Fractals and Iterated function systems

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Language: French or English

Description: Fractals are intuitively known as geometric shapes containing detailed structures at arbitrarily small scales such that they appear similar at various scales. This "definition" is a bit restrictive and, mathematically, we say that a set F is fractal if *its fractal dimension* strictly exceeds its *topological dimension*. Nevertheless, many fractals can be described using *iterated function systems* in a way closer to the intuitive notion of fractals. Let D be a closed subset of $\mathbb{R}^n$, a contraction $S : D \to D$ is a mapping such that there exists $0 < c < 1$ for which, for all $x, y \in D$, $|S(x) - S(y)| \leq c|x - y|$. An iterated function system (IFS) is a finite family $\{S_1, \dots, S_m\}$, with $m \geq 2$, of contractions.

Let $\{S_1, \dots, S_m\}$ be an IFS with, for all $1 \leq j \leq m$, $0 < c_j < 1$ such that

$$|S_j(x) - S_j(y)| \leq c_j |x - y| \quad \forall\, x, y \in D. \tag{1}$$

For all non empty compact subset K of D, we set

$$S(K) := \bigcup_{j=1}^{m} S_j(K).$$

It is possible to prove that

- there exists a unique compact subset F of D such that

  $$F = S(F).$$

  We call F the attractor of the IFS.

- For all non empty compact subset K of D, the sequence $(S^j(K))_j$ (with $S^j(K) = S(S^{j-1}(K)))$ converges in the so-called *Hausdorff metric* to F.

- If the contractions are similarities, meaning that the inequality in (1) is in fact an equality, the fractal dimension of the attractor F is given by the unique positive solution s to the equation

  $$\sum_{j=1}^{m} c_j^s = 1. \tag{2}$$

  In general, the fractal dimension is always bounded from above by the solution s to equation (2). It is possible to find a bound from below in a similar way.

In particular, the attractor F of an IFS made of similarities is a fractal if the solution s to the equation (2) is less than the topological dimension of F. For instance, the well-known (modified) von Koch Curve is a fractal which is obtained as an attractor of IFS, see Figure 1 below.

The objectives of this project are listed as follows:

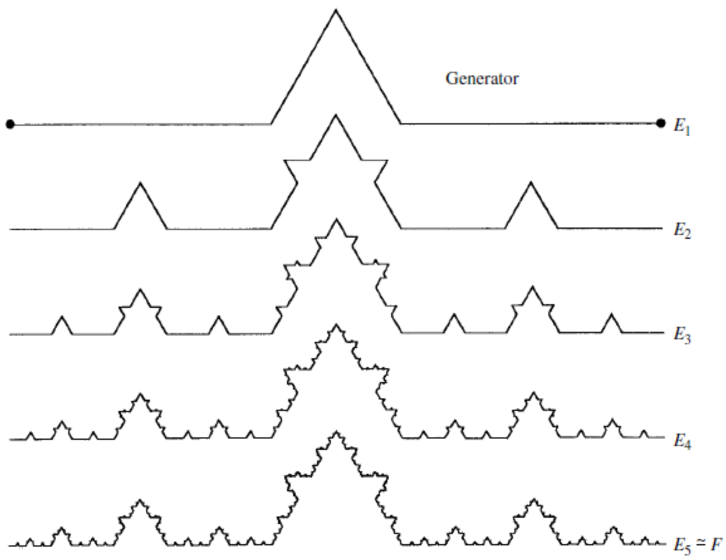

Figure 1: The von Koch curve is the attractor of the contractions that map $[0, 1]$ onto each intervals in E_1

1. **Theoretical part:** the report must show that the students have understood properly the following notions:
    - fractal and topological dimensions
    - IFS
    - Hausdorff metric
2. **Programming part:** the program must be able, given a finite family of functions on a closed set D
    - to check if the family of functions is an IFS and if it is made of similarities,
    - if the family is an IFS, to draw the resulting attractor (or at least a good approximation of it),
    - if the family is an IFS, to estimate (or even determine in the case of similarities) the fractal dimension of the attractor and to decide whether it is a fractal or not.

Depending on the progression of the project and the skills of the students, other subjects can be discussed, such as, for instance, self-affine sets, creating a program to test the convergence in Hausdorff metric,...