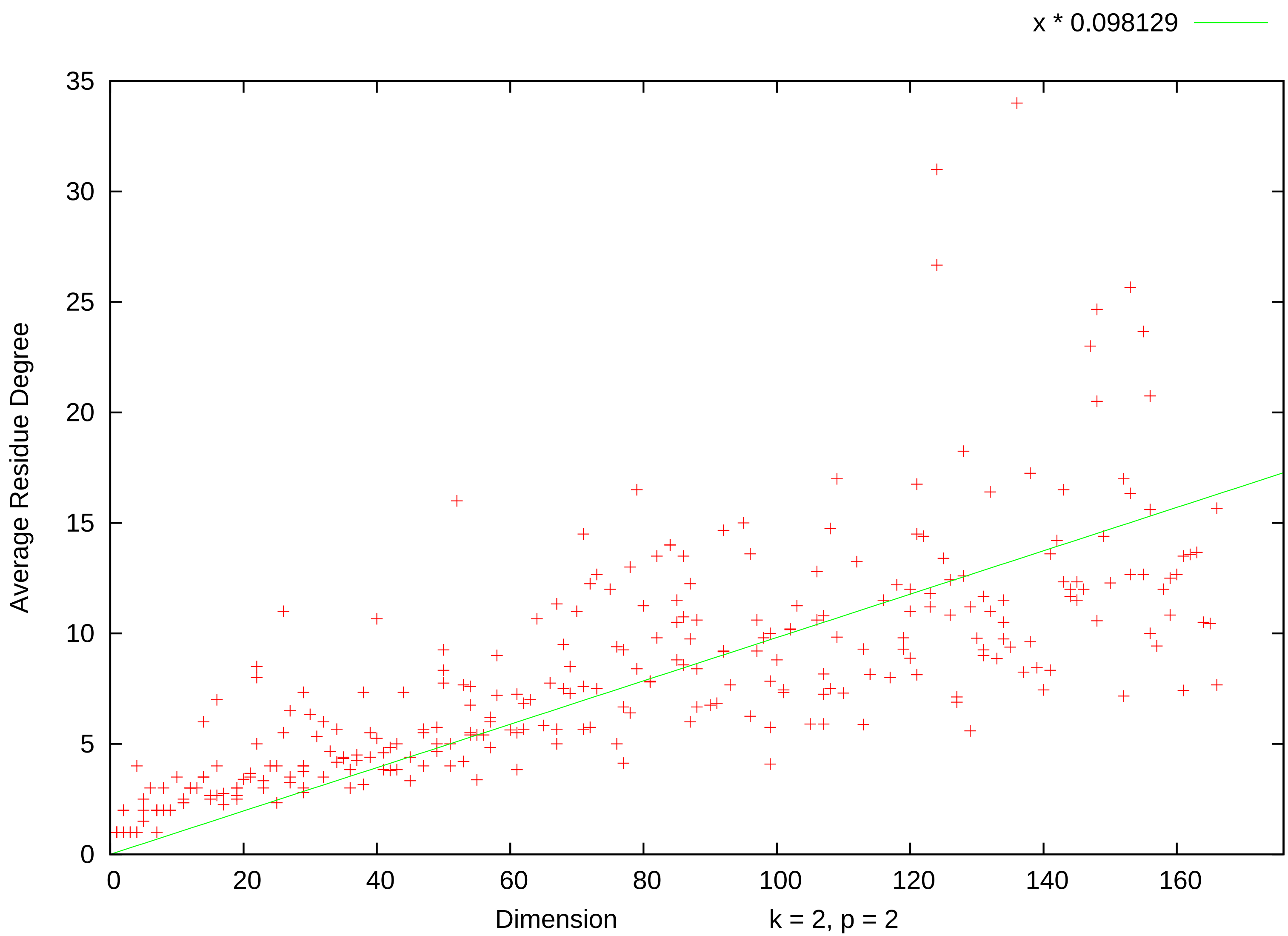
Average degree mod p



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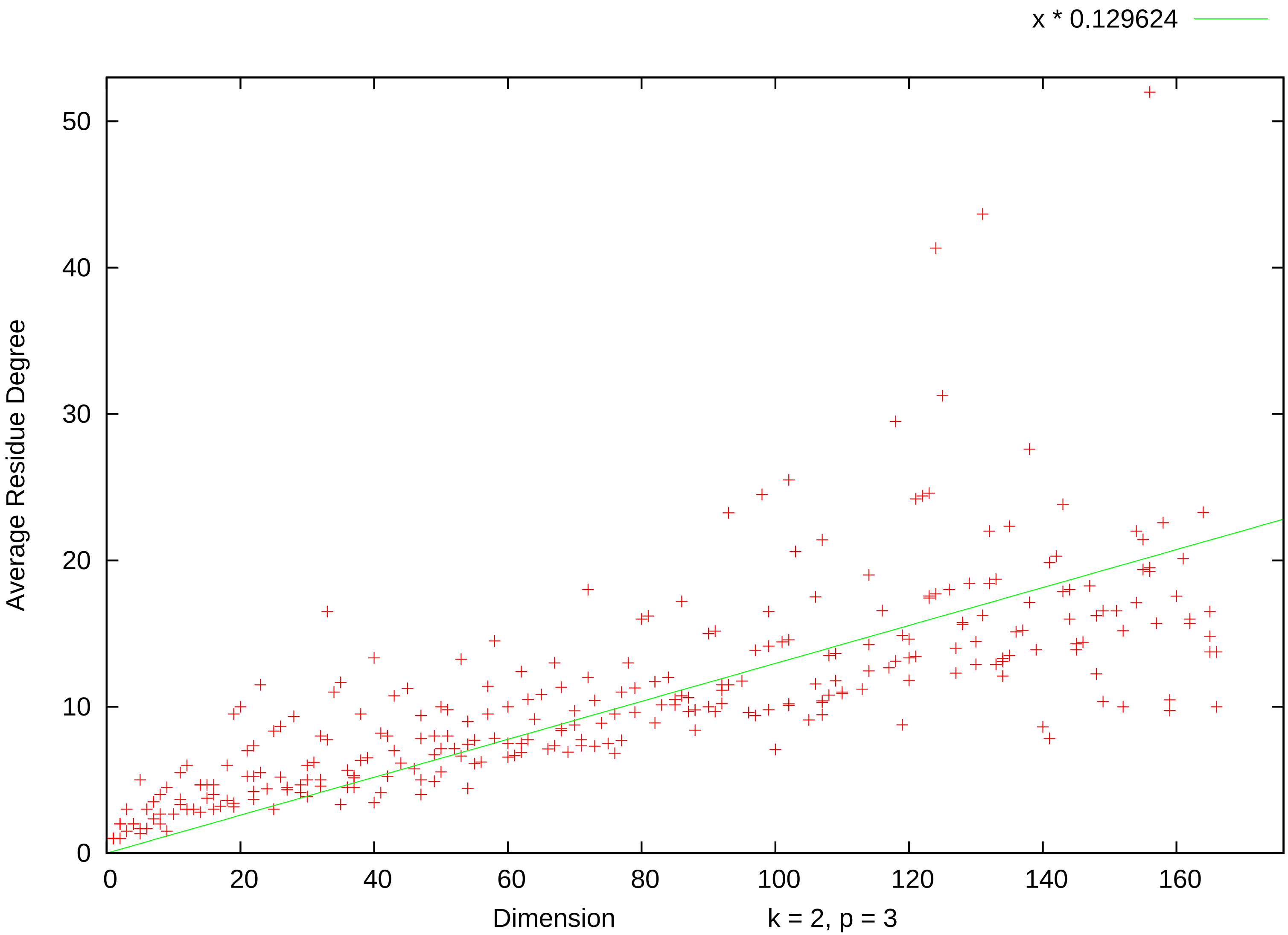

