Hypo contact and Sasakian structures on Lie groups

**SU(2)-structures in 5-dimensions**

**Definition**

An *SU(2)*-structure \((\eta, \omega_1, \omega_2, \omega_3)\) on \(N^5\) is given by a 1-form \(\eta\) and by three 2-forms \(\omega_i\) such that

\[
\omega_i \wedge \omega_j = \delta_{ij} \nu, \quad \nu \wedge \eta \neq 0, \\
i_X \omega_3 = i_Y \omega_1 \Rightarrow \omega_2(X, Y) \geq 0,
\]

where \(i_X\) denotes the contraction by \(X\).

**Remark**

The pair \((\eta, \omega_3)\) defines a *U(2)*-structure or an almost contact metric structure on \(N^5\), i.e. \((\eta, \xi, \varphi, g)\) such that

\[
\eta(\xi) = 1, \quad \varphi^2 = -\text{Id} + \xi \otimes \eta, \\
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).
\]
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\[
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\]
Sasakian structures

An almost contact metric structure \((\eta, \xi, \varphi, g)\) on \(N^{2n+1}\) is said contact metric if \(2g(X, \varphi Y) = d\eta(X, Y)\). On \(N^5\) the pair \((\eta, \omega_3)\) defines a contact metric structure if \(d\eta = -2\omega_3\).

\((\eta, \xi, \varphi, g)\) is called normal if

\[ N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \]

satisfies the condition \(N_\varphi = -d\eta \otimes \xi\).

**Definition (Sasaki)**

A Sasakian structure on \(N^{2n+1}\) is a normal contact metric structure.

**Theorem (Boyer, Galicki)**

A Riemannian manifold \((N^{2n+1}, g)\) has a compatible Sasakian structure if and only if the cone \(N^{2n+1} \times \mathbb{R}^+\) equipped with the conic metric \(\tilde{g} = dr^2 + r^2 g\) is Kähler.
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Sasaki-Einstein structures

Example

**Sasaki-Einstein structure on** $N^5$

\[
d\eta = -2\omega_3, \ d\omega_1 = 3\eta \wedge \omega_2, \ d\omega_2 = -3\eta \wedge \omega_1.\]

On $S^2 \times S^3$ there exist an infinite family of explicit Sasaki-Einstein metrics [Gauntlett, Martelli, Sparks, Waldram, ...].

**Definition (Boyer, Galicki)**

$(N^{2n+1}, g, \eta)$ is Sasaki-Einstein if the conic metric $\tilde{g} = dr^2 + r^2 g$ on the symplectic cone $N^{2n+1} \times \mathbb{R}^+$ is Kähler and Ricci-flat (CY).

- $N^{2n+1} \times \mathbb{R}^+$ has an integrable $SU(n+1)$-structure, i.e. an Hermitian structure $(J, \tilde{g})$, with $F = d(r^2 \eta)$, and a $(n+1, 0)$-form $\psi = \psi_+ + i\psi_-$ of length 1 such that $dF = d\psi = 0 \Rightarrow \tilde{g}$ has holonomy in $SU(n+1)$.

- $N^{2n+1}$ has a real Killing spinor, i.e. the restriction of a parallel spinor on the Riemannian cone [Friedrich, Kath].
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**SU(2)-structures in 5-dimensions**

Sasakian structures

**Sasaki-Einstein structures**

η-Einstein structures

**Hypo-contact structures**

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Consequences

New metrics with holonomy $SU(3)$

Sasakian structures on Lie groups

3-dimensional Lie groups

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Generalized Killing spinors

Contact $SU(n)$-structures

Examples

Contact reduction
Remark

An $SU(2)$-structure $P$ on $N^5$ induces a spin structure on $N^5$ and $P$ extends to $P_{Spin(5)} = P \times_{SU(2)} Spin(5)$.

