Exercises in Commutative Algebra

Winter Term 2011/2012

Université du Luxembourg
Prof. Dr. Gabor Wiese

Sheet 4
10/10/2011

The aim of this sheet is to prove the following famous theorem of Gauß: *The polynomial ring over a factorial ring is factorial.*

1. (*Fraction field.*) Just do this exercise if you feel unfamiliar with fraction fields. Otherwise skip it!

   Let R be an integral domain.

   (a) Show that the relation
   $$(r_1, s_1) \sim (r_2, s_2) \Leftrightarrow r_1 s_2 = r_2 s_1$$
   defines an equivalence relation on $R \times (R \setminus \{0\})$. Denote the equivalence class of an element (r, s) by $\frac{r}{s}$. Let $\mathrm{Frac}(R)$ denote the set of equivalence classes.

   (b) Define $+$ and $\cdot$ on $\mathrm{Frac}(R)$ by
   $$\frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} := \frac{r_1 r_2}{s_1 s_2}.$$
   Show that $\mathrm{Frac}(R)$ is a field with respect to $+$ and $\cdot$ with $0 = \frac{0}{1}$ and $1 = \frac{1}{1}$.

   We call $\mathrm{Frac}(R)$ the *fraction field* (or *field of fractions*) of R. Note that it is essential that R is an integral domain. We will later in the lecture identify the fraction field with the localisation of R at the prime ideal (0).

2. Let R be a factorial ring with field of fractions $\mathrm{Frac}(R) =: K$. Choose a set $\mathcal{P}$ of representatives of the prime elements of R up to associates. Then any $0 \neq r \in R$ can be represented as $r = \epsilon \cdot \prod_{p \in \mathcal{P}} p^{v_p(r)}$ with a unit $\epsilon \in R^\times$ and unique $v_p(r) \in \mathbb{N}$. Note that all of them are 0, except for finitely many. Define $v_p(0) := \infty$. (Compute with ∞ in the usual way $n + \infty = \infty$ for $n \in \mathbb{N}$, etc.)

   (a) Let $x = \frac{r}{s} \in K$ and $p \in \mathcal{P}$. Show that
   $$v_p(x) := v_p(r) - v_p(s)$$
   is well-defined, i.e. independent of the choice of representative of the equivalence class $\frac{r}{s}$.

   (b) Show: $x \in R \Leftrightarrow v_p(x) \geq 0$ for all $p \in \mathcal{P}$.

   (c) Show that $v_p(xy) = v_p(x) + v_p(y)$ for all $x, y \in K$ and all $p \in \mathcal{P}$.

   (d) Show that $v_p(x) = 0$ for all $p \in \mathcal{P}$ if and only if $x \in R^\times$.

3. (R, $\mathcal{P}$ and K as in 2.) Let $f = \sum_{i=0}^{d} a_i X^i \in K[X]$ be a polynomial and $p \in \mathcal{P}$. Define
   $$v_p(f) := \min_{i=0,\ldots,d}(v_p(a_i)).$$
   (Note that in the case of $a_i \in R$ for all i, this is v_p of the greatest common divisor of $a_0, \ldots, a_d$.)

   (a) Show that $v_p(f) \geq 0$ for all $p \in \mathcal{P}$ if and only if $f \in R[X]$.

(b) Show that $v_p(fg) = v_p(f) + v_p(g)$ for all $p \in \mathcal{P}$ and all $f, g \in K[X]$.

(c) Let $h \in R[X]$ and $f, g \in K[X]$ be monic polynomials such that $h = fg$. Show that $f, g \in R[X]$.

4. (R, $\mathcal{P}$ and K as in 2.) We call a polynomial $f \in R[X]$ *primitive* if $v_p(f) = 0$ for all $p \in \mathcal{P}$.

(a) Show that monic polynomials in $R[X]$ are primitive.

(b) Let $0 \neq f \in K[X]$. Show that there is $a \in K \setminus \{0\}$ such that $\frac{1}{a} f$ is a primitive polynomial in $R[X]$.

(c) Let $f \in R[X]$ be primitive and assume that f (considered as an element of $K[X]$) is a prime element in $K[X]$. Show that f is a prime element of $R[X]$. Let us call these prime elements of *Type I*.

(d) Let $f \in R$ be a prime element (of the ring, not the polynomial ring!). Show that f considered as a constant polynomial in $R[X]$ is a prime element of $R[X]$. Let us call these prime elements of *Type II*.

(e) Show that every $f \in R[X] \setminus (\{0\} \cup R^\times)$ is a product of prime elements of $R[X]$ of Type I or Type II.

[Hint: Use (b) to make the polynomial primitive: $g = \frac{1}{a} f$. Next, factor a into prime elements of R and g into prime elements of $K[X]$ (recall $K[X]$ is factorial!). Apply (b) again to make the factors of g into polynomials in $R[X]$. Conclude that the constant appearing in the latter process lies in $R^\times$.]

(f) Conclude from (e) that in $R[X]$ every irreducible element is a prime element.

[Hint: This comes down to the following very simple statement: We have a ring S (our $R[X]$) such that every element in $S \setminus (\{0\} \cup S^\times)$ is a product of prime elements. Then every irreducible element is prime.]

Later in the lecture we will prove **Hilbert's Basis Theorem**:
If R is a Noetherian ring, then $R[X]$ is also a Noetherian ring.

Thus we obtain **Gauß' Theorem**:
If R is a factorial ring, then $R[X]$ is also a factorial ring.

(g) Conclude from (e) and (f) that all prime elements $f \in R[X]$ are either of Type I or of Type II.

(h) Let $f \in R[X] \setminus R$ be primitive. Show:

$$f \text{ is a prime element of } R[X] \quad \Leftrightarrow \quad f \text{ is a prime element of } K[X].$$