Exercises in Commutative Algebra

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1. *Tensor product as an R-module.*

   Let R be a commutative ring and let M, N be R-modules. Recall that since R is commutative we can see M and N as both right and left modules. We only use left notation in this exercise.

   Show that the tensor product $M \otimes_R N$ is an R-module if we define the R-scalar multiplication as $r(m \otimes n) := rm \otimes n$, which is equal to $m \otimes rn$.

2. *Tensor product of algebras.*

   Let R be a commutative ring. Recall the definition of an R-algebra A: A is a ring together with a (fixed) ring homomorphism $\varphi : R \to A$. This makes A into an R-module by defining the R-scalar multiplication as $r.a := \varphi(r)a$ for $r \in R$ and $a \in A$.

   Let now A, B be R-algebras and form the tensor product $A \otimes_R B$.

   Show that $A \otimes_R B$ is an R-algebra by defining multiplication on the generators as follows:

   $$(a \otimes b) \cdot (a' \otimes b') := (aa') \otimes (bb'),$$

   for $a, a' \in A$ and $b, b' \in B$, and extending it linearly:

   $$\Big(\sum_{i=1}^{n} r_i(a_i \otimes b_i)\Big)\Big(\sum_{j=1}^{m} r'_j(a'_j \otimes b'_j)\Big) := \sum_{i=1}^{n}\sum_{j=1}^{m} r_i r'_j (a_i a'_j \otimes b_i b'_j),$$

   for $r_i, r'_j \in R$, $a_i, a'_j \in A$ and $b_i, b'_j \in B$.

3. Let R be a ring and $S \subseteq R$ a multiplicatively closed subset with $1 \in S$. Let $\mu : R \to S^{-1}R$, given by $r \mapsto \frac{r}{1}$.

   Show the following statements.

   (a) The map
   $$\{\mathfrak{b} \lhd S^{-1}R \text{ ideal}\} \longrightarrow \{\mathfrak{a} \lhd R \text{ ideal}\}, \quad \mathfrak{b} \mapsto \mu^{-1}(\mathfrak{b}) \lhd R$$
   is an injection, which preserves inclusions and intersections. Moreover, if $\mathfrak{b} \lhd S^{-1}R$ is a prime ideal, then so is $\mu^{-1}(\mathfrak{b}) \lhd R$.

   (b) Let $\mathfrak{a} \lhd R$ be an ideal. Then the following statements are equivalent:

      (i) $\mathfrak{a} = \mu^{-1}(\mathfrak{b})$ for some $\mathfrak{b} \lhd S^{-1}R$ (i.e. $\mathfrak{a}$ is in the image of the map in (a)).

      (ii) $\mathfrak{a} = \mu^{-1}(\mathfrak{a}S^{-1}R)$ (here $\mathfrak{a}S^{-1}R$ is short for the ideal of $S^{-1}R$ generated by $\mu(\mathfrak{a})$, i.e. by all elements of the form $\frac{a}{1}$ for $a \in \mathfrak{a}$).

      (iii) Every $s \in S$ is a non-zero divisor modulo $\mathfrak{a}$, meaning that if $r \in R$ and $rs \in \mathfrak{a}$, then $r \in \mathfrak{a}$.

   (c) The map in (a) defines a bijection between the prime ideals of $S^{-1}R$ and the prime ideals $\mathfrak{p}$ of R such that $S \cap \mathfrak{p} = \emptyset$.

   Hint: Use (b) (iii).

4. In this exercise you prove *Nakayama's Lemma*:

   Let R be a ring, $\mathfrak{a} \lhd R$ an ideal and M a finitely generated R-module.

   (a) Assume that $\mathfrak{a}M = M$. Show that there is $a \in \mathfrak{a}$ such that $(1-a)M = 0$.

   Hint: Choose a set of generators $\{m_1, \ldots, m_n\}$ of M as an R-module. As in Proposition 4.6 of the lecture, write each m_i as an $\mathfrak{a}$-linear combination of $m_1, \ldots, m_n$. This leads to a matrix $A = (a_{i,j})$ with entries in $\mathfrak{a}$. Similarly to Proposition 4.6, you can obtain the desired $1-a$ as the determinant of the identity matrix minus A.

   (b) Assume again $\mathfrak{a}M = M$ and assume additionally that $\mathfrak{a} \lhd R$ is contained in the Jacobson radical $J(R)$ of R.

   Show that $M = 0$.

5. This exercise proves a very useful corollary to Nakayama's Lemma, which is sometimes itself called Nakayama's Lemma.

   Let R be a local ring with unique maximal ideal $\mathfrak{m}$ and let M be a finitely generated R-module. Let $m_1, \ldots, m_n \in M$ be elements such that their images $\overline{m}_i := m_i + \mathfrak{m}M$ are generators of the quotient module $M/\mathfrak{m}M$.

   Show that $m_1, \ldots, m_n$ generate M as an R-module.