The spinor bundle is $P \times_{SU(2)} \Sigma$, where $\Sigma \cong \mathbb{C}^4$ and $Spin(5)$ acts transitively on the sphere in $\Sigma$ with stabilizer $SU(2)$ in a fixed unit spinor $u_0 \in \Sigma$.

Then the $SU(2)$-structures are in one-to-one correspondence with the pairs $(P_{Spin(5)}, \psi)$, with $\psi$ a unit spinor such that $\psi = [u, u_0]$ for any local section $u$ of $P$, i.e. $\psi \in P \times_{Spin(5)} (Spin(5)u_0)$. 
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Hypo structures

Definition
An $SU(2)$-structure on $N^5$ is hypo if
\[ d\omega_3 = 0, \quad d(\eta \wedge \omega_1) = 0, \quad d(\eta \wedge \omega_2) = 0. \]

Proposition (Conti, Salamon)
An $SU(2)$-structure $P$ on $N^5$ is hypo if and only if the spinor $\psi$ (defined by $P$) is generalized Killing (in the sense of Bär, Gauduchon, Moroianu), i.e.
\[ \nabla_X \psi = \frac{1}{2} O(X) \cdot \psi, \]
where $O$ is a section of $\text{Sym}(TN^5)$ and $\cdot$ is the Clifford multiplication.

If $N^5$ is simply connected and Sasaki-Einstein, then $O = \pm \text{Id}$ [Friedrich, Kath].
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If $N^5$ is simply connected and Sasaki-Einstein, then $O = \pm \text{Id}$ [Friedrich, Kath].
• Any oriented hypersurface $N^5$ of $(M^6, F, \psi)$ with an integrable $SU(3)$-structure $(F, \psi)$ has in a natural way a hypo structure. The generalized Killing spinor $\psi$ on $N^5$ is the restriction of the parallel spinor on $M^6$ and $O$ is just given by the Weingarten operator. If $\psi$ is the restriction of a parallel spinor over the Riemannian cone then $O$ is a constant multiple of the identity.

• Nilmanifolds cannot admit Sasaki-Einstein structures but they can admit hypo structures.

**Theorem (Conti, Salamon)**

The nilpotent Lie algebras admitting a hypo structure are

$$(0, 0, 12, 13, 14), \quad (0, 0, 0, 12, 13 + 24),$$

$$(0, 0, 0, 12, 13), \quad (0, 0, 0, 0, 12 + 34),$$

$$(0, 0, 0, 0, 12), \quad (0, 0, 0, 0, 0).$$
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- $(0, 0, 0, 0, 12 + 34)$,
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- $(0, 0, 0, 0, 0)$. 
Hypo evolution equations

**Theorem (Conti, Salamon)**

A real analytic hypo structure \((\eta, \omega_i)\) on \(N^5\) determines an integrable \(SU(3)\)-structure on \(N^5 \times I\), with \(I\) some open interval, if \((\eta, \omega_i)\) belongs to a one-parameter family of hypo structures \((\eta(t), \omega_i(t))\) which satisfy the evolution equations

\[
\begin{align*}
\partial_t \omega_3(t) &= -\hat{d}\eta(t), \\
\partial_t (\omega_2(t) \wedge \eta(t)) &= \hat{d}\omega_1(t), \\
\partial_t (\omega_1(t) \wedge \eta(t)) &= -\hat{d}\omega_2(t).
\end{align*}
\]

The \(SU(3)\)-structure on \(N^5 \times I\) is given by

\[
F = \omega_3(t) + \eta(t) \wedge dt,
\]

\[
\Psi = (\omega_1(t) + i\omega_2(t)) \wedge (\eta(t) + idt).
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**η-Einstein structures**

**Definition**

An almost contact metric manifold \((N^{2n+1}, \eta, \xi, \varphi, g)\) is **η-Einstein** if there exist \(a, b \in C^\infty(N^{2n+1})\) such that

\[
\text{Ric}_g(X, Y) = ag(X, Y) + b \eta(X)\eta(Y),
\]

where \(\text{scal}_g = a(2n+1) + b\) and \(\text{Ric}_g(\xi, \xi) = a + b\).