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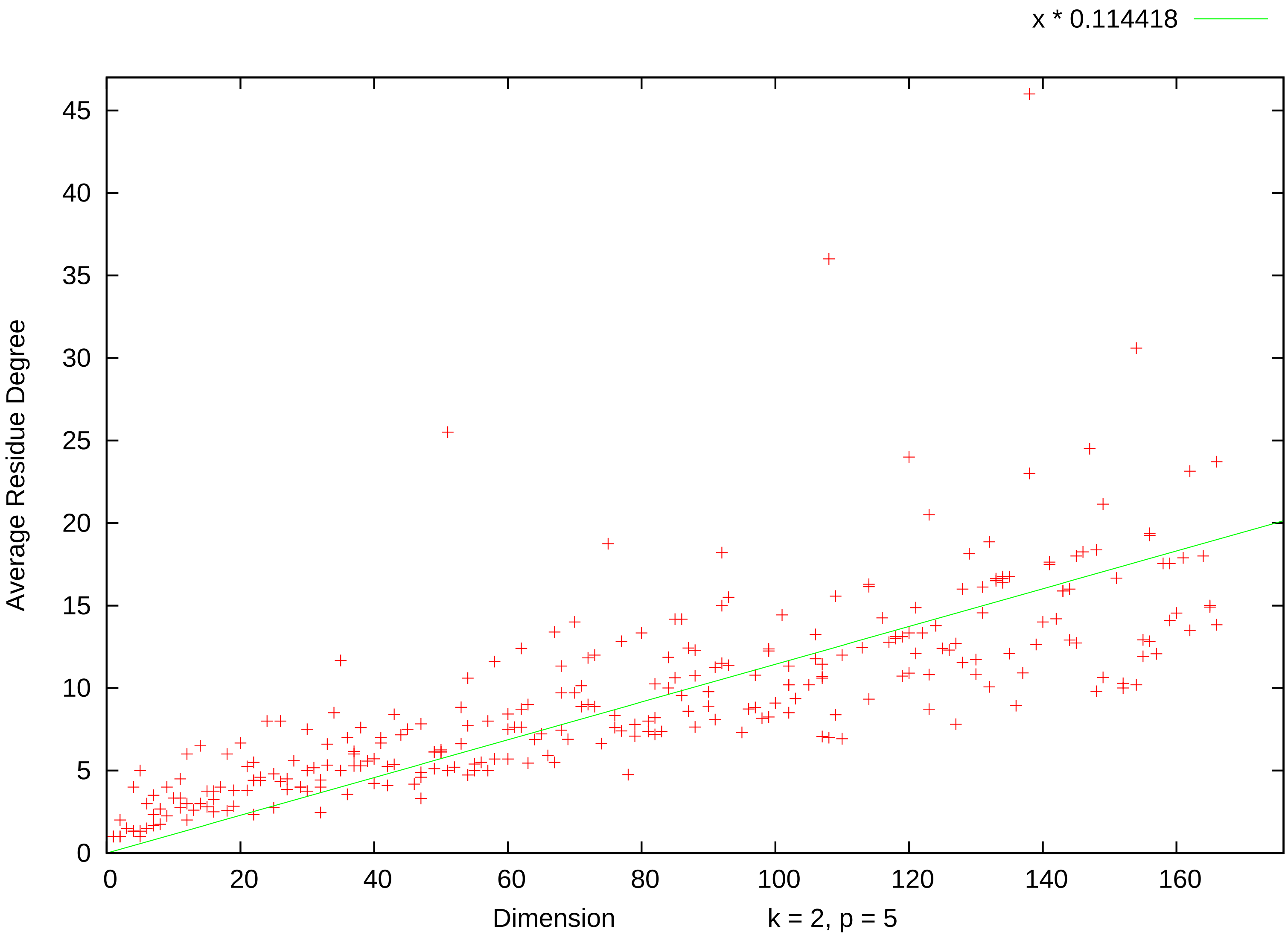

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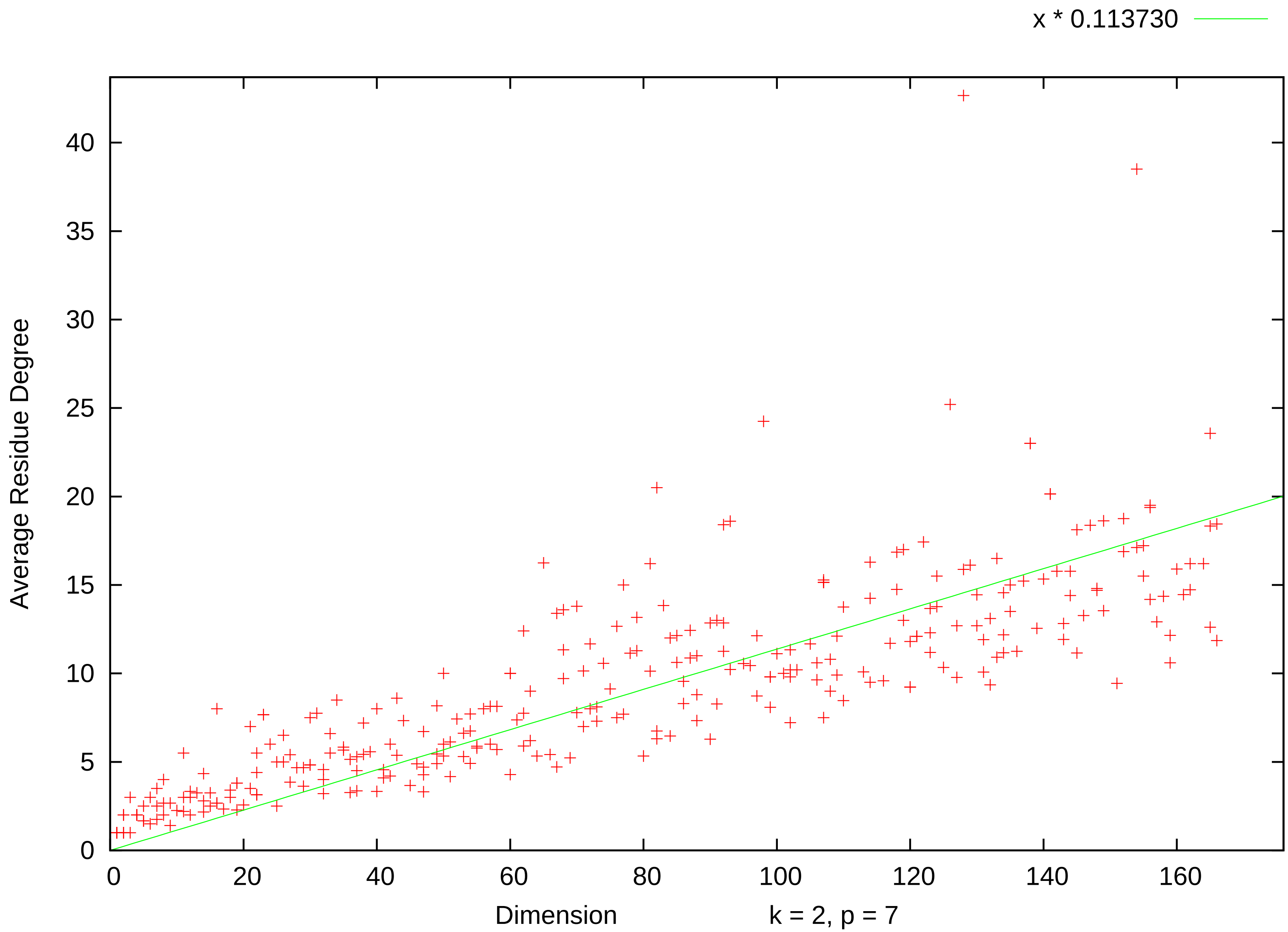

