Exercises in Commutative Algebra

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Assignments like the ones below could appear in the exam. You will not be allowed to bring any notes to the exam, so that you will not be able to look up any definitions or theorems.

1. Let $\varphi : R \to S$ be a ring homomorphism (R and S are rings). Formulate and prove the homomorphism theorem.

2. Let R be a ring.

   (a) When is R called a factorial ring?

   (b) When is R called a principal ideal domain?

   (c) When is R called a Euclidean ring?

   (d) Prove that every Euclidean ring is a principal ideal domain.

   (e) Prove that every factorial ring is integrally closed in its field of fractions.

   (f) Is $\mathbb{Z}[X]$ a Euclidean ring? Prove your answer.

3. Let K be a field. Consider the polynomial ring in countably many variables, i.e. $S := K[X_1, X_2, X_3, \dots]$.

   (a) When is a ring R called an integral domain?

   (b) Is S an integral domain?

   (c) Is S a Euclidean ring?

   (d) Is S a principal ideal domain?

   (e) When is a ring R called Noetherian?

   (f) Is S a Noetherian ring?

4. Let R be a ring and $\mathfrak{p} \lhd R$ an ideal.

   (a) When is $\mathfrak{p}$ called a prime ideal?

   (b) Show that the following two statements are equivalent:

      (i) $\mathfrak{p}$ is a prime ideal.

      (ii) $R/\mathfrak{p}$ is an integral domain.

   (c) Let $\varphi : S \to R$ be a ring homomorphism. Prove that $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of S if $\mathfrak{p}$ is a prime ideal of R.

      (Hint: Use the definition.)

5. Let $R \subseteq S$ be a ring extension.

   (a) Let $s \in S$. When is s called integral over R?

   (b) When is the ring extension $R \subseteq S$ called integral?

   (c) Assume now that $R \subseteq S$ is integral. Let $\mathfrak{b} \lhd S$ be an ideal and $\mathfrak{a} := \mathfrak{b} \cap R \lhd R$.
   Show that the inclusion $\iota : R \to S$ induces an injective ring homomorphism $R/\mathfrak{a} \hookrightarrow S/\mathfrak{b}$, which is an integral ring extension.

   (d) Keeping the notation of (c) and assuming that $\mathfrak{b}$ is a prime ideal, show the following:
   $\mathfrak{a}$ is maximal $\Leftrightarrow$ $\mathfrak{b}$ is maximal.

   (e) Let $R \subseteq S$ be an integral ring extension and assume in addition that S is an integral domain. Show:
   R is a field $\Leftrightarrow$ S is a field.

6. Let R be a ring.

   (a) Let $T \subseteq R$ be a multiplicatively closed subset containing 1. What is the definition of $T^{-1}R$?

   (b) Let $\mathfrak{p} \lhd R$ be a prime ideal. How is the localisation $R_\mathfrak{p}$ of R at $\mathfrak{p}$ defined?

   (c) Let R be an integral domain. Describe the localisation of R at (0). Which other name does it have?

   (d) Let $R \subseteq S$ be an integral ring extension. Show that $T^{-1}R \subseteq T^{-1}S$ is an integral ring extension.

7. Let R be a ring. An element $x \in R$ is called nilpotent if there is $k \in \mathbb{N}$ such that $x^k = 0$. Let $\mathrm{Nil}(R)$ be the subset of R consisting of the nilpotent elements.

   (a) Show that $\mathrm{Nil}(R)$ is an ideal of R, which is contained in all prime ideals of R.

   (b) Show that $\mathrm{Nil}(R/\mathrm{Nil}(R)) = (0)$.

   (c) Let $x \in R$ be nilpotent. Show that $1 - x$ is a unit in R.

8. Let R be a ring.

   (a) What is the universal property of a free R-module over a set I?

   (b) Show, using the universal property of a free module over a set (as in (a)), that every R-module M is a quotient module of a free module.

   (c) Let M be an R-module. A free resolution of M is an exact sequence

   $$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0$$

   consisting of free R-modules F_n for $n \in \mathbb{N}$.
   Show that every R-module M admits a free resolution.
   Hint: Use (b) repeatedly.