The diameter of totally sum-free sets

Michael A. Daas

For a set X, we let $\mathcal{P}(X) = \{A \mid A \subset X\}$ denote its power set.

Definition 1. A subset $X \subset \mathbb{Z}$ of the integers is called *totally sum-free* if the map

$$\Sigma : \mathcal{P}(X) \to \mathbb{Z} \quad \text{given by} \quad \Sigma(A) = \sum_{a \in A} a$$

is injective. In more plain terms, a subset $X \subset \mathbb{Z}$ is called totally sum-free if no two distinct subsets of X have the property that their elements have the same total sum.

For notational convenience, we abbreviate

$$\max(X) = \max\{x \mid x \in X\} \quad \text{and} \quad \min(X) = \min\{x \mid x \in X\}.$$

The diameter of a finite set $X \subset \mathbb{Z}$ is defined as

$$\Delta(X) = \max(X) - \min(X).$$

The question that this project aims to study is as follows:

Question 2. Given a positive integer n, what is the mimimal value of $\Delta(X)$ among all totally sum-free sets $X \subset \mathbb{Z}$ with $\#X = n$?

Let us denote this value as $a(n)$, yielding an infinite sequence of integers. It is clear that $a(n) \geq n - 1$ for all $n \geq 1$, and because $X = \{1, 2, 3\}$ is totally sum-free, equality holds for $n = 1, 2, 3$. On the other hand, we have the following.

Lemma 3. *The value* $a(4)$ *is equal to* 4.

Proof. Suppose that $X \subset \mathbb{Z}$ is a set with 4 elements such that $\Delta(X) = 3$. Then for some $m \in \mathbb{Z}$, we must have $X = \{m, m+1, m+2, m+3\}$. But now the subsets $\{m, m+3\}$ and $\{m+1, m+2\}$ of X have the same sum $2m + 3$; a contradiction. Therefore $a(4) \geq 4$. We invite the reader to verify that the set $X = \{3, 5, 6, 7\}$ is in fact totally sum-free, completing the proof. □

Concretely, the goals of the project would be as follows:

- Compute more values of $a(n)$ using a combination of programming and mathematical insights to trim down the exponentially growing search space. Can you compute $a(10)$?
- Find good lower bounds for $a(n)$ by identifying mathematical obstructions preventing totally sum-free sets of small diameter. I personally do not know how to do this well!
- Find good upper bounds for $a(n)$ by coming up with novel and creative constructions of totally sum-free sets, with or without computer assistance.
- Can you find literature about this problem?