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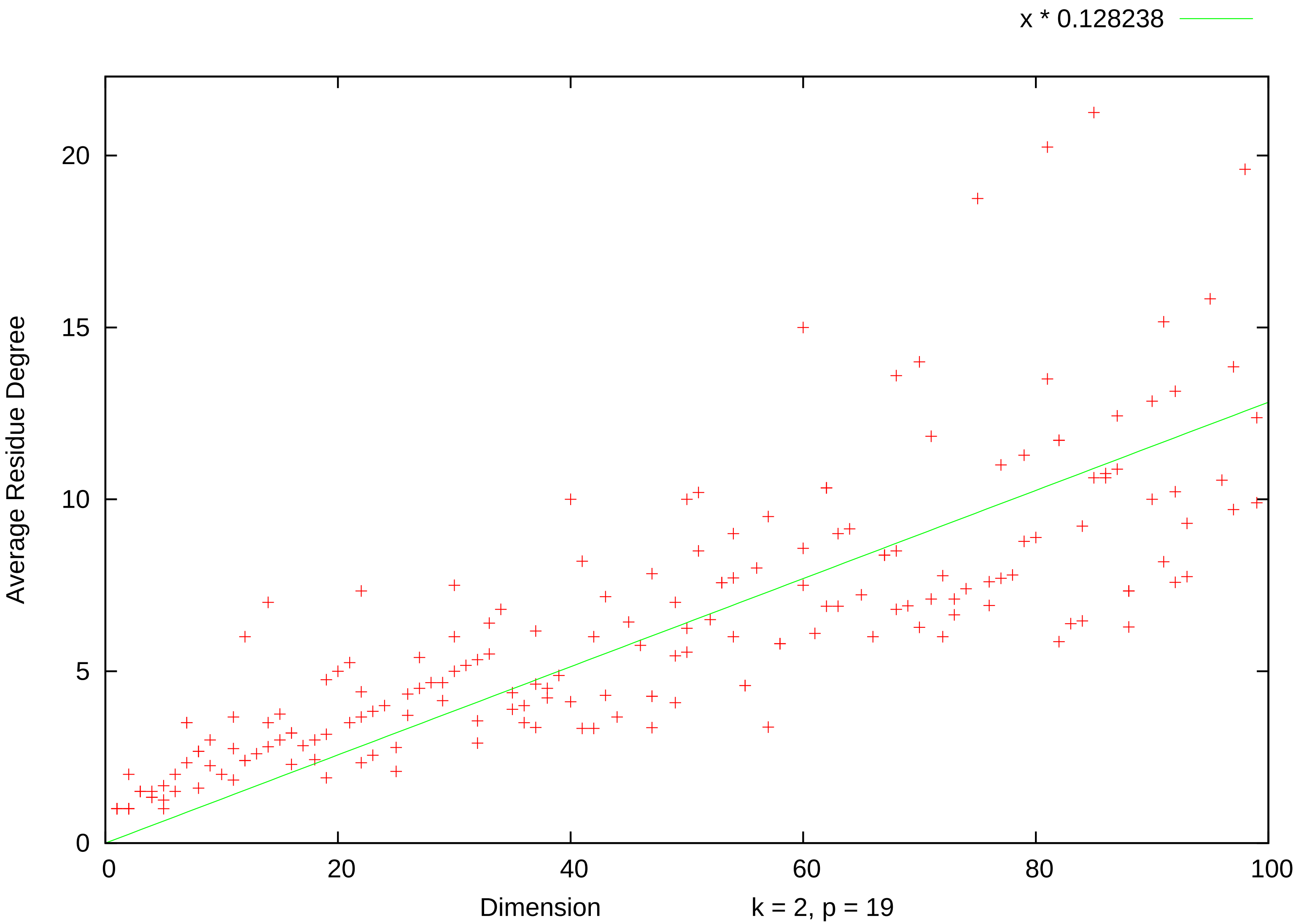

