Exercises in Commutative Algebra

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The aim of this exercise sheet is to recall some terminology and results from previous lectures.
If you do not know this terminology, then ask! *We will then include it in the lecture.*

1. Let K be a field. Convince yourself (by recalling to yourself the necessary definitions) that K-modules are K-vector spaces and vice versa.
2. Let K be a field. By $K[X]$ we denote the polynomial ring over K in one variable. Show that $K[X]$ is a principal ideal domain, i.e. that every ideal of $K[X]$ is a principal ideal. Is the same assertion true for $\mathbb{Z}[X]$?
3. Let K be a field, V a finite dimensional K-vector space, and $\varphi \in \mathrm{End}_K(V)$ a K-endomorphism of V (i.e. by definition a K-linear map $V \to V$). Show that V is a (left) $K[X]$-module by the action
$$f.v := f(\varphi)(v) := \sum_{i=0}^{d} a_i \varphi^i(v)$$
for any $f(X) = \sum_{i=0}^{d} a_i X^i \in K[X]$ and any $v \in V$.
4. Show that one obtains a ring homomorphism
$$\Phi : K[X] \to \mathrm{End}_K(V), \quad f \mapsto (v \mapsto f.v).$$
5. Translate the previous two items into matrices by choosing a K-basis of V.
6. Relate the minimal polynomial of φ (or of the associated matrix) to the ring homomorphism Φ.
7. Let R be a ring and $N \leq M$ be R-modules. Show that the relation $x \sim y :\Leftrightarrow x - y \in N$ defines an equivalence relation on M.

   Show that the equivalence classes $\overline{x} = x + N$ form an R-module (denoted M/N) with respect to

   - $+ : M/N \times M/N \to M/N, \quad (x + N, y + N) \mapsto x + y + N,$
   - $0 = \overline{0} = 0 + N = N$ as neutral element w.r.t. $+$,
   - $. : R \times M/N \to M/N, \quad (r, x + N) \mapsto rx + N.$

   In particular, check that $+$ and . are well-defined maps.
8. Let R be a commutative ring and $I \trianglelefteq R$ be an ideal. Show that the quotient module R/I is a commutative ring with multiplication
$$\cdot : R/I \times R/I \to R/I, \quad (r + I, s + I) \mapsto rs + I.$$
9. Let R be a <u>commutative</u> ring and M a left R-module. Show how one can consider M as a right R-module.

10. Let R be a ring, M, N be R-modules and $\varphi : M \to N$ an R-homomorphism. Show that the *kernel* $\ker(\varphi) := \{m \in M \mid \varphi(m) = 0\}$ is an R-submodule of M. Also show that the *image* $\mathrm{im}(\varphi) := \{\varphi(m) \mid m \in M\}$ is an R-submodule of N.
11. Let R be a ring. An abelian group $(M, +, 0)$ is an R-module if and only if the map

$$R \to \mathrm{End}(M), \quad r \mapsto (x \mapsto r.x)$$

is a ring homomorphism. Here $\mathrm{End}(M)$ denotes the endomorphism ring of M as an abelian group.
12. Let $n, m \in \mathbb{Z}$ and let g be the greatest common divisor of n and m. Show that $(n, m) = (g)$.

Don't hesitate to contact me with any problems, questions or remarks!

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