Lagrangian submanifolds of $\mathbb{C}P^n$.

Franki Dillen

April 24, 2009

Introduction

Let $f: M^n \to \widetilde{M}^m$ be an isometric immersion of a Riemannian n-manifold M^n into a Kaehler m-manifold $\widetilde{M}^m$ of complex dimension m.

Then f is said to be *totally real* if the complex structure J of $\widetilde{M}^m$ carries each tangent space of M^n into its corresponding normal space, i.e. $J(f_* T_p M) \subset T_p^\perp M$.

If in addition $n = m$, then M^n is said to be *Lagrangian* , i.e. $J(f_* T_p M) = T_p^\perp M$.

If no confusion is possible, we will assume M^n embedded and we identify M^n and $f(M^n)$.

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From now on: $n = m$, $\widetilde{M}^n = \mathbb{C}P^n(4)$

$$M^n \subset \mathbb{C}P^n(4) \quad \text{Lagrangian Submanifold}$$

Contents:

- Lift to $S^{2n+1}(1)$, fundamental theorem.
- Examples of Lagrangian submanifolds
- Curvature inequalities
- Pseudo-parallel Lagrangian submanifolds

Joint work with John Bolton, Bang-Yen Chen, Johan Fastenakels, Joeri Van der Veken, Pol Verstraelen, Luc Vrancken

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Basic equations for Lagrangian Submanifolds

Formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

Relations: $J(A_{JX} Y) = h(X, Y)$ and $\nabla^{\perp}_X JY = J\nabla_X Y$

The equation of Gauss:

$$R(X, Y) = X \wedge Y + [A_{JX}, A_{JY}].$$

The equation of Codazzi:

$$(\overline{\nabla} h)(X, Y, Z) := \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

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The cubic form C is totally symmetric, where C is defined by

$$C(X,Y,Z) = g(h(X,Y),JZ) = g(A_{JX}Y,Z),$$

Theorem

Let $x^1, x^2 : M^n \to \mathbb{C}P^n$ be two Lagrangian isometric immersions of a connected manifold M^n. If

$$C^1(X,Y,Z) = C^2(X,Y,Z)$$

for all vector fields X, Y, Z tangent to M, then there exists an isometry F of $\mathbb{C}P^n$ such that $x^1 = F(x^2)$.

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C-totally real and Legendrian submanifolds of $S^{2m+1}(1)$

Consider $S^{2m+1}(1) \subset \mathbb{C}^{m+1}$ with standard Sasakian metric.

$f: M^n \to S^{2m+1}$ is called *C-totally real* if for $p \in M$, $i f(p)$ is normal to M

If $n = m$, then a C-totally real immersion $f: M^n \to S^{2n+1}$ is called *Legendrian*

Consider the Hopf fibration $\pi : S^{2m+1}(1) \to \mathbb{C}P^m(4)$. If $f: M^n \to S^{2m+1}$ is C-totally real (resp. Legendrian), then $\pi(f): M^n \to \mathbb{C}P^m(4)$ is totally real (resp. Lagrangian), and conversely (at least locally).

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Lifting up and down

More precisely: let $f : M^n \to S^{2m+1}(1)$ be a C-totally real isometric immersion. Then $g = \pi(f) : M^n \to \mathbb{C}P^m(4)$ is again an isometric immersion, which is totally real. Under this correspondence, the second fundamental forms h^f and h^g of f and g satisfy $\pi_* h^f = h^g$. Moreover, h^f is horizontal w.r.t. π.

Conversely, let $g : M^n \to \mathbb{C}P^m(4)$ be a totally real isometric immersion. Then there exists an isometric covering map $\tau : \widehat{M}^n \to M^n$, and a C-totally real isometric immersion $f : \widehat{M}^n \to S^{2m+1}$ such that $g(\tau) = \pi(f)$.

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Theorem

Let (M^n, g) be an n-dimensional simply connected Riemannian manifold. Let α be a symmetric bilinear TM-valued form on M satisfying

- $g(\alpha(X, Y), Z)$ *is totally symmetric,*
- $(\nabla\alpha)(X, Y, Z)$ *is totally symmetric,*
- $R(X, Y)Z = X \wedge Y(Z) + \alpha(\alpha(Y, Z), X) - \alpha(\alpha(X, Z), Y)$,

then there exists a Legendrian immersion $x : M \to S^{2n+1}(1)$ such that the second fundamental form h satisfies $h(X, Y) = \varphi\alpha(X, Y)$, where φ is the standard Sasakian structure on $S^{2n+1}(1)$.

Fundamental theorem for Lagrangian submanifolds of $\mathbb{C}P^n$

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then there exists a Lagrangian immersion $x : M \to \mathbb{C}P^n(4)$ such that the second fundamental form h satisfies $h(X, Y) = J\alpha(X, Y)$, where J is the complex structure on $\mathbb{C}P^n(4)$.

