3D-Printing Hyperbolic Polyhedra

Quirijn Boeren Carl Lutz

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Description

The five Platonic solids (tetrahedron, cube, octahedron, dodecahedron, icosahedron) are the regular polyhedra of ordinary Euclidean space. Each is described by a *Schläfli symbol* $\{p, q\}$: p-sided regular faces, q of them meeting at every vertex. Because of the angle-sum condition $1/p+1/q > 1/2$, there are only five such symbols for which a finite polyhedron exists in Euclidean 3-space.

What happens for the combinations $\{p, q\}$ that do *not* satisfy this inequality? They do not disappear — they reappear as the cells of regular tilings ("honeycombs") of hyperbolic 3-space. A striking feature of hyperbolic geometry is that, unlike in Euclidean space, a regular polyhedron of a given combinatorial type $\{p, q\}$ is not rigid: as it grows larger, its dihedral angles shrink continuously. This gives, for a fixed shape such as the dodecahedron, an entire family of "hyperbolic dodecahedra," each with a different dihedral angle, rather than a single rigid shape.

Two historical examples:

- A dodecahedron with dihedral angle exactly 90° is a *right-angled* hyperbolic dodecahedron; a right-angled polyhedron was used by Löbell in 1931 to build the very first known examples of compact hyperbolic 3-manifolds.
- A dodecahedron with dihedral angle exactly 72° is the building block of the Seifert–Weber dodecahedral space (1933), obtained by gluing its opposite faces together.

Turning such an object into a physical 3D print requires choosing a model of hyperbolic space that embeds it into ordinary Euclidean 3-space. We shall use the Poincaré ball model, in which edges become circular arcs and faces become curved. The aim of this project is to compute this embedding explicitly and to turn it into a mesh a 3D printer can handle. See figure 1 for an example of a shape we might aim to 3D print. Before tackling this, we will first practice the whole modeling-to-printing pipeline on the (easier, flat-faced) Platonic and Archimedean solids.

Goal

- Practice by modeling and 3D printing a few Platonic solids, and one or two Archimedean solids.
- Get acquainted with the basics of hyperbolic geometry needed for the project: the model of hyperbolic space (Poincaré ball), Schläfli symbols $\{p, q, r\}$ for regular honeycombs, and why hyperbolic regular polyhedra come in continuous families of dihedral angles rather than a single rigid shape.
- Choose one or two hyperbolic regular polyhedra; parametrize them explicitly, and 3D print the result.

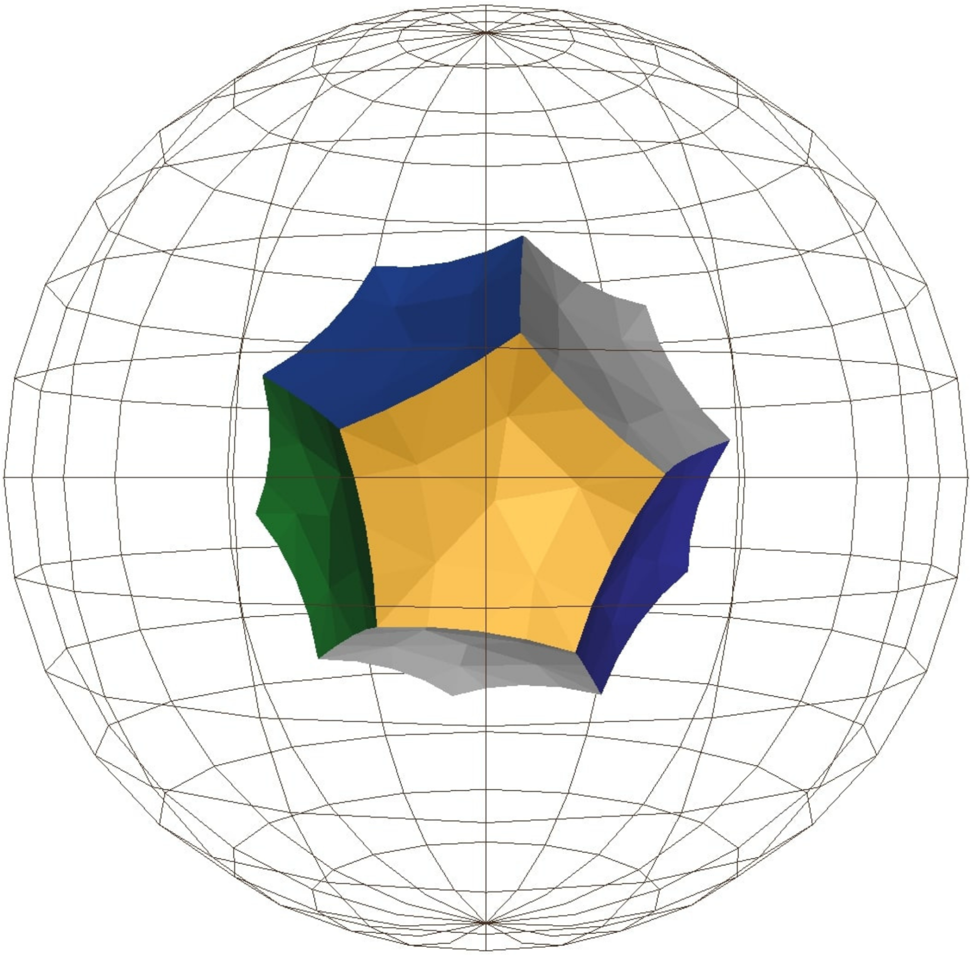

Figure 1: (Polygonized) example of a right-angled dodecahedron in the Poincaré ball, taken from this paper.

Tools / Prerequisites

- No prior knowledge of hyperbolic geometry is required; the necessary background (models of hyperbolic space, isometries, Schläfli symbols) will be introduced during the project.
- Willingness to program is required.
- The math department has a 3D printer we can use.