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Experimental Mathematics

Numerical simulation of a harmonic field around a slit-screen with one aperture

BMATH Year 2, Semester 4

In this project we study a two-dimensional "screen with one hole" modelled by two slits on the imaginary axis. We build and test a numerical solver for a harmonic problem in an exterior domain, and we visualise the associated harmonic field. A central experimental observable is the circulation of the harmonic field around a slit.

Geometry

The screen is the union of two line segments on the imaginary axis:

$$\Gamma_+ = \{(0, y) : y \text{ in } [1, 2]\}, \qquad \Gamma_- = \{(0, y) : y \text{ in } [-2, -1]\}, \qquad \Gamma = \Gamma_+ \cup \Gamma_-. \tag{1}$$

The aperture is the gap between $y = -1$ and $y = 1$. The computational domain is the plane without the screen:

$$\Omega = \mathbb{R}^2 \setminus \Gamma. \tag{2}$$

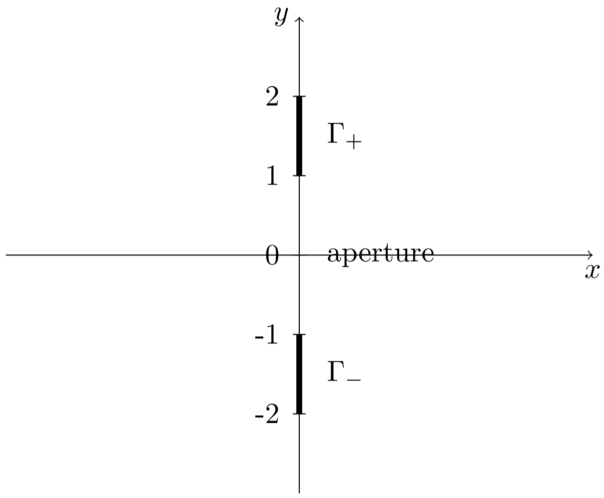

Figure 1: The slit-screen Γ on the imaginary axis (two segments), with an aperture in the middle.

Main goals

- Define precisely the mathematical object: a bounded harmonic function u in Ω with prescribed constant values on Γ_+ and Γ_-, and the associated harmonic vector field H.
- Implement a finite-difference solver on a truncated domain (large box), using sparse linear algebra, and produce robust visualisations for u and H.
- Develop a theoretical benchmark using conformal mapping ideas (elliptic integrals / Jacobi functions), and check quantitatively that the theoretical prediction matches the numerical measurements of circulation.

1 Continuous model

1.1 Harmonic in the exterior of the slit-screen

We introduce a scalar potential u defined on Ω. We prescribe constant values on each connected component of the screen:

$$u = 1 \text{ on } \Gamma_+, \qquad u = 0 \text{ on } \Gamma_-. \tag{3}$$

Inside the domain Ω, the potential is harmonic:

$$\Delta u = 0 \quad \text{in } \Omega. \tag{4}$$

Because the geometry is symmetric with respect to the origin, the solution satisfies the symmetry

$$u(-x, -y) = 1 - u(x, y). \tag{5}$$

In particular, the limit at infinity (if it exists) must be 1/2. For the purposes of the project we aim for a bounded solution (the values remain between 0 and 1).

1.2 Associated harmonic field

From u we define a planar vector field H by a 90° rotation of the gradient:

$$H = \nabla^{\perp} u = (\partial_y u, -\partial_x u). \tag{6}$$

If u is harmonic, then H is both divergence-free and curl-free in Ω, and it is tangent to the screen (no normal component on Γ).

2 Numerical method (finite differences)

Since Ω is unbounded, we work on a large square box:

$$\Omega_L = [-L, L] \times [-L, L] \setminus \Gamma, \tag{7}$$

with L chosen "large enough".

We discretise Ω_L with a uniform Cartesian grid of step h. On interior grid points we use the standard 5-point stencil:

$$(\Delta_h u)_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}. \tag{8}$$

Boundary conditions in the code

On grid points that approximate Γ_+ we set $u = 1$ (Dirichlet). On grid points that approximate Γ_- we set $u = 0$ (Dirichlet). On the outer boundary of the box we set $u = 1/2$ (Dirichlet) to mimic the far-field symmetry. We assemble a sparse linear system $AU = b$ and we solve it using a sparse direct solver or a Krylov method (CG/GMRES).

Discrete harmonic field

After solving for the grid values of u, we approximate derivatives by centered differences and define a discrete field $H_h = \nabla_h^{\perp} u$. We visualise: contours of u (equipotential lines), streamlines of H_h, and the magnitude $|H_h|$.

Sanity checks

- **Symmetry.** The quantity $u(-x,-y)+u(x,y)$ is close to 1 everywhere (away from the screen and the outer boundary).
- **Maximum principle (discrete).** The computed values satisfy $0 \le u \le 1$ up to numerical error.
- **Mesh refinement.** Repeating the computation with smaller h stabilises the plots and reduces residuals.
- **Independence of L.** Increasing the box size L has a diminishing effect near the screen.

3 Experimental questions

We explore numerically how the geometry controls the field intensity near the aperture. We can vary the endpoints, for instance by replacing $[1,2]$ and $[-2,-1]$ with $[a,b]$ and $[-b,-a]$, and by moving the segments slightly away from the axis.

Suggested experiments

- Measure $\max|H_h|$ in a fixed neighbourhood of the aperture and study its dependence on the gap size.
- Compute the discrete Dirichlet energy $E_h = \sum |\nabla_h u|^2 h^2$ and study convergence with h.
- Compare two outer boundary conditions (for instance $u = 1/2$ versus a two-step domain enlargement) and quantify the difference near the aperture.
- Replace the square box by a disk (disk mask) and check whether the results change.

4 Analytic benchmark (conformal mapping)

After a rotation $\zeta = -iz$, the screen becomes two real slits:

$$[-2,-1] \cup [1,2] \subset \mathbb{R}. \tag{9}$$

There is an explicit conformal map from the two-slit exterior domain to a rectangle. This map can be written in terms of elliptic integrals, and its inverse can be expressed using Jacobi elliptic functions.

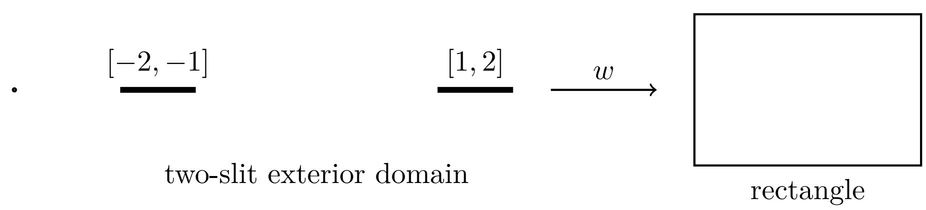

Figure 2: After rotation, the screen becomes two real slits. Conformal mapping theory turns the exterior domain into a rectangle, where harmonic functions and harmonic conjugates become linear coordinates.

The analytic component aims at producing a quantitative benchmark for the circulation of H around a slit. In the rectangle picture, this circulation becomes a period jump of a harmonic conjugate. The report includes:

- a clear derivation of the conformal map for the present endpoints,
- the resulting closed-form expression for the target circulation constant (in terms of complete elliptic integrals),
- a numerical evaluation of this constant and a comparison with the computed circulation.