If \(b = 0\), a Sasaki \(\eta\)-Einstein is Sasaki- Einstein.

**Theorem (Conti, Salamon)**

A hypo structure on \(N^5\) is **η-Einstein** ⇔ it is Sasakian.

For a Sasaki \(\eta\)-Einstein structure on \(N^5\) we have

\[
d\eta = -2\omega_3, \quad d\omega_1 = \lambda \omega_2 \wedge \eta, \quad d\omega_2 = -\lambda \omega_1 \wedge \eta
\]

and for the associated generalized Killing spinor

\[
O = a \text{Id} + b \eta \otimes \xi,
\]

with \(a\) and \(b\) constants [Friedrich, Kim].
**η-Einstein structures**

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An almost contact metric manifold \((N^{2n+1}, \eta, \xi, \varphi, g)\) is **η-Einstein** if there exist \(a, b \in C^\infty(N^{2n+1})\) such that

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\text{Ric}_g(X, Y) = a \cdot g(X, Y) + b \cdot \eta(X)\eta(Y),
\]

where \(\text{scal}_g = a(2n + 1) + b\) and \(\text{Ric}_g(\xi, \xi) = a + b\).

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A hypo structure on \(N^5\) is **η-Einstein** \(\iff\) it is Sasakian.

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### $\eta$-Einstein structures

**Definition**

An almost contact metric manifold $(N^{2n+1}, \eta, \xi, \varphi, g)$ is $\eta$-Einstein if there exist $a, b \in C^\infty(N^{2n+1})$ such that

$$\text{Ric}_g(X, Y) = a g(X, Y) + b \eta(X)\eta(Y),$$

where $\text{scal}_g = a(2n+1) + b$ and $\text{Ric}_g(\xi, \xi) = a + b$.

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*A hypo structure on $N^5$ is $\eta$-Einstein $\iff$ it is Sasakian.*

For a Sasaki $\eta$-Einstein structure on $N^5$ we have

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η-Einstein structures

**Definition**

An almost contact metric manifold \((\mathcal{N}^{2n+1}, \eta, \xi, \varphi, g)\) is **η-Einstein** if there exist \(a, b \in C^\infty(\mathcal{N}^{2n+1})\) such that

\[
\text{Ric}_g(X, Y) = a g(X, Y) + b \eta(X)\eta(Y),
\]

where \(\text{scal}_g = a(2n + 1) + b\) and \(\text{Ric}_g(\xi, \xi) = a + b\).

If \(b = 0\), a Sasaki η-Einstein is Sasaki- Einstein.

**Theorem (Conti, Salamon)**

A hypo structure on \(\mathcal{N}^5\) is **η-Einstein** \(\iff\) it is Sasakian.

For a Sasaki η-Einstein structure on \(\mathcal{N}^5\) we have

\[
d\eta = -2\omega_3, \quad d\omega_1 = \lambda \omega_2 \wedge \eta, \quad d\omega_2 = -\lambda \omega_1 \wedge \eta
\]

and for the associated generalized Killing spinor

\[O = a \text{Id} + b \eta \otimes \xi,\]

with \(a\) and \(b\) constants [Friedrich, Kim].
**Hypo-contact structures**

In general, for a hypo structure the 1-form $\eta$ is **not** a contact form.

A hypo structure is **contact** if and only if $d\eta = -2\omega_3$.

**Problem**

*Find examples of manifolds $N^5$ with a hypo-contact structure.*

**Examples**

- Sasaki $\eta$-Einstein manifolds.