Consider the unit sphere

$$S^3 = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 \mid y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1\}$$

in $\mathbb{R}^4$. Let X_1, X_2 and X_3 be the vector fields defined by

$$\begin{aligned} X_1(y_1, y_2, y_3, y_4) &= (y_2, -y_1, y_4, -y_3), \\ X_2(y_1, y_2, y_3, y_4) &= (y_3, -y_4, -y_1, y_2), \\ X_3(y_1, y_2, y_3, y_4) &= (y_4, y_3, -y_2, -y_1). \end{aligned}$$

Define a metric g on S^3 such that X_1, X_2 and X_3 are orthogonal and such that

$$g(X_1, X_1) = g(X_2, X_2) = 3,\ g(X_3, X_3) = 9.$$

Define a symmetric bilinear form α by

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An exotic Lagrangian immersion of S^3 into $\mathbb{C}P^3$

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Then (S^3, g) and α satisfy all conditions of the fundamental theorem, such that there exists a Lagrangian immersion

$$f : (S^3, g) \to \mathbb{C}P^3(4).$$

Then f is minimal and (S^3, g) has constant scalar curvature $\frac{1}{3}$.

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Lagrangian immersions of $M^n(1)$ into $\mathbb{C}P^n(4)$.

Observation: $\mathbb{R}P^n(1) \subset \mathbb{C}P^n(4)$ is totally geodesic Lagrangian submanifolds.

Problem

Besides totally geodesic Lagrangian submanifolds, do there exist many Lagrangian isometric immersions of a real-space-form of constant sectional curvature 1 into $\mathbb{C}P^n(4)$?

Problem

How to explicitly construct Lagrangian isometric immersions of a real-space-form of constant sectional curvature 1 into $\mathbb{C}P^n(4)$

Twisted products

Definition

Let $(M_1, g_1), \dots, (M_m, g_m)$ be m Riemannian manifolds, f_i a positive function on $M_1 \times \cdots \times M_m$ and $\pi_i : M_1 \times \ldots \times M_m \to M_i$ the i-th canonical projection for $i = 1, \dots, m$. Then the *twisted product*

$$_{f_1}M_1 \times \cdots \times_{f_m} M_m$$

of $(M_1, g_1), \dots, (M_m, g_m)$ is the differentiable manifold $M_1 \times \ldots \times M_k$ equipped with the twisted product metric g defined by

$$g(X, Y) = f_1 \cdot g_1(\pi_{1*}X, \pi_{1*}Y) + \cdots + f_m \cdot g_m(\pi_{m*}X, \pi_{m*}Y)$$

for all vector fields X and Y of $M_1 \times \cdots \times M_m$.

Twisted product decomposition

Consider

$$ {}_{f_1}I_1 \times \cdots \times_{f_k} I_k \times_1 S^{n-k}(1), $$

where $I_1, \ldots, I_k$ are open intervals.
Assume the twisted product has constant sectional curvature 1, then we call it a *twisted product decomposition* of a space of curvature $M^n(1)$. Notation: $M^n(1) = TP^n_{f_1 \cdots f_k}(1)$.
Let x_j be the standard coordinate on I_j for $j = 1, \ldots, k$. We define the *twistor form* $\Phi(TP)$ by

$$ \Phi(TP) = f_1 dx_1 + \cdots + f_k dx_k. $$

$\Phi(TP)$ is said to be *twisted closed* if

$$ \sum_{i,j=1}^{k} \frac{\partial f_i}{\partial x_j} dx_j \wedge dx_i = 0. $$

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Existence part

Theorem

Let $TP^n_{f_1\cdots f_k}(1)$, $1 \leq k \leq n$, be a twisted product decomposition of a simply-connected real-space-form $M^n(1)$. If the twistor form $\Phi(TP)$ of $TP^n_{f_1\cdots f_k}(1)$ is twisted closed, then, up to rigid motions of $\mathbb{C}P^n(4)$, there is a unique Lagrangian isometric immersion:

$$L_{f_1\cdots f_k}: \;\; TP^n_{f_1\cdots f_k}(1) \to \mathbb{C}P^n(4),$$

whose second fundamental form satisfies

$$h(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j}) = J\frac{\partial}{\partial x_j}, \quad j = 1, \ldots, k,$$

and all other components vanish.

Classification

Theorem

Let M be a Lagrangian submanifold of $\mathbb{C}P^n(4)$. Then M is of constant sectional curvature 1 if and only if, for each point $p \in M$, the second fundamental form at p satisfies

$$h(e_1, e_1) = \lambda_1 Je_1, \ldots, h(e_n, e_n) = \lambda_n Je_n,$$

$$h(e_i, e_j) = 0, \quad 1 \leq i \neq j \leq n$$

for some $\lambda_1, \ldots, \lambda_n$ with respect to some orthonormal basis $e_1, \ldots, e_n$ of T_pM.

