Exercises in Commutative Algebra

Winter Term 2011/2012

Université du Luxembourg
Prof. Dr. Gabor Wiese

Sheet 3
03/10/2011

1. Let $K \subseteq L \subseteq M$ be finite field extensions. Prove *multiplicativity of degrees*, i.e. prove the formula

$$[M : K] = [M : L][L : K]$$

(in other words: $\dim_K M = (\dim_K L)(\dim_L M)$.). Also show that this formula even holds if the field extensions are allowed to be infinite with the usual rules $n\infty = \infty$ for any $n > 0$ and $\infty\infty = \infty$.

2. Let K be a field, L/K a field extension and $a \in L$. Show that the *evaluation map*

$$\Phi_a : K[X] \to L, \quad f \mapsto f(a)$$

is a homomorphism of rings.

3. Let R be an integral domain. Show that $R[X]^\times = R^\times$. In words, show that the unit group of the polynomial ring over R is equal to the unit group of R.

4. (*Homomorphism theorem for rings*) Let R, S be rings and $\varphi : R \to S$ a ring homomorphism. Show that the map

$$R/\ker(\varphi) \to \operatorname{im}(\varphi), \quad r + \ker(\varphi) \mapsto \varphi(r)$$

is well-defined and an isomorphism of rings.

5. Let $\alpha := \frac{1+\sqrt{5}}{2} \in \mathbb{Q}(\sqrt{5})$. Compute the minimal polynomial of α over $\mathbb{Q}$.

Note that your answer is (should be!) a monic polynomial in $\mathbb{Z}[X]$, although α seems to have a denominator. This kind of phenomenon will be discussed in the lecture.

6. Let $f(X) = X^3 + 3X - 3 \in \mathbb{Q}[X]$. This is an irreducible polynomial (How can one prove this?), so $K := \mathbb{Q}[X]/(f)$ is a field extension of $\mathbb{Q}$ of degree 3. Let $\alpha := X + (f) \in K$. Then the set $B := \{1, \alpha, \alpha^2\}$ is a $\mathbb{Q}$-basis of K.

   (a) Represent α^{-1} and $(1 + \alpha)^{-1}$ in terms of the basis B, i.e. as $\mathbb{Q}$-linear combination of 1, α and α^2.

   (b) Compute the minimal polynomial of $\beta := \alpha^2 - \alpha + 2$ over $\mathbb{Q}$.

7. Let L/K be a field extension (possibly of infinite degree). Show that the following statements are equivalent:

   (i) L/K is algebraic.

   (ii) L can be generated over K by (possibly infinitely many) elements of L that are algebraic over K.

8. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Prove that $\overline{\mathbb{Q}}$ is countable.