Exercises in Commutative Algebra

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1. *Uniqueness of direct sums.* Let R be a ring and M_i for $i \in I$ (some set) R-modules. Let S together with $\epsilon_i : M_i \to S$ and S' together with $\epsilon'_i : M_i \to S'$ be two direct sums of the M_i, $i \in I$.

   Show that there is a unique R-isomorphism $S \to S'$.

2. *Existence and uniqueness of free modules over a set.* Let R be a ring and I a set. Define $F_I := \bigoplus_{i \in I} R$ and $\epsilon : I \to F_I$ by sending $j \in I$ to the element $(m_i)_{i \in I}$ such that $m_j = 1$ and $m_i = 0$ for all $i \in I \setminus \{j\}$.

   (a) Show that F_I is a free R-module over I.

   (b) Show that if G is any other free R-module over I, then there is a unique R-isomorphism $F_I \to G$.

3. Let R be a ring and N, M_i for $i = 1, 2, 3$ be R-modules. Show that the functor $\mathrm{Hom}_R(\cdot, N)$ is contravariant (reverses directions of arrows) and left-exact. That is, show the following statement:

   If
   $$M_1 \xrightarrow{\psi_2} M_2 \xrightarrow{\psi_3} M_3 \to 0$$
   is an exact sequence, then
   $$0 \to \mathrm{Hom}_R(M_3, N) \xrightarrow{\tilde{\psi}_3} \mathrm{Hom}_R(M_2, N) \xrightarrow{\tilde{\psi}_2} \mathrm{Hom}_R(M_1, N)$$
   is also exact, where $\tilde{\psi}_i$ sends $\alpha \in \mathrm{Hom}_R(M_i, N)$ to $\alpha \circ \psi_i \in \mathrm{Hom}_R(M_{i-1}, N)$ for $i = 2, 3$.

4. Let R be a ring, N and M_i for $i \in I$ (some set) be R-modules. Show that there is an R-isomorphism:
   $$\Psi : \mathrm{Hom}_R(\bigoplus_{i \in I} M_i, N) \to \prod_{i \in I} \mathrm{Hom}_R(M_i, N).$$

5. Let R be a ring and $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ a short exact sequence. Show that the following statements are equivalent:

   (i) There is an R-homomorphism $s : C \to B$ such that $\beta \circ s = \mathrm{id}_C$ (s is called a *split*).

   (ii) There is an R-homomorphism $t : B \to A$ such that $t \circ \alpha = \mathrm{id}_A$ (t is also called a *split*).

   (iii) There is an R-isomorphism $A \oplus C \to B$.

   Please turn over.

6. Let R be a ring.

   (a) Let $M_1, \dots, M_n$ be R-modules and put $M := \prod_{i=1}^n M_i$. Show that there are R-homomorphisms $e_i : M \to M$ for $i = 1, \dots, n$ such that

      (1) $e_i \circ e_i = e_i$ for all $i = 1, \dots, n$ (a homomorphism with this property is called an *idempotent*).

      (2) $e_i \circ e_j = 0$ for all $1 \leq i, j \leq n$ and $i \neq j$ (one says that the idempotents e_i, $i = 1, \dots, n$ are *orthogonal*).

      (3) $\mathrm{id}_M = e_1 + \cdots + e_n$ (one says that the e_i, $i = 1, \dots, n$ are a *complete set of orthogonal idempotents of* M).

   (b) Let M be an R-module and $e_1, \dots, e_n \in \mathrm{Hom}_R(M, M)$ a complete set of orthogonal idempotents of M, i.e. they satisfy (1), (2) and (3). Let $M_i := e_i(M)$.

      Show that there is an R-isomorphism $M \to \prod_{i=1}^n M_i$.