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Let $L\colon\ M^n(1) \to \mathbb{C}P^n(4)$ be a Lagrangian isometric immersion of a real-space-form $M^n(1)$ into $\mathbb{C}P^n(4)$. If there is an integer $k \in \{1, \ldots, n\}$ such that the second fundamental form of L satisfies

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for some nowhere vanishing functions $\lambda_1, \ldots, \lambda_k$, with respect to an orthonormal frame field $e_1, \ldots, e_n$, then $M^n(1) = TP^n_{\lambda_1^{-2} \cdots \lambda_k^{-2}}(1)$ such that the twistor form is twisted closed. Moreover, the Lagrangian immersion L is congruent to the Lagrangian isometric immersion $L_{f_1 \cdots f_k}$.

Explicit expressions for special cases.

Further applications: Hamiltonian stationary Lagrangian submanifols.

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Chen's δ-invariants

For any Riemannian manifold M^n, we can define:

$$\delta_M(p) = \tau(p) - \inf K(p),$$

where $\inf K$ is the function assigning to each $p \in M^n$ the infimum of $K(\pi)$, where π runs over all planes in T_pM and τ is defined by $\tau = \sum_{i<j} K(e_i \wedge e_j)$.

If M^n is Lagrangian submanifold of $\mathbb{C}P^n$, then

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}(n+1)(n-2),$$

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Let $x : M^n \to \mathbb{C}P^n$ and $n \geq 3$, be a Lagrangian isometric immersion. If M^n realizes equality in Chen's inequality, then M is minimal.

Theorem

Let $x : M^n \to \mathbb{C}P^n$ $(n \geq 3)$ be a Lagrangian immersion with constant scalar curvature. If M^n realizes equality in Chen's inequality then either M^n is a totally geodesic immersion or $n = 3$ and x is congruent to the immersion $(S^3, g) \to \mathbb{C}P^3$ above.

General classification results by Bolton, Scharlach, Vrancken, Woodward,...

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Let (M^n, g) be a Lagrangian submanifold of $\mathbb{C}P^n(4)$. Then we have

$$\delta(p) \leq \frac{n^2(2n-3)}{2(2n+3)}\|H\|^2 + \frac{1}{2}(n-2)(n+1)$$

This inequality is the best possible.

Theorem

Let M^n be a Lagrangian submanifold of a $\mathbb{C}P^n(4)$ attaining equality in the improved Chen inequality at every point. If $n \geq 4$, then M is minimal.

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Pseudo-parallel Lagrangian submanifolds

A submanifold M^n of a Riemannian manifold $(\widetilde{M}^m, g)$ is called semi-parallel if its second fundamental form h satisfies

$$\overline{R} \cdot h = 0,$$

where $\overline{R}$ acts on h as a derivation:

$$(\overline{R} \cdot h)(X, Y, Z, W) = R^{\perp}(X, Y)h(Z, W) - h(R(X, Y)Z, W) - h(Z, R(X, Y)W)$$

M^n is called pseudo-parallel if its second fundamental form h satisfies

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for some function ϕ on M, where $Q(g, h)$ is defined by

$$Q(g, h)(X, Y, U, V) = -((X \wedge Y) \cdot h)(U, V) = h((X \wedge Y)U, V) + h(U, (X \wedge Y)V)$$

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Pseudo parallel Lagrangian submanifolds

Chacon and Lobos (DGA 2008) prove that a Lagrangian surface M^2 of a complex space form $\widetilde{M}^2(4c)$ is pseudo-parallel if and only if it is (locally) flat or minimal. They conjecture that for $n \geq 3$ pseudo-parallel implies semi-parallel.

Theorem

A Lagrangian pseudo-parallel submanifold M^n of a complex space form $\widetilde{M}^n(4c)$ of dimension $n \geq 3$ is semi-parallel.

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Better notion for pseudo-parallel Lagrangian submanifolds:

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References

- B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Lagrangian isometric immersions of a real-space-form* $M^n(c)$ *into a complex-space-form* $\tilde{M}^n(4c)$, Math. Proc. Cambridge Philo. Soc., **124**, 107–125 (1998).
- B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken *An exotic totally real minimal immersion of* S^3 *in* $\mathbb{C}P^3$ *and its characterization* Proc. Royal Soc. Edinburgh Sect. A, Math. **126** (1996) 153–165.
- B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Totally real submanifolds of* CP^n *satisfying a basic equality,* Arch. Math. (Basel) 63 (1994), 553–564

References

- F. Dillen, J. Van der Veken, L. Vrancken, *Pseudo-parallel Lagrangian submanifolds are semi-parallel.* Differential geometry and its applications (2009)
- J. Bolton, F. Dillen, J. Fastenakels, L. Vrancken, *A best possible inequality for curvature-like tensor fields,* Mathematical Inequalities and Applications (2009)
- B. Y. Chen, F. Dillen, *Warped product decompositions of real space forms and Hamiltonian-stationary Lagrangian submanifolds,* Nonlinear analysis, 69, 3462-3494 (2008)