Exercises in Commutative Algebra

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1. Let R be a ring and $\mathfrak{m}$ be a maximal ideal. Recall that $\mathfrak{m}R_{\mathfrak{m}}$ is the unique maximal ideal of the localisation $R_{\mathfrak{m}}$ of R at $\mathfrak{m}$.

   (a) Show that the natural map $\mu : R \to R_{\mathfrak{m}}$, $r \mapsto \frac{r}{1}$ induces a ring isomorphism
   $$R/\mathfrak{m} \cong R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}.$$

   (b) Let M be an R-module and denote by $M_{\mathfrak{m}}$ its localisation at $\mathfrak{m}$. Conclude from (a):
   $$M/\mathfrak{m}M \cong M_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}M_{\mathfrak{m}}.$$

2. Let R be a ring and M_i for $i \in I$ be R-modules. (For simplicity, we assume R to be commutative. But, the result holds in general with the same proof.)

   Show that the following statements are equivalent:

   (i) M_i is flat over R for all $i \in I$.

   (ii) $\bigoplus_{i \in I} M_i$ is flat over R.

   Hint: Use Proposition 8.7 and the injectivity of the direct sum of injective homomorphisms.

3. Let R be a ring. Let M be a flat and T a faithfully flat R-module. Show that $R \oplus T$ is faithfully flat over R.

4. Let $\varphi : R \to S$ be a homomorphism of rings and let M be an S-module and N an R-module. Recall that M is an R-module via $r.m := \varphi(r).m$ for $r \in R$ and $m \in M$ (you need not prove this!). Recall also that $S \otimes_R N$ is an S-module via $s.(t \otimes n) := (st) \otimes n$ for $s, t \in S$ and $n \in N$ (you need not prove this either).

   Show the following statements:

   (a) If ϕ is a flat ring homomorphism and M is a flat S-module, then M is a flat R-module.

   (b) If ϕ is a faithfully flat ring homomorphism and M is a faithfully flat S-module, then M is a faithfully flat R-module.

   (c) If ϕ is a flat ring homomorphism and N is a flat R-module, then $S \otimes_R N$ is a flat S-module.

   (d) If ϕ is a faithfully flat ring homomorphism and N is a faithfully flat R-module, then $S \otimes_R N$ is a faithfully flat S-module.

   Hint: Use Lemma 8.10.