Exercises in Algebraic Number Theory

Winter term 2009/2010

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Sheet 3

To be handed in by: Friday, 6 November 2009, 2 pm.

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1. (4 points)

   (a) Show that the fractional ideals of $\mathbb{Z}$ form an abelian group (with respect to the multiplication of ideals), which is isomorphic to $\mathbb{Q}^\times/\mathbb{Z}^\times$.

   (b) Consider the ring $R = \mathbb{Z}[\sqrt{-61}]$. Show that $(2, 3+\sqrt{-61})$ and $(5, 3+\sqrt{-61})$ are invertible ideals in R and determine their order in $\mathrm{Pic}(R)$.

2. (4 points) Consider the ring $R = \mathbb{Z}[\sqrt{-19}]$. Use for this exercise that $\mathrm{Pic}(R)$ is a finite group of order 3. Determine all integral solutions of the equation $x^2 + 19 = y^5$.

3. (4 points) Show the following variant of Theorem 3.12 of the lecture: Let R be an integral domain and I a fractional R-ideal. Then the following are equivalent:

   (i) I is invertible.

   (ii) I is finitely generated and for all prime ideals $\mathfrak{P} \lhd R$ the localisation $I_{\mathfrak{P}}$ is a principal ideal in $R_{\mathfrak{P}}$.

   You may use Theorem 3.12.

4. (4 points) Let R be a commutative ring.

   (a) Show that every R module is a quotient of a free R-module. You can, for instance, make use of the universal mapping property of free modules.

   (b) Let M be an R-module. A *free resolution* of M is an exact sequence

   $$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0$$

   consisting of free R-modules F_n for $n \in \mathbb{N}$.

   Show that every R-module M admits a free resolution.

   Hint: Iterate (a).