Exercises in Commutative Algebra

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Sheet 2
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1. Finish Sheet 1.
2. Let R be a ring and N, M be R-modules. Show that $\mathrm{Hom}_R(M, N)$ is an R-module with respect to pointwise defined $+$ and ., i.e. $(f + g)(m) := f(m) + g(m)$ and $(r.f)(m) := r.(f(m))$ for all $f, g \in \mathrm{Hom}_R(M, N)$, all $m \in M$ and all $r \in R$.
3. Let R be a ring different from the zero-ring (i.e. $0 \neq 1$). Show that the following statements are equivalent:
   (i) R is a field.
   (ii) The only ideals of R are (1) and (0).
   (iii) Every homomorphism of R into a non-zero ring S is injective.
4. Let $i = \sqrt{-1} \in \mathbb{C}$. Convince yourself that the ring of Gaussian integers $\mathbb{Z}[i] := \{a+bi \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$ with $+$ and $\cdot$ is a subring of $\mathbb{C}$.

   Show that it is a Euclidean ring with respect to the *norm*:

$$N(a + ib) := (a + ib)(a - ib) = (a + ib)\overline{(a + ib)} = a^2 + b^2.$$