Billiards in a cube and tetrahedral packings

EML project, winter semester 2024

Branch: Geometry
Language: English
Difficulty: adjustable
Trailer: https://www.youtube.com/watch?v=5gXBIww-wxA
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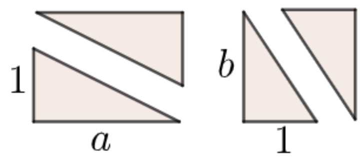
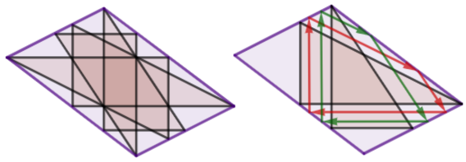
Two-dimensional teaser

You are given these four triangles in the plane ⟶
You are allowed to parallel translate given shapes (slide them around without rotating). The goal is to minimize the area of the convex hull [1], which is the smallest polygon containing the triangles.



The answer (spoiler alert) is a parallelogram. ⟶
It turns out that some pairs of triangles bound certain parallel families of billiard trajectories (see on the right), and the problem is closely connected to billiard dynamics.



Three-dimensional goal

Investigate a similar problem in 3D, and visualize the billiard dynamics in a cube in a normed space (for example, in Geogebra or Mathematica). Optionally, we could also do a harder optimization problem about minimizing the volume of the convex hull of a few tetrahedra.

Motivation

This problem is cool because it allows to solve special cases of *Viterbo's conjecture*, which is a fancy problem in *symplectic geometry*. The conjecture is reduced to the question of minimal convex hull using a very neat *billiard* approach [2].

References

[1] https://en.wikipedia.org/wiki/Convex_hull.

[2] Balitskiy, A. "Equality cases in Viterbo's conjecture and isoperimetric billiard inequalities." International Mathematics Research Notices 2020.7 (2020): 1957-1978. Arxiv version: https://arxiv.org/abs/1512.01657.