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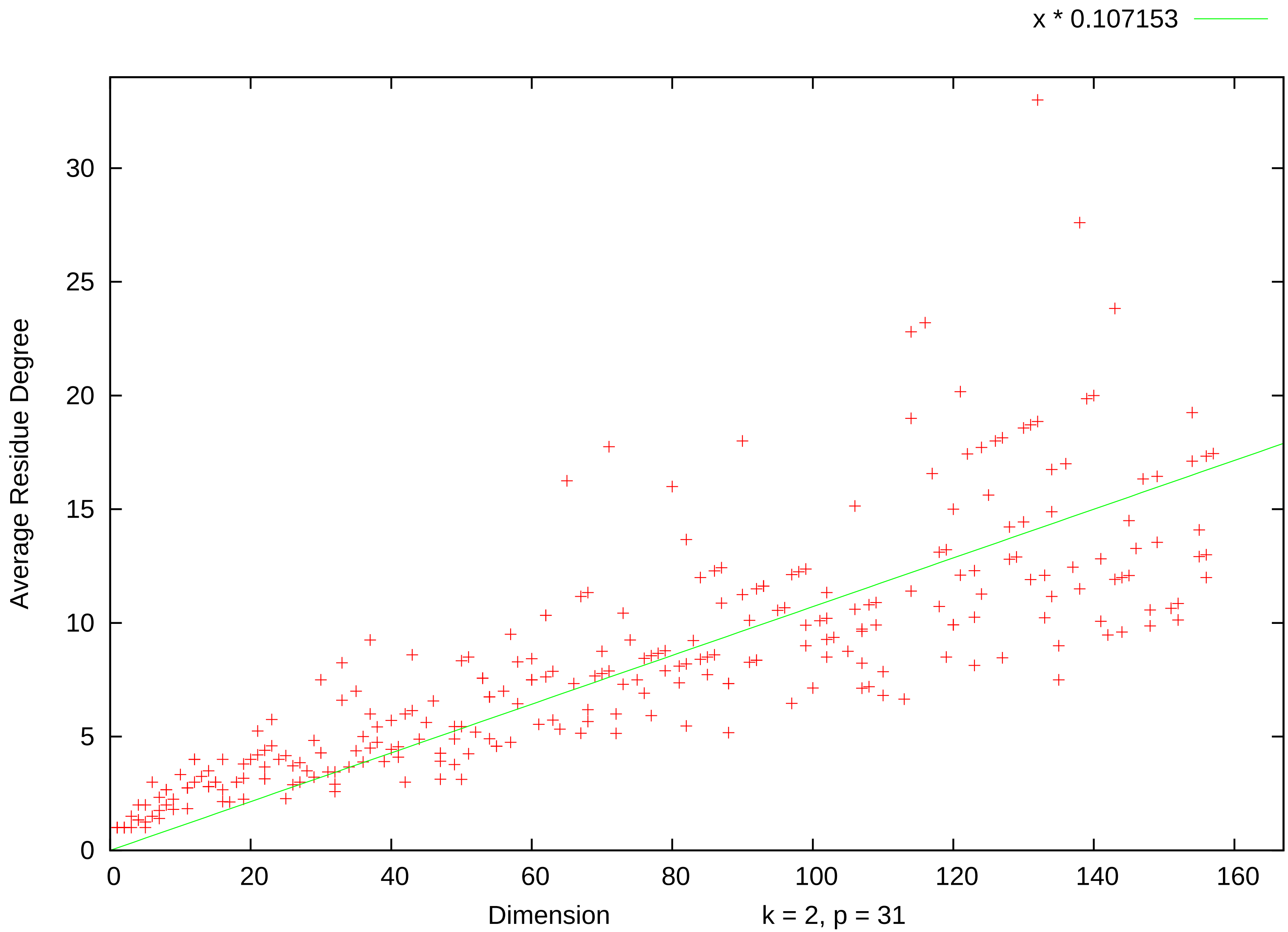
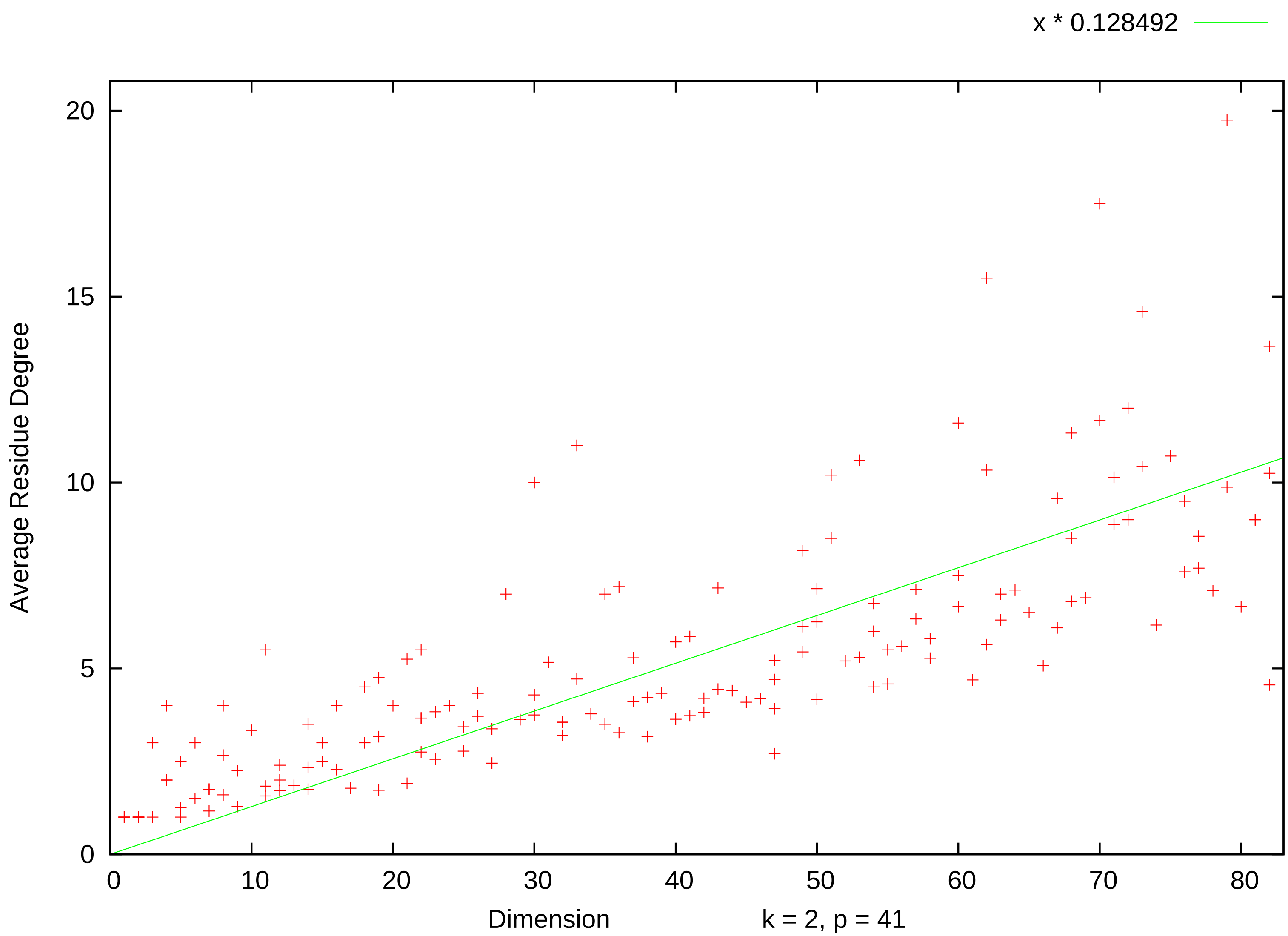
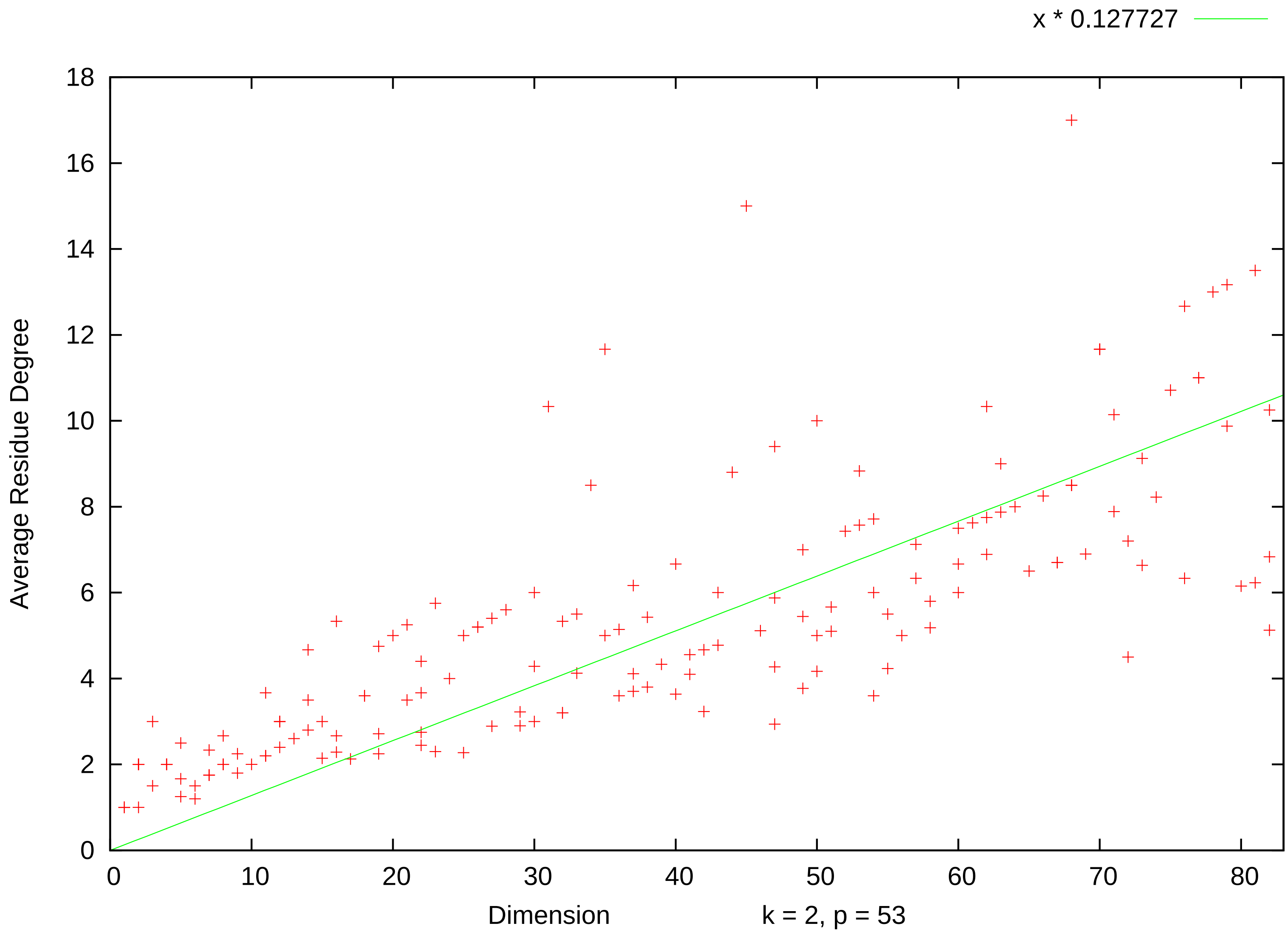


