Exercises in Commutative Algebra

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1. (a) Let R be a ring and M be an R-module. Let further M_i for $i = 1, \dots, n$ be submodules of M such that M is generated by the M_i for $i = 1, \dots, n$. Show that the following two statements are equivalent:

   (i) M is Noetherian (resp. Artinian).

   (ii) M_i is Noetherian (resp. Artinian) for all $i = 1, \dots, n$.

   Hint: You may use Lemma 11.4.

   (b) Let R be a Noetherian (resp. Artinian) ring. Conclude from (a) that every finitely generated R-module is Noetherian (resp. Artinian).

2. Let R be a Noetherian local ring and $\mathfrak{m} \lhd R$ its maximal ideal. Show the following assertions:

   (a) $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an $R/\mathfrak{m}$-vector space for the natural operation.

   (b) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ is the minimal number of generators of the ideal $\mathfrak{m}$.
   Hint: Use the corollary of Nakayama's Lemma.

   (c) If $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 1$, then $\mathfrak{m}$ is a principal ideal and there are no ideals $\mathfrak{a} \lhd R$ such that $\mathfrak{m}^{n+1} \subsetneq \mathfrak{a} \subsetneq \mathfrak{m}^n$ for any $n \in \mathbb{N}$.

3. Let R be a local Noetherian integral domain of Krull dimension 1 with maximal ideal $\mathfrak{m}$. Let $(0) \subsetneq I \lhd R$ be an ideal.

   Show that there is $n \in \mathbb{N}$ such that $\mathfrak{m}^n \subseteq I$.

   Hint: Let Σ be the set of all ideals $I \lhd R$ such that $\mathfrak{m}^n \not\subseteq I$ for all $n \in \mathbb{N}$. This set is non-empty and contains a maximal element I. Show that $I = (0)$. Otherwise, I is not a prime ideal, so it contains a product xy without containing x and y individually. Now consider (I, x) and (I, y).

Please turn over.

4. Let K be a field, $f \in K[X,Y]$ a non-constant irreducible polynomial and $C = \mathcal{V}_{(f)}(K)$ the associated plane curve.

   Let $(a,b) \in C$ be a point. The *tangent equation to C at (a,b)* is defined as

   $$T_{C,(a,b)}(X,Y) = \frac{\partial f}{\partial X}|_{(a,b)}(X-a) + \frac{\partial f}{\partial Y}|_{(a,b)}(Y-b) \in K[X,Y].$$

   If $T_{C,(a,b)}(X,Y)$ is the zero polynomial, then we call (a,b) a *singular point of C*.

   If (a,b) is non-singular (also called: *smooth*), then $\mathcal{V}_{T_{C,(a,b)}}(K)$ is a line (instead of $\mathbb{A}^2(K)$), called the *tangent line to C at (a,b)*.

   (a) Let $f(X,Y) = Y^2 - g(X) \in K[X,Y]$, where $g(X) \in K[X]$. Determine all the singularities of the associated curve C by relating them to the zeros of $g(X)$.

   (b) Let $f(X,Y) = Y^2 - X^3 \in \mathbb{R}[X,Y]$.

   Make a sketch of the associated curve C. Find all its singularities. Describe the behaviour of the tangent lines at points on any of the two branches close to the singularity, when they approach the singularity.

   (c) Let $f(X,Y) = Y^2 - X^3 - X^2 \in \mathbb{R}[X,Y]$.

   Make a sketch of the associated curve C. Find all its singularities. Describe the behaviour of the tangent lines at points on any of the two branches close to the singularity, when they approach the singularity.

   (d) Let $f(X,Y) = Y(Y-X)(Y+X) + X^6 - Y^7 \in \mathbb{R}[X]$.

   Make a sketch of the associated curve C. Find all its singularities.