Exercises in Commutative Algebra

Winter Term 2011/2012

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Sheet 8
7/11/2011

1. Let R be a ring, M a right R-module and N a left R-module. Let (P, f) and (Q, g) be tensor products of M and N over R.

   Show that there is a unique group isomorphism $\phi : P \to Q$ such that $g = \phi \circ f$.

2. Let R be a commutative ring , and M, N R-modules. Show that $M \otimes_R N$ and $N \otimes_R M$ are isomorphic.

3. Let R and S be rings. Let M be a right R-module, P a left S-module, N a right S-module and a left R-module such that $(rn)s = r(ns)$ for all $r \in R$, all $s \in S$ and all $n \in N$.

   Show the following statements.

   (a) $M \otimes_R N$ is a right S-module via $(m \otimes n).s = m \otimes (ns)$.

   (b) $N \otimes_S P$ is a left R-module via $r(n \otimes p) = (rn) \otimes p$.

   (c) There is an isomorphism
   $$(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P).$$

4. Let R be a ring, M a right R-module, N a left R-module and P a $\mathbb{Z}$-module.

   Show the following statements.

   (a) $\mathrm{Hom}_{\mathbb{Z}}(N, P)$ is a right R-module via $(\varphi.r)(n) := \varphi(rn)$ for $r \in R$, $n \in N$, $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(N, P)$ (you can skip this if you feel sure about it).

   (b) The map
   $$\{\text{Balanced maps } f : M \times N \to P\} \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N, P)),$$
   which is given by
   $$f \mapsto \big(m \mapsto (n \mapsto f(m, n))\big)$$
   is a bijection with inverse
   $$\varphi \mapsto \big((m, n) \mapsto (\varphi(m))(n)\big).$$

   (c) There is an isomorphism of abelian groups:
   $$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N, P)) \cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, P).$$