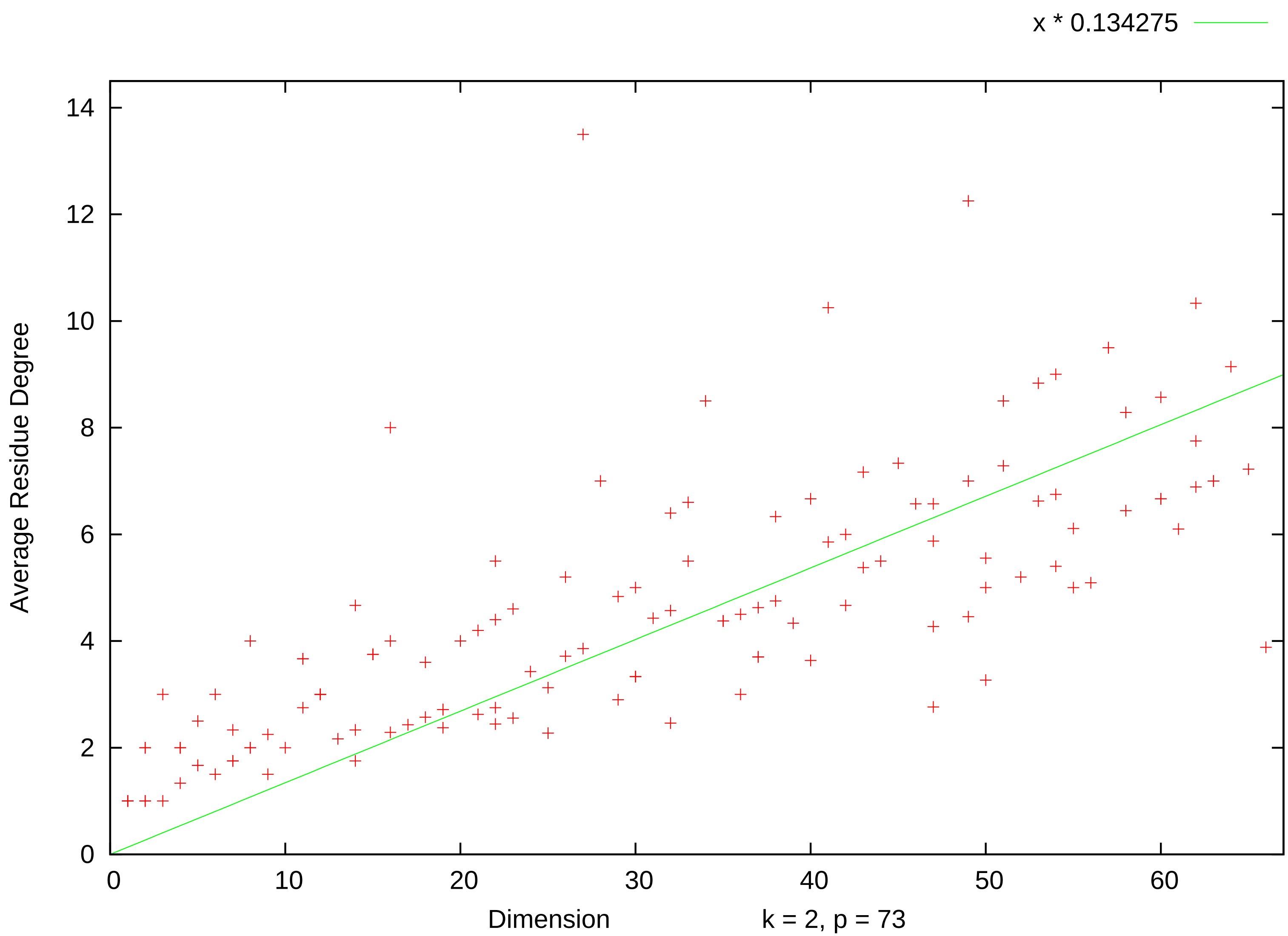
Average degree mod p



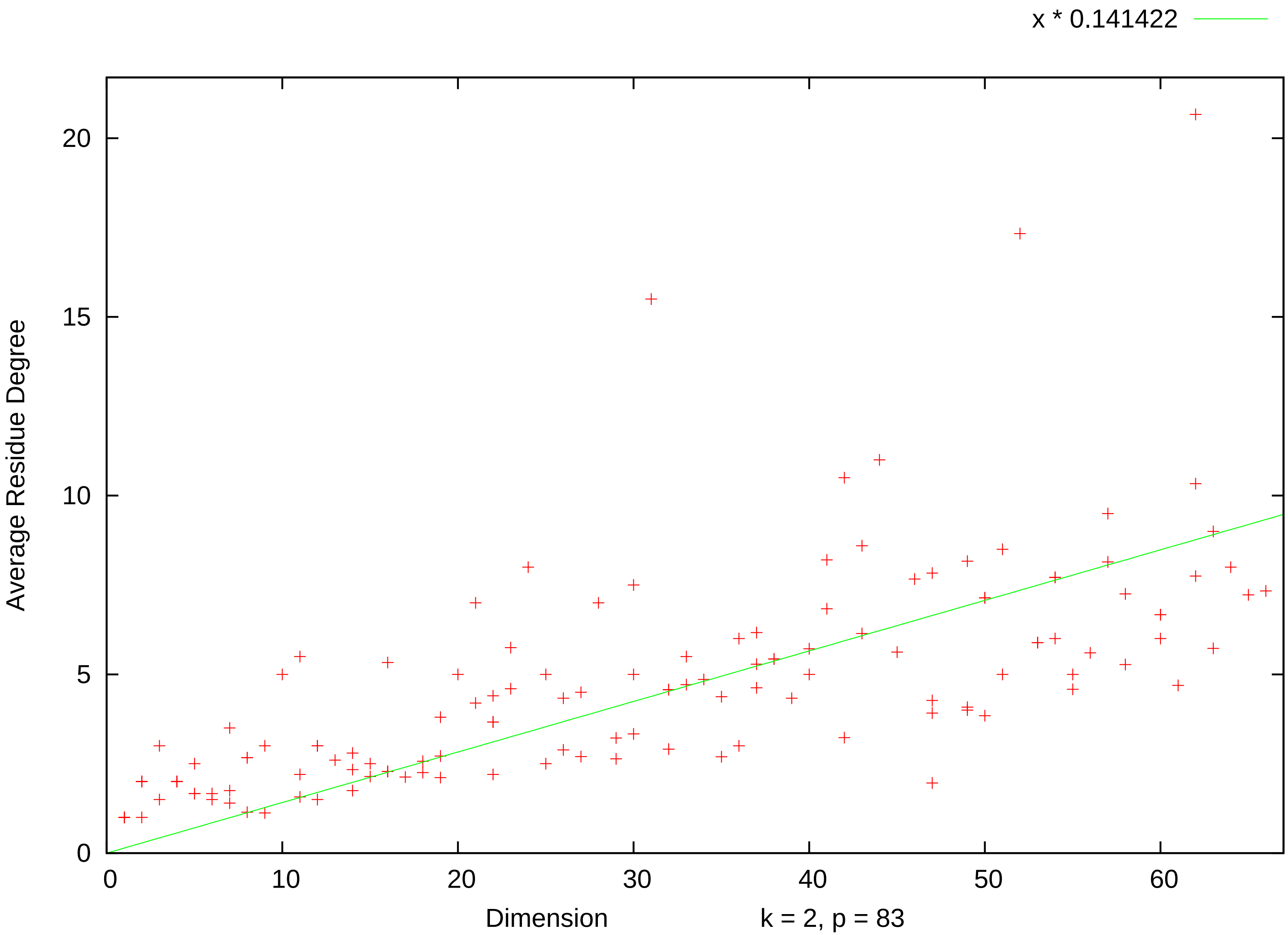
Average degree mod p



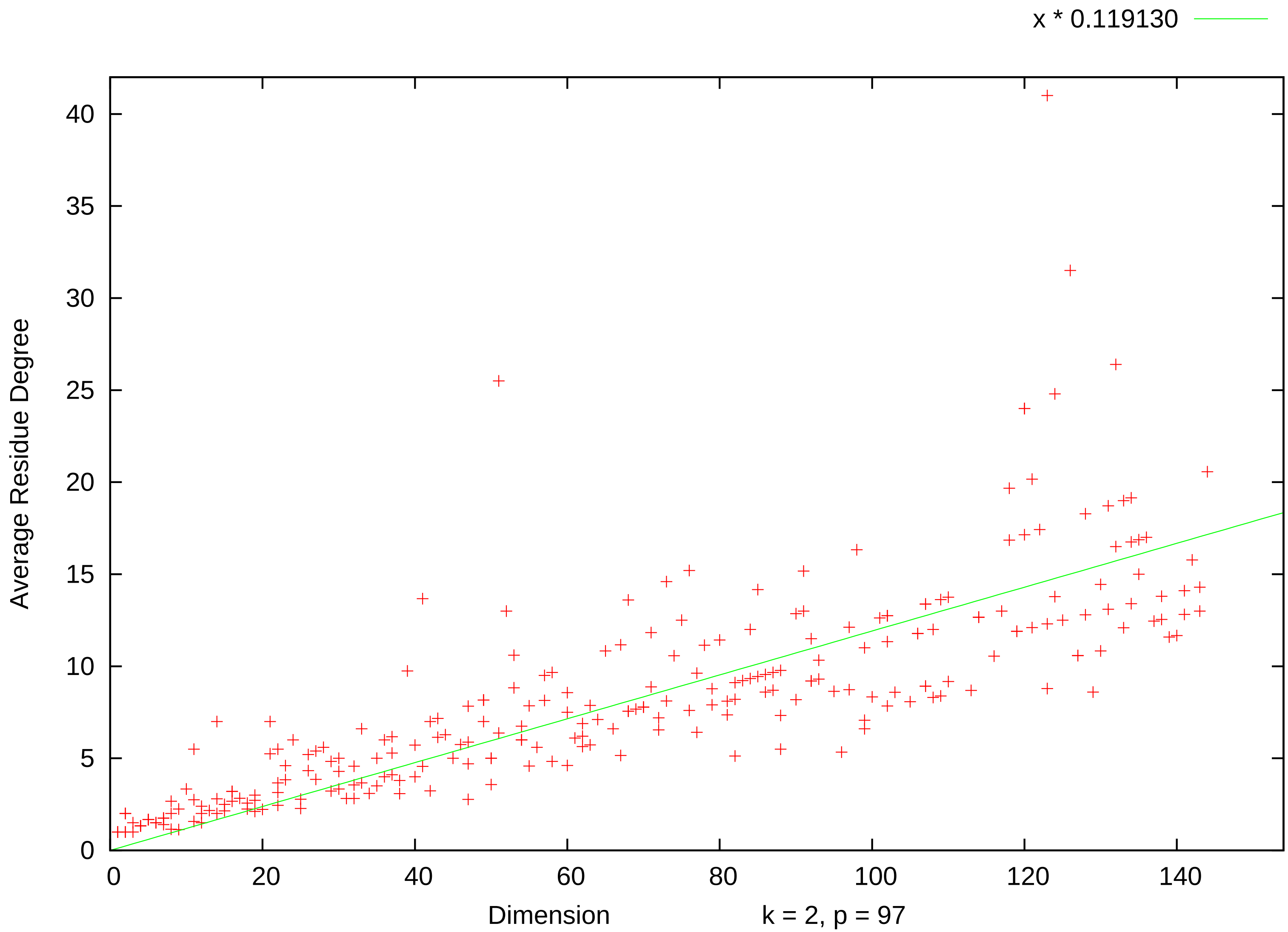
Average degree mod p



Average degree mod p



Average degree mod p

