COMBINATORICS & LLM

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Project type

This EML project is *undergraduate research*. Even more, we are proud to offer *a joint research project* between the supervisors and the students.

The project is very accessible, meaning there are no prerequisites beyond basic notions seen at the beginning of the Bachelor. Nevertheless, the project is a serious mathematical research project.

- The problem can be approached *theoretically*, meaning writing with pen and paper results, remarks, examples.
- The problem can also be approached *experimentally*, with mathematical softwares like the Python-based software Sagemath: the software will generate examples and it is then possible to inspect the data to make conjectures or construct counterexamples to some initial guesses.
- The problem can also be approached *with AI*. AI use is here welcome: Claude, ChatGPT, Gemini,... can all be used for inspiration and will surely provide useful insights. AI can produce and analyze data, formulate conjectures, and prove results (of course, anything coming from AI must be thoroughly verified).

We welcome students from the Bachelor and from the Master, from any year (there are no specific prerequisites). It is not necessary to have used mathematical softwares before, nor to have used AI for mathematics before. The plan is *working as a team*, and each team member is expected to contribute to the best of their capacities. Some students will be better in proving results, others in discussing with AI...

The collaboration is planned in the typical EML format, meaning that meetings every 10 days are foreseen (better in person, and remotely if necessary). Such meetings allow to monitor the project's progress and make research plans. They are also brainstorming sessions on the mathematical questions themselves.

Joint documents containing the obtained results/examples and AI chats will be shared with Overleaf. The supervisors can work on the shared files and answer questions (or, more generally, work for the project) not only during the meetings.

We are aware of and we will respect students' time constraints during the semester (many courses, midterms, personal reasons). However, we will distinguish students who do their best and put some effort from those who don't (an unsatisfactory participation will not be rewarded with a passing grade nor with authorship of eventual joint papers).

We encourage good and very good students to apply to this project because this is, in one word, a *cool* project that combines the research experience and the AI experience, and also, as mentioned, it is planned as a genuine collaboration with the supervisors. On top of that, we will have the kind assistance from Prof. Andrea Caranti (University of Trento).

It is never possible to promise mathematical results or papers in advance but the nature of this problem (various levels of generality; many concrete questions; several directions) and the commitment of the supervisors are the best possible conditions towards original results and, possibly, a publication. For these reasons, being a serious and motivated student is a necessary requirement that we put for admission to this project.

Project description

The project aims at addressing a problem in combinatorics that can be considered to be a natural question (namely, it is a basic and important question on the given subject).

Interested participants can be explained in full the applications to number theory of the results. In a nutshell, the one application we have in mind (there could be more, and not only in number theory) is seeing the given groups as Galois groups of cyclotomic extensions of number fields.

The mathematical objects under considerations are groups of tuples of integers modulo n ($n \geq 2$ being a given integer), where the operation is the addition of tuples (made component-wise).

A concrete example: Let $n = 2$ and consider triples. We are then considering elements of the form e.g.

$$(1, 1, 0) \qquad \text{or} \qquad (1, 0, 1)\,.$$

The full group of triples modulo 2 has eight elements, as there are eight possible triples made with zeroes and ones.

A tuple can be written as the sum of tuples that have many zeroes, for example

$$(1, 1, 0) = (1, 0, 0) + (0, 1, 0)\,.$$

The question we want to address is simple to state in its essence:

> If we restrict to consider the tuples only in a given subgroup, to which extent can we write a tuple as a sum of tuples with more zeroes?

For example, consider the group consisting of these four elements:

$$(1, 1, 0) \qquad (0, 0, 1) \qquad (1, 1, 1) \qquad (0, 0, 0)\,.$$

Then it is not possible to decompose $(1, 1, 0)$ as a sum of tuples with more zeroes that belong to this group.

Our starting point is what we call *level 2*, namely decomposing a tuple that has two non-zero entries as a sum of tuples that have one non-zero entry.

Chi Wa Chan (a former PhD student of Perucca) has obtained a very general criterion, where n is a prime power and one considers a group of m-tuples for some fixed positive integer m. Beyond some straight-forward constraints, there is just one additional condition in the problem, that we now explain. We can look at three components at the time (to ease notation, we now call their indices $1, 2, 3$). Then we define (in a straight-forward way) a parameter $c_{1,2}$ which measures the number of tuples that we have in the given group that have non-zero entries precisely at positions 1 and 2 (with a similar notation for the other entries). Then the condition is that the three numbers

$$c_{1,2} \qquad c_{1,3} \qquad c_{2,3}$$

are either all equal or there is precisely one maximum while the other two numbers are equal.

The supervisors look forward to this fun (and useful) project, and welcome students to share this mathematical adventure. For any questions, feel free to send us an email.