Frobenius Problem

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February 17, 2023

The Nugget Problem

Imagine that at your favourite restaurant, you can buy nuggets to go in packs of 6, 9 and of 20. It is therefore possible to order for instance precisely $15 = 6 + 9$ nuggets, but impossible to buy precisely 14 nuggets. Now, what number of nuggets can you not buy in precise amounts? And what is the biggest one? You can find the answer to both questions in this Numberphile video.

A More General Version

The goal of this project will be to understand and partially solve the above problem in a more general context. More precisely, for integers $p_1, \dots, p_r \geqslant 1$, we consider the set

$$A(p_1, \dots, p_r) \coloneqq \mathbb{N} - \sum_{i=1}^{r} p_i \mathbb{N}$$

of positive integers which cannot be represented as a sum of the p_i, and we are interested in answering the following two questions:

- What is
  $$g(p_1, \dots, p_r) \coloneqq \sup A(p_1, \dots, p_r),$$
  the *biggest* positive integer which cannot be expressed as a sum of the p_i?
- What is
  $$a(p_1, \dots, p_r) \coloneqq \# A(p_1, \dots, p_r),$$
  that is, *how many* positive integers exist which cannot be expressed as a sum of the p_i?

It may also help to have a look at the corresponding Wikipedia page.

Some Special Cases

In case the p_i share a common factor, one easily sees that

$$g(p_1, \dots, p_r) = \infty, \qquad a(p_1, \dots, p_r) = \infty.$$

Further, for coprime integers $p, q \geqslant 1$, it is not hard to prove that

$$g(p, q) = (p-1)(q-1) - 1, \qquad a(p, q) = \frac{(p-1)(q-1)}{2}.$$

Unfortunately, there is little hope to find a general formula for $g(p_1, \ldots, p_r)$ and for $a(p_1, \ldots, p_r)$. However, there are many special cases waiting to be discovered that *can* be solved! For instance, one can find formulas for $g(p, q_1, \ldots, q_{p-1})$ and for $a(p, q_1, \ldots, q_{p-1})$ in case $p, q_1, \ldots, q_{p-1}$ are coprime positive integers, none of them being a sum of the remaining ones.

Project Goals

Students working on this project will in particular:

- Write a program computing the numbers $g(p_1, \ldots, p_r)$ and $a(p_1, \ldots, p_r)$.
- Understand all the special cases that were mentioned above.
- Come up with additional special cases for which exact formulas exist.