An example is given by the nilmanifold associated to

$$(0, 0, 0, 0, 12 + 34) \cong \mathfrak{h}_5.$$

- Contact Calabi-Yau structures, defined by the equations $d\eta = -2\omega_3$, $d\omega_1 = d\omega_2 = 0$ [Tomassini, Vezzoni].
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Classification in the hypo-contact case

Theorem (De Andres, Fernandez, –, Ugarte)

A 5-dimensional solvable Lie algebra \( \mathfrak{g} \) has a hypo-contact structure \( \Leftrightarrow \mathfrak{g} \) is isomorphic to one of the following:

\[
\begin{align*}
\mathfrak{g}_1 & : [e_1, e_4] = [e_2, e_3] = e_5 \ (\text{nilpotent and } \eta\text{-Einstein}); \\
\mathfrak{g}_2 & : \frac{1}{2}[e_1, e_5] = [e_2, e_3] = e_1, \ [e_2, e_5] = e_2, \\
& \quad [e_3, e_5] = e_3, \ [e_4, e_5] = -3e_4; \\
\mathfrak{g}_3 & : \frac{1}{2}[e_1, e_4] = [e_2, e_3] = e_1, \ [e_2, e_4] = [e_3, e_5] = e_2, \\
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\mathfrak{g}_5 & : [e_1, e_5] = [e_2, e_4] = e_1, \\
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\]

\( \Rightarrow \) Description of the 5-dimensional solvable Lie algebras which admit a hypo-contact structure.
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- $\mathfrak{g}_2 : \frac{1}{2}[e_1, e_5] = [e_2, e_3] = e_1, [e_2, e_5] = e_2, [e_3, e_5] = e_3, [e_4, e_5] = -3e_4$;
- $\mathfrak{g}_3 : \frac{1}{2}[e_1, e_4] = [e_2, e_3] = e_1, [e_2, e_4] = [e_3, e_5] = e_2, [e_2, e_5] = -[e_3, e_4] = -e_3$ ($\eta$-Einstein);
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\[\Rightarrow\] Description of the 5-dimensional solvable Lie algebras which admit a hypo-contact structure.
Consequences

- All the 5-dimensional solvable Lie algebras with a hypo-contact structure are irreducible.
- \( g_1 \cong \mathfrak{h}_5 \) is the unique nilpotent Lie algebra with a hypo-contact structure.
- The Lie algebras of the classification cannot be Einstein since they are contact [Diatta].
- The unique 5-dimensional solvable Lie algebras with a \( \eta \)-Einstein hypo-contact structure are \( g_1 \) and \( g_3 \).
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New metrics with holonomy $SU(3)$

Studying the Conti-Salamon evolution equations for the left-invariant hypo-contact structures on the simply-connected solvable Lie groups $G_i$ ($1 \leq i \leq 5$) with Lie algebra $g_i$:

**Theorem (De Andres, Fernandez, –, Ugarte)**

Any left-invariant hypo-contact structure on any $G_i$ ($1 \leq i \leq 5$) determines a Riemannian metric with holonomy $SU(3)$ on $G_i \times I$, for some open interval $I$.

For the nilpotent Lie group $G_1$ we get the metric found by Gibbons, Lü, Pope and Stelle.
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A homogeneous 3-dimensional Sasakian manifold has to be a Lie group endowed with a left-invariant Sasakian structure.

**Theorem (Geiges, Cho-Chung)**

Any 3-dimensional Sasakian Lie algebra is isomorphic to one of the following: \(\mathfrak{su}(2), \mathfrak{sl}(2, \mathbb{R}), \mathfrak{aff}(\mathbb{R}) \times \mathbb{R}, \mathfrak{h}_3\), where \(\mathfrak{aff}(\mathbb{R})\) is the Lie algebra of the Lie group of affine motions of \(\mathbb{R}\).

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Classify 5-dimensional Lie groups with a left-invariant Sasakian structure.
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**General results**

**Proposition (Andrada,–, Vezzoni)**

Let \((g, \eta, \xi)\) be a contact Lie algebra. Then \(\dim \mathfrak{z}(g) \leq 1\).