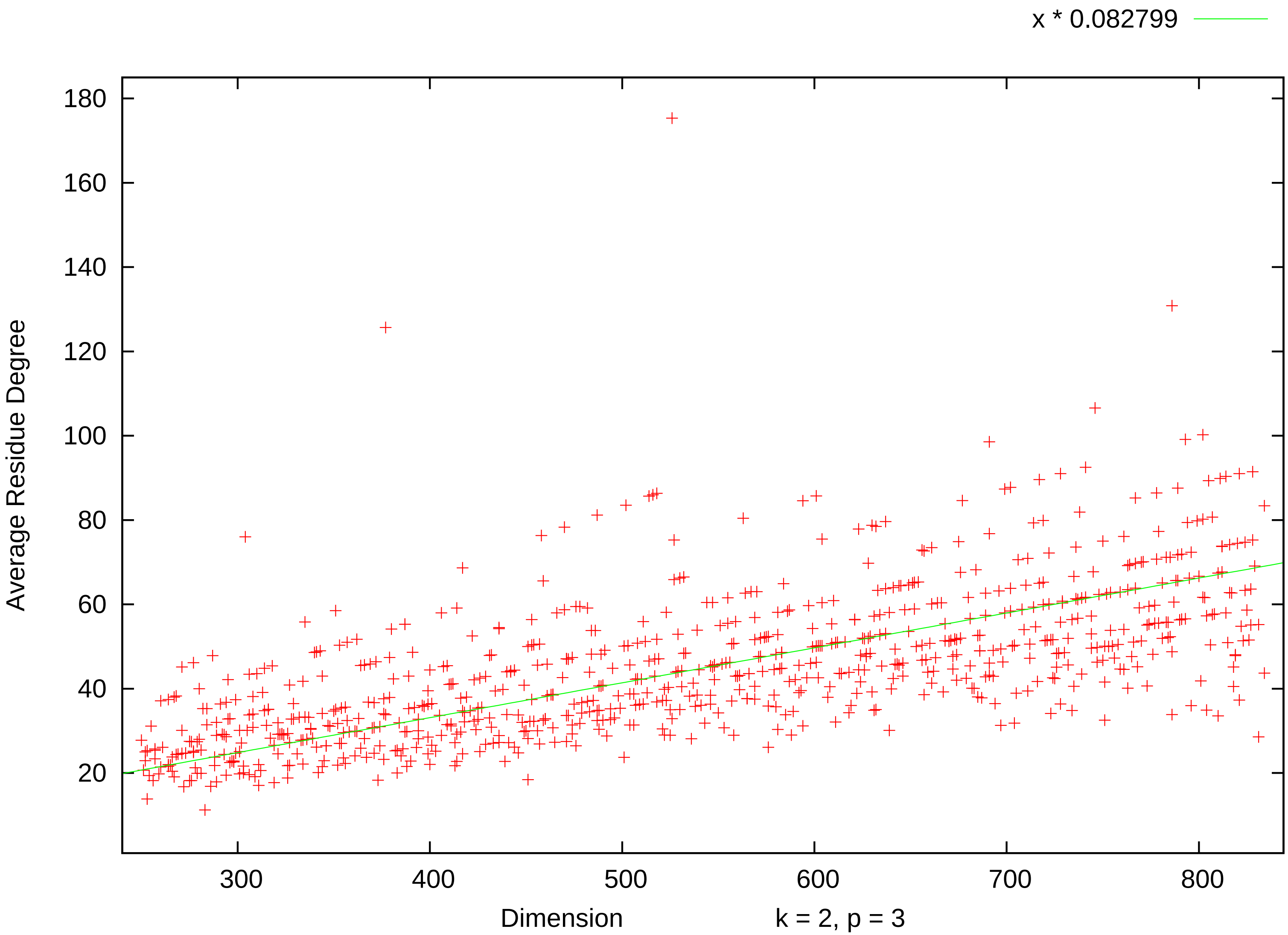


Average degree mod p

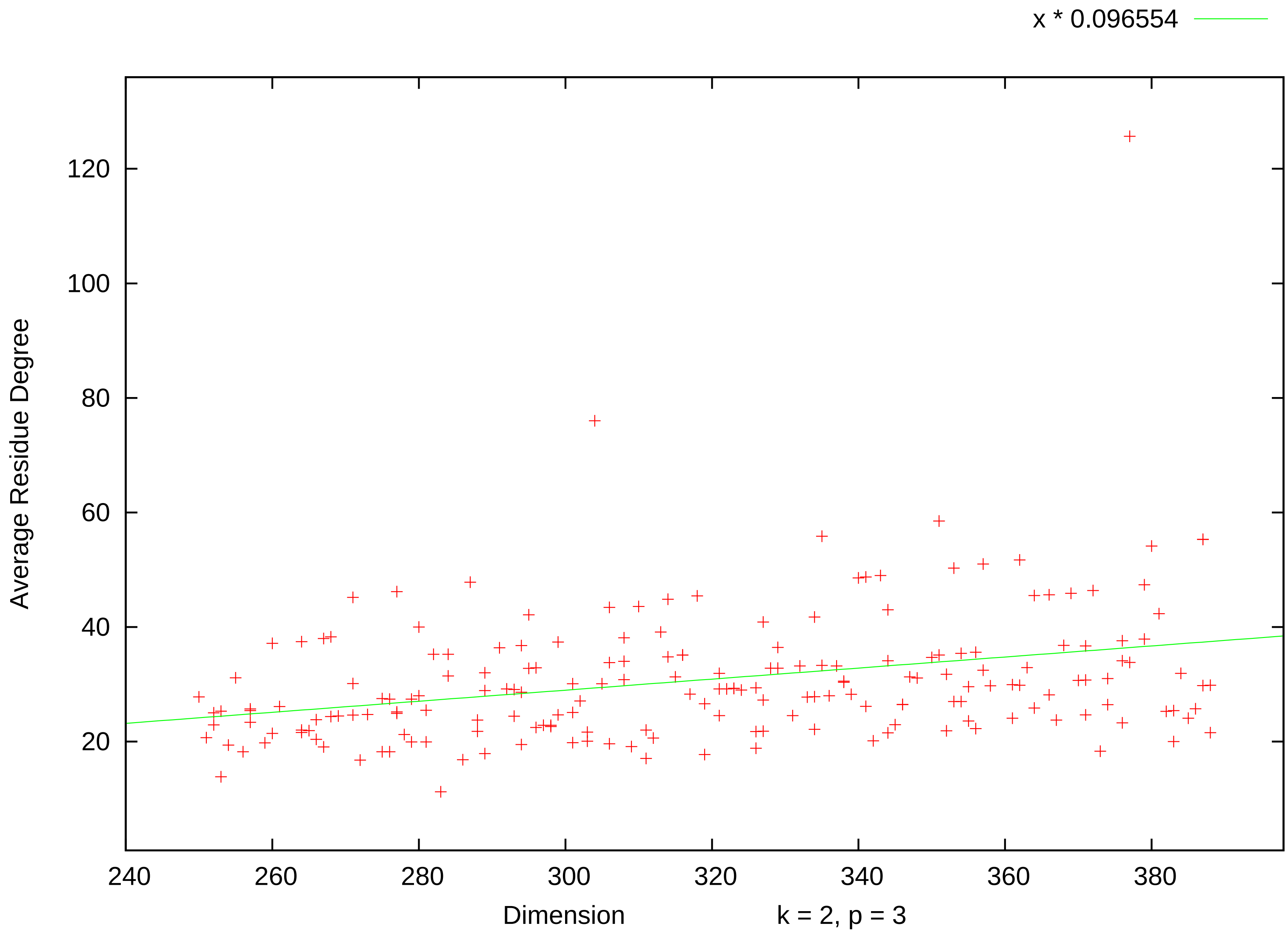
Consider $p = 3$ and weight $k = 2$ in a bigger range.

Plot $\mathrm{average}_k^{(p)}(N)$ as a function of $\dim S_k(N)$.

