

# Union of Subgroups

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EML project (Bachelor or Master)

Consider an abelian group  $G$ .

Is  $G$  the set theoretical union of proper subgroups?

If  $G$  is cyclic, this is not possible because  $G$  is the only subgroup containing an element of maximal order. If  $G$  is not cyclic, then  $G$  is the union of its cyclic subgroups, which are all proper subgroups.

Suppose that  $G$  has order  $p^n$ , where  $p$  is a prime power and  $n$  is a positive integer. Moreover, suppose w.l.o.g. that  $G$  is not cyclic.

How many proper (non-trivial) subgroups are needed to cover  $G$ ?

It turns out that only groups of small size can be covered by few subgroups. There are estimates and general considerations in the mathematical literature, which will serve as a starting point. Perucca has recently considered the union of up to 4 subgroups for an application to Artin's conjecture (the groups being Galois groups of cyclotomic-Kummer extensions of a number field).

The aim of this project, after an initial phase aimed at understanding the known results, is to investigate what happens for the union of 5 (or slightly more) subgroups, and making further general considerations.

The project is accessible without specific prerequisites beyond a minimum of group theory, and we encourage good and very good students to apply for it. New examples of union of subgroups may lead to new examples of zero density in Artin's conjecture (more precisely, to new types of obstructions to the positivity of the density).

The problem can be approached theoretically, computationally, and also with AI (asking small questions, and checking the output carefully). Quality results are potentially publishable, let's see what we will achieve...

Precise references will be provided to the participants, and interested students may contact [antonella.perucca@uni.lu](mailto:antonella.perucca@uni.lu) for any question.