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Let \((g, \eta, \xi, \varphi, g)\) be a Sasakian Lie algebra.
- If \(\dim \mathfrak{z}(g) = 1\), then \(\mathfrak{z}(g) = \mathbb{R} \xi\) and \((\ker \eta, \theta, \varphi, g)\) is a Kähler Lie algebra, where \(\theta\) is the component of the Lie bracket of \(g\) on \(\ker \eta\).
- If \(\mathfrak{z}(g) = \{0\}\), then \(\text{ad}_\xi \varphi = \varphi \text{ad}_\xi\), and one has the orthogonal decomposition
  \[ g = \ker \text{ad}_\xi \oplus \text{Im} \text{ad}_\xi. \]

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- $\mathfrak{g}$ is either solvable or a direct sum.
- A 5-dimensional Sasakian solvmanifold is either a compact quotient of $H_5$ or of $\mathbb{R} \ltimes (H_3 \times \mathbb{R})$ with structure equations

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Anna Fino

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**SU(n)-structures in (2n + 1)-dimensions**

**Definition**

An $SU(n)$-structure $(\eta, \phi, \Omega)$ on $N^{2n+1}$ is determined by the forms

\[
\eta = e^{2n+1}, \quad \phi = e^{1} \wedge e^{2} + \ldots + e^{2n-1} \wedge e^{2n}, \\
\Omega = (e^{1} + ie^{2}) \wedge \ldots \wedge (e^{2n-1} + ie^{2n}).
\]

As for the case of $SU(2)$-structures in dimensions 5 we have that an $SU(n)$-structure $P_{SU}$ on $N^{2n+1}$ induces a spin structure $P_{Spin}$ and if we fix a unit element $u_0 \in \Sigma = (\mathbb{C}^2)^{\otimes 2n}$ we have that

\[
P_{SU} = \{ u \in P_{Spin} \mid [u, u_0] = \psi \}.
\]

The pair $(\eta, \phi)$ defines a $U(n)$-structure or an almost contact metric structure on $N^{2n+1}$.

The $U(n)$-structure is a contact metric structure if $d\eta = -2\phi$. 

**SU(2)-structures in 5-dimensions**

- Sasakian structures
- Sasaki-Einstein structures
- Hypo structures
- $\eta$-Einstein structures

**Hypo-contact structures**

- Classification
- Consequences
- New metrics with holonomy $SU(3)$

**Sasakian structures on Lie groups**

- 3-dimensional Lie groups
- General results
- 5-dimensional Lie groups

**SU(n)-structures in (2n + 1)-dimensions**

- Generalized Killing spinors
- Contact $SU(n)$-structures
- Examples
- Contact reduction
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\eta = e^{2n+1}, \quad \phi = e^1 \wedge e^2 + \ldots + e^{2n-1} \wedge e^{2n}, \\
\Omega = (e^1 + i e^2) \wedge \ldots \wedge (e^{2n-1} + i e^{2n}).
\]

As for the case of \( SU(2) \)-structures in dimensions 5 we have that an \( SU(n) \)-structure \( P_{SU} \) on \( N^{2n+1} \) induces a spin structure \( P_{Spin} \) and if we fix a unit element \( u_0 \in \Sigma = (\mathbb{C}^2)^{\otimes 2n} \) we have that

\[
P_{SU} = \{ u \in P_{Spin} \mid [u, u_0] = \psi \}.
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The pair \((\eta, \phi)\) defines a \( U(n) \)-structure or an almost contact metric structure on \( N^{2n+1} \).

The \( U(n) \)-structure is a contact metric structure if \( d\eta = -2\phi \).
**SU(n)-structures in (2n + 1)-dimensions**

**Definition**

An $SU(n)$-structure $(\eta, \phi, \Omega)$ on $N^{2n+1}$ is determined by the forms

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The pair $(\eta, \phi)$ defines a $U(n)$-structure or an almost contact metric structure on $N^{2n+1}$.