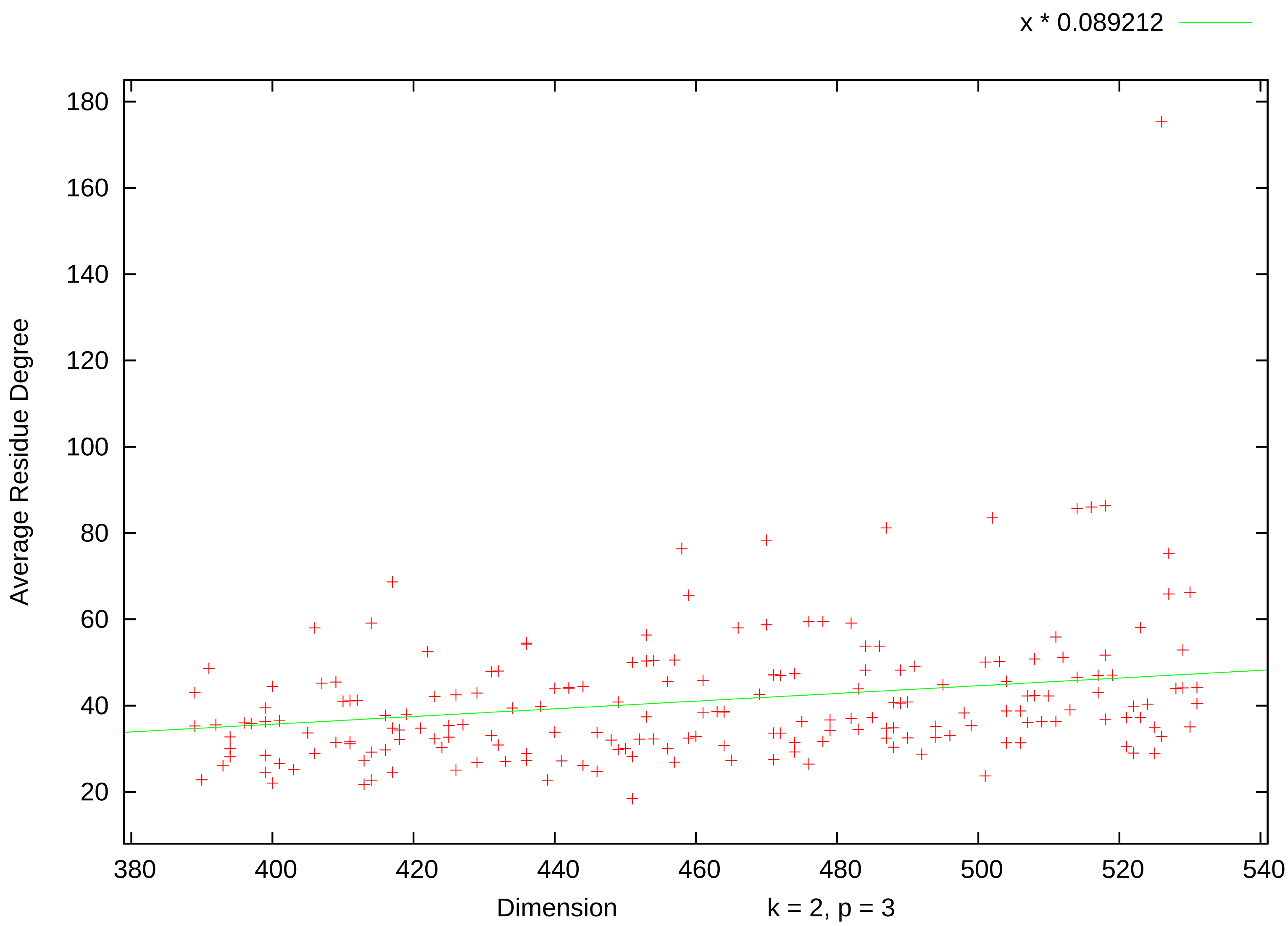
Maximum degree mod p



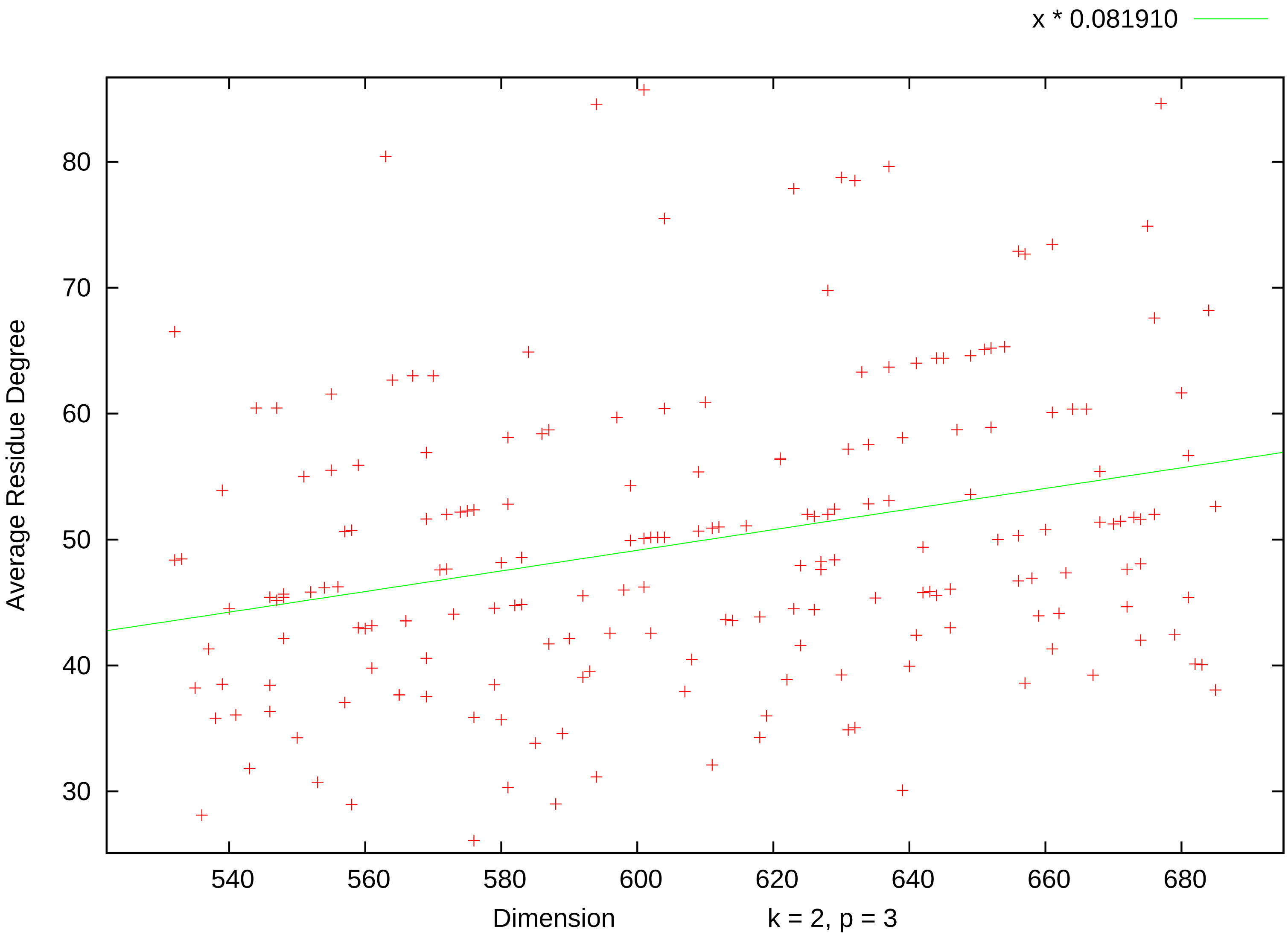
Maximum degree mod p



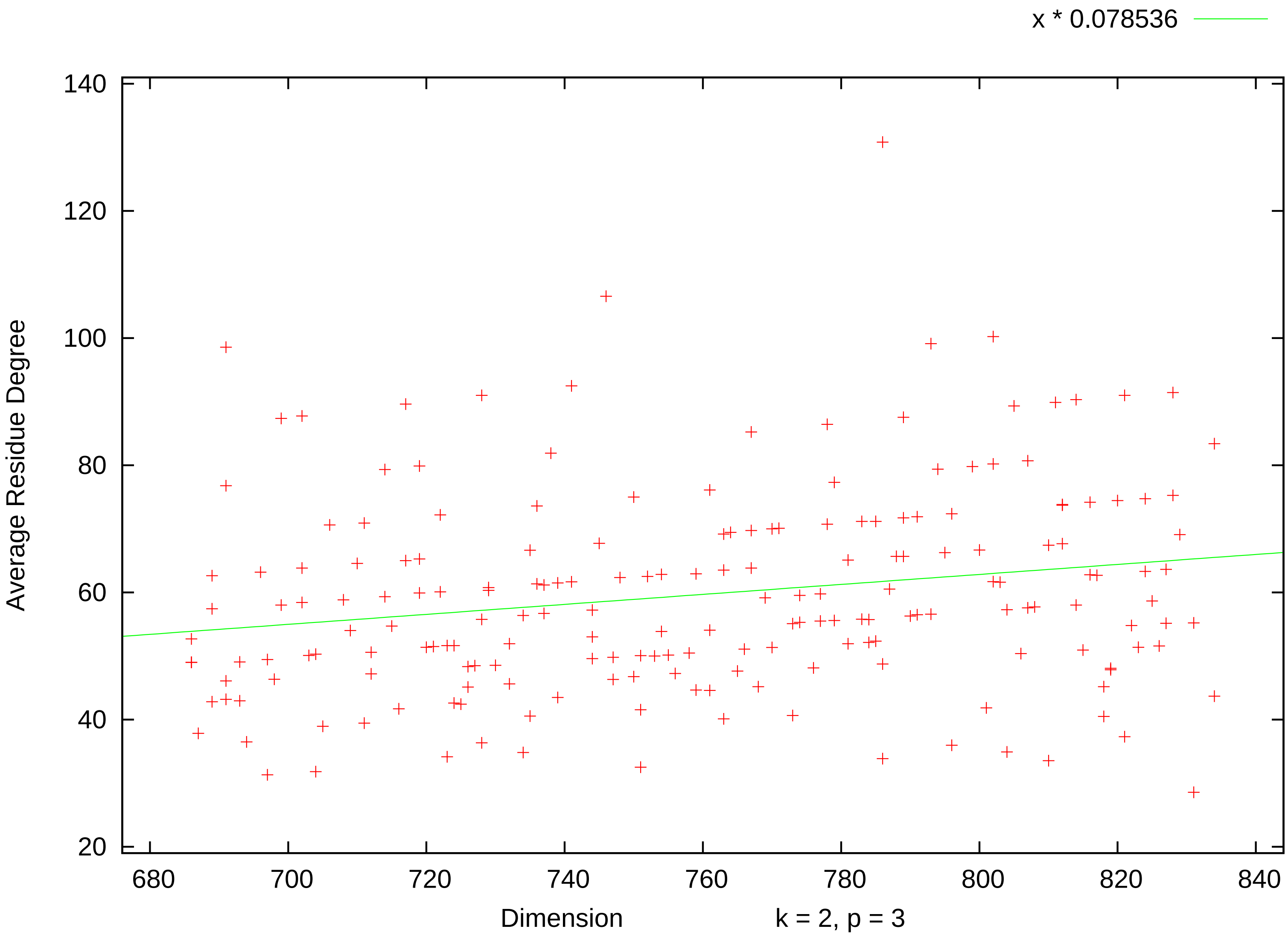
Maximum degree mod p



Maximum degree mod p



Maximum degree mod p



Average degree mod p

Question: *Fix p and the weight $k \geq 2$.*

Are there $0 < \alpha \leq \beta < 1$ and $C, D > 0$ s.t.

$$\alpha \dim S_k(N)/\log(\dim S_k(N)) - C \leq \mathrm{average}_k^{(p)}(N) \leq \beta \dim S_k(N) + D \quad ?$$

THE END

INSTITUT FÜR EXPERIMENTELLE MATHEMATIK