Exercises in Commutative Algebra

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Sheet 6
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1. Let K be a field and $n \in \mathbb{N}$. Show the following statements:
   (a) Let $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathbb{A}^n(K)$ be subsets. Then $\mathcal{I}_{\mathfrak{X}} \supseteq \mathcal{I}_{\mathfrak{Y}}$.
   (b) $\mathcal{I}_\emptyset = K[\underline{X}]$.
   (c) If K has infinitely many elements, then $\mathcal{I}_{\mathbb{A}^n(K)} = (0)$.
   (d) Let $S \subseteq K[\underline{X}]$ be a subset. Then $\mathcal{I}_{\mathcal{V}_S(K)} \supseteq S$.
   (e) Let $\mathcal{X} \subseteq \mathbb{A}^n(K)$ be a subset. Then $\mathcal{V}_{\mathcal{I}_\mathcal{X}}(K) \supseteq \mathcal{X}$.
   (f) Let $S \subseteq K[\underline{X}]$ be a subset. Then $\mathcal{V}_{\mathcal{I}_{\mathcal{V}_S(K)}}(K) = \mathcal{V}_S(K)$.
   (g) Let $\mathcal{X} \subseteq \mathbb{A}^n(K)$ be a subset. Then $\mathcal{I}_{\mathcal{V}_{(\mathcal{I}_\mathcal{X})}(K)} = \mathcal{I}_\mathcal{X}$.

2. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a topological space and $\mathcal{Y} \subseteq \mathcal{X}$ be a subset. Define $\mathcal{O}_\mathcal{Y} := \{U \cap \mathcal{Y} \mid U \in \mathcal{O}_\mathcal{X}\}$.
   Show that $\mathcal{O}_\mathcal{Y}$ is a topology on $\mathcal{Y}$. It is called the *relative topology* or the *subset topology*.

3. Let K be a field. Show that the closed subsets of $\mathbb{A}^1(K)$ are $\emptyset$, $\mathbb{A}^1(K)$ and finite sets of points.

4. Let K be a field, $n \in \mathbb{N}$ and $\mathcal{X} \subseteq \mathbb{A}^n(K)$ a subset.
   With $f \in K[X_1, \dots, X_n]$ associate (as in the lecture) the map
   $$\varphi : \mathcal{X} \to \mathbb{A}^1(K), \quad \underline{x} \mapsto f(\underline{x}).$$
   Show that φ is a continuous map, when we consider $\mathcal{X}$ with the relative topology from $\mathbb{A}^n(K)$. Of course, $\mathbb{A}^n(K)$ and $\mathbb{A}^1(K)$ are equipped with the Zariski topology.
   [By definition a map between topological spaces is continuous if the preimage of any open set is an open set.
   Hint: Use the previous exercise.]

5. Let R be a ring.
   (a) *Homomorphism theorem for modules*. Let M and N be R-modules and $\varphi : M \to N$ an R-homomorphism. Prove that the map
   $$M/\ker(\varphi) \to \mathrm{im}(N), \quad m + ker(\varphi) \mapsto \varphi(m)$$
   is a well defined R-isomorphism.
   If you have done Exercise 4 on Sheet 3, you can skip this exercise, as the proof is exactly the same.
   (b) *Isomorphism theorems*. Let M be an R-module and let N_1 and N_2 be R-submodules of M. Conclude from (a) that there are the R-isomorphisms
   $$(M/N_1)/(N_2/N_1) \cong M/N_2 \quad \text{(assuming here also } N_1 \subseteq N_2\text{)}$$
   and
   $$(N_1 + N_2)/N_1 \cong N_2/(N_1 \cap N_2).$$