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Generalized Killing spinors

Example

$N^{2n+1} \hookrightarrow M^{2n+2}$ (with holonomy in $SU(n + 1)$).
Then the restriction of the parallel spinor defines an $SU(n)$-structure $(\eta, \phi, \Omega)$ where the forms $\phi$ and $\Omega \wedge \eta$ are the pull-back of the Kähler form and the complex volume form on the CY manifold $M^{2n+2}$.

Proposition (Conti, –)

Let $N^{2n+1}$ be a real analytic manifold with a real analytic $SU(n)$-structure $P_{SU}$ defined by $(\eta, \phi, \Omega)$. The following are equivalent:

1. The spinor $\psi$ associated to $P_{SU}$ is a generalized Killing spinor, i.e. $\nabla_X \psi = \frac{1}{2} O(X) \cdot \psi$.
2. $d\phi = 0$ and $d(\eta \wedge \Omega) = 0$.
3. A neighbourhood of $M \times \{0\}$ in $M \times \mathbb{R}$ has a CY structure which restricts to $P_{SU}$.
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\( \mathcal{N}^{2n+1} \hookrightarrow \mathcal{M}^{2n+2} \) (with holonomy in \( SU(n+1) \)).
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The assumption of real analyticity is certainly necessary to prove that (1) or (2) implies (3), but the fact that (1) implies (2) does not require this hypothesis.

(2) $\Rightarrow$ (3) can be described in terms of evolution equations in the sense of Hitchin. Indeed, suppose that there is a family $(\eta(t), \phi(t), \Omega(t))$ of $SU(n)$-structures on $N^{2n+1}$, with $t$ in some interval $I$, then the forms

$$\eta(t) \wedge dt + \phi(t), \quad (\eta(t) + idt) \wedge \Omega(t)$$

define a CY structure on $N^{2n+1} \times I$ if and only if (2) holds for $t = 0$ and the evolution equations

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Contact $SU(n)$-structures

**Definition**

An $SU(n)$-structure $(\eta, \phi, \Omega)$ on $N^{2n+1}$ is contact if $d\eta = -2\phi$.

In this case $N^{2n+1}$ is contact metric with contact form $\eta$ and we may consider the symplectic cone over $(N^{2n+1}, \eta)$ as the symplectic manifold $(N^{2n+1} \times \mathbb{R}^+, -\frac{1}{2} d(r^2\eta))$.

If $N^{2n+1}$ is Sasaki-Einstein, we know that the symplectic cone is CY with the cone metric $r^2 g + dr^2$ and the Kähler form equal to the conical symplectic form.

**Problem**

If one thinks the form $\phi$ as the pullback to $N^{2n+1} \cong N^{2n+1} \times \{1\}$ of the conical symplectic form, which types of contact $SU(n)$-structures give rise to a CY symplectic cone but not necessarily with respect to the cone metric?
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The answer is given by the following

**Proposition (Conti, –)**

Let $N^{2n+1}$ be a real analytic manifold with a real analytic contact $SU(n)$-structure $P_{SU}$ defined by $(\eta, \phi, \Omega)$. The following are equivalent:

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Examples

- 5-dimensional hypo-contact solvable Lie groups [De Andres, Fernandez, –, Ugarte].
- The $(2n + 1)$-dimensional real Heisenberg Lie group $H_{2n+1}$
  \[
d e^i = 0, \quad i = 1, \ldots, 2n, \\
d e^{2n+1} = e^1 \wedge e^2 + \ldots + e^{2n-1} \wedge e^2.
\]
- A two-parameter family of examples in the sphere bundle in $T\mathbb{C}P^2$ [Conti].
- A 7-dimensional compact example, quotient of the Lie group $SU(2) \ltimes \mathbb{R}^4$, which has a weakly integrable generalized $G_2$-structure [–, Tomassini].
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Hypo contact

