Exercises in Algebraic Number Theory

Winter term 2009/2010

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Sheet 7

To be handed in by: Friday, 4 December 2009, 2 pm.

1. (4 points) Let K be a number field and L/K and M/K finite extensions. Fix a prime $\mathfrak{p}$ of K. Show:

   (a) If $\mathfrak{p}$ is unramified in L and in M, then it is unramified in LM.

   (b) If $\mathfrak{p}$ is completely split in L and in M, then it is completely split in LM.

   Hint: You can use the inertia field and the decomposition field.

2. (4 points) Compute the ramified primes and the inertia fields in the splitting field of the polynomial $X^4 - 19 \in \mathbb{Z}[X]$.

   Hint: Use Proposition 7.2 from the lecture.

3. (4 points) Let α be a zero of the polynomial $f(X) = X^3 - X - 1 \in \mathbb{Z}[X]$ and put $R = \mathbb{Z}[\alpha]$. Show that R is a Dedekind ring. Determine all prime ideals of R of norm at most 30. Show that the unit group $R^\times$ is infinite.

   Hint: Use Theorem 4.13 and Proposition 7.2 from the lecture. For the factorisation of f modulo small primes you may use a computer and just state the result.

4. (4 points) For $n \in \mathbb{N}$ with $n \geq 2$ let ϕ_n be the n-th cyclotomic polynomial in $\mathbb{Z}[X]$. Compute $\phi_n(1)$.

   Hint: Look at prime powers $n = p^k$ first and for general n consider $X^n - 1 = \prod_{d|n} \phi_d(X)$.