Anna Fino

\(SU(2)\)-structures in 5-dimensions
- Sasakian structures
- Sasaki-Einstein structures
- Hypo structures
- \(\gamma\)-Einstein structures

Hypo-contact structures
- Classification
- Consequences
- New metrics with holonomy \(SU(3)\)

Sasaki structures on Lie groups
- 3-dimensional Lie groups
- General results
- 5-dimensional Lie groups

\(SU(n)\)-structures in \((2n + 1)\)-dimensions
- Generalized Killing spinors
- Contact \(SU(n)\)-structures

Examples
- Contact reduction
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Proposition (Conti, −)

\( H: \) compact Lie group
\( \rho \) a representation of \( H \) on \( V \).
Then \( H \ltimes \rho \ V \) has a left-invariant contact structure if and only if \( H \ltimes \rho \ V \) is either \( SU(2) \ltimes \mathbb{R}^4 \) or \( U(1) \ltimes \mathbb{C} \).

Then, if \( H \) is compact, the example \( SU(2) \ltimes \mathbb{R}^4 \) is unique in dimensions \( > 3 \).
If \( H \) is solvable we have

Proposition (Conti, −)

\( H : \) 3-dimensional solvable Lie group.
There exists \( H \ltimes \mathbb{R}^4 \) admitting a contact \( SU(3) \)-structure whose associated spinor is generalized Killing if and only if the Lie algebra of \( H \) is isomorphic to one of the following

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$(0, 0, 0)$, $(0, \pm 13, 12)$, $(0, 12, 13)$, $(0, 0, 13)$. 
Contact reduction

Let $N^{2n+1}$ be a $(2n + 1)$-dimensional manifold endowed with a contact metric structure $(\eta, \phi, g)$ and a spin structure compatible with the metric $g$ and the orientation. We say that a spinor $\psi$ on $N^{2n+1}$ is compatible if

$$\eta \cdot \psi = i^{2n+1} \psi, \quad \phi \cdot \psi = -ni\psi.$$

Suppose that $S^1$ acts on $N^{2n+1}$ preserving both metric and contact form, so that the fundamental vector field $X$ satisfies

$$\mathcal{L}_X \eta = 0 = \mathcal{L}_X \phi.$$

and denote by $t$ its norm. The moment map is given by $\mu = \eta(X)$.
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The moment map is given by $\mu = \eta(X)$. 
Assume that 0 is a regular value of \( \mu \) and consider the hypersurface \( \iota : \mu^{-1}(0) \to N^{2n+1} \).

Then the contact reduction is given by \( N^{2n+1} // S^1 = \mu^{-1}(0)/S^1 \) [Geiges, Willett].

- The contact \( U(n) \)-structure on \( N^{2n+1} \) induces a contact \( U(n-1) \)-structure on \( N^{2n+1} // S^1 \).

Let \( \nu \) be the unit normal vector field, dual to the 1-form \( i_{\iota^{-1}X} \phi \).

- The choice of an invariant compatible spinor \( \psi \) on \( N^{2n+1} \) determines a spinor

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\psi^\pi = i^* \psi + i \nu \cdot i^* \psi.
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where $X$ is the fundamental vector field associated to the $S^1$-action, and $t$ is the norm of $X$.

Example

If we apply the previous theorem to $SU(2) \ltimes \mathbb{R}^4$ we get a new hypo-contact structure on $S^2 \times T^3$. 
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**Corollary (Conti, −)**

$N^{2n+1}$ with an $\eta$-Einstein-Sasaki structure $(g, \eta, \phi, \psi)$, and let $S^1$ act on $M$ preserving the structure in such a way that $0$ is a regular value for the moment map $\mu$ and $S^1$ acts freely on $\mu^{-1}(0)$. Then the Sasaki quotient $M // S^1$ is also $\eta$-Einstein if and only if

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