

Arithmetic Billiards

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Abstract

Arithmetic billiards are a mathematical model for some (idealized) billiards. We start off by giving a short motivation for the use of arithmetic billiards followed by developing and explaining the mathematical background. We consider the usual 2-dimensional billiard and extend some property to its n -dimensional generalization. We provide several examples and also study other variations of the classical setting.

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1 Introduction

Arithmetic billiards model what we can experience on a regular billiard table. The same rules are applied for arithmetic billiards as for ordinary billiards but with some assumptions:

1. We only consider integer coordinates
2. The ball is shot from one corner making a 45° angle with the sides
3. The ball has no mass and is as large as a point
4. There is no friction
5. There are no pockets in the 4 corners, the trajectory stops after hitting a corner
6. The ball can bounce indefinitely many times on the sides of the billiard table
7. The ball reflects at a 90° angle

1.1 Motivation

What is the most interesting about arithmetic billiards is that they provide a geometrical help to determine the greatest common divisor and the least common multiple of two natural numbers. In the picture below you can try to figure how the construction works.

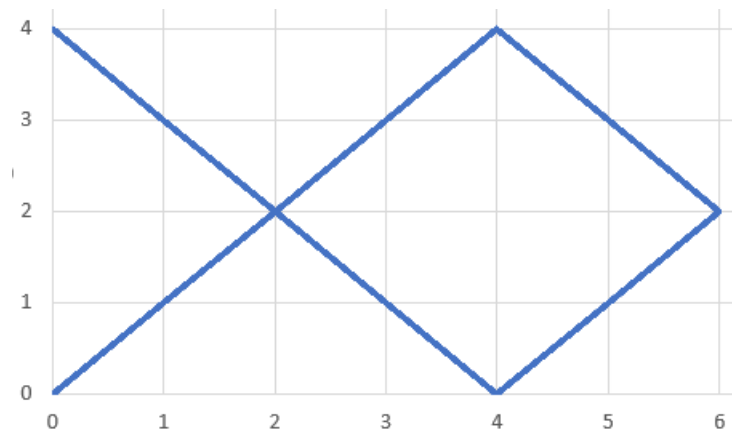


Figure 1: The greatest common divisor of 4 and 6 is 2 and the least common multiple of 4 and 6 is 12.

The least common multiple of two given positive integers is equal to the sum of all unit squares which the path crosses, more precisely it is the total length of the path divided by $\sqrt{2}$.

The greatest common divisor of two given positive integers is equal to the sum of all unit squares crossed by the first segment of the path up to the first intersection point or bouncing point, more precisely it's the total length from the starting point to the first intersection point or bouncing point divided by $\sqrt{2}$.

2 Bouncing points

In this section we want to find an easy way to calculate how many bouncing points the path of an arithmetic billiard has. Without loss of generality, we suppose to shoot the ball from the bottom left corner with respect to the sides of the rectangle. We also want to look at the bouncing points if we start shooting the ball from other coordinates and we hope to find some formula which holds for every point. We will only change the values of the y coordinate for the starting point because the ball bounces on the right side (including the corners) at some point and the path is periodic. Later we will see that it doesn't matter if we start at the coordinates $(t, 0)$ or at the coordinates $(0, t)$ for t a natural number.

Definition 1. (Bouncing point)

The point where the path touches one side of the rectangle and is reflected is called a bouncing point. The corners will not be counted as bouncing points.

Definition 2. (Periodic path)

The path of the arithmetic billiard is called periodic if the ball goes back to the starting point, in other words the path contains a loop, repeats itself and crosses itself.

Example 1. Here are two examples of periodic paths.

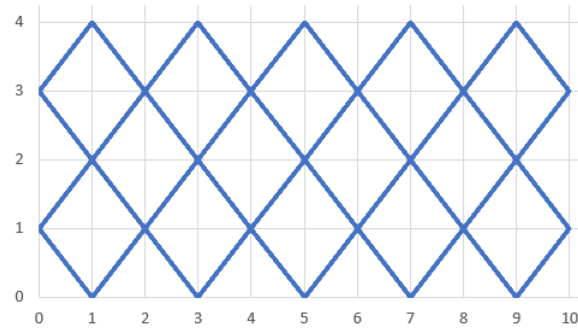


Figure 2: Periodic path starting at the point $(1, 0)$.

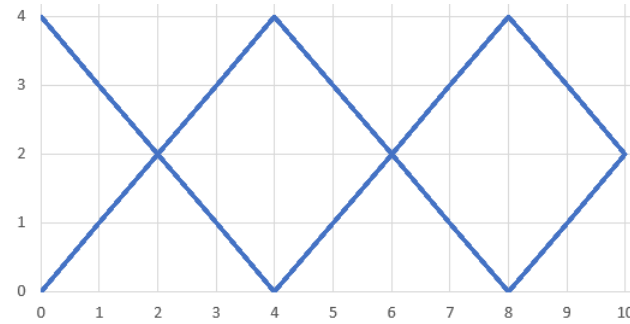


Figure 3: Periodic path starting at the point $(0, 0)$.

Note that the path ends in a corner if and only if it starts in a corner.

2.1 Bouncing points starting at the point $(0, 0)$

We will start by writing a table where we write at the top the length of one side of the rectangle and on the left the length of the other side of the rectangle. In the middle we will put the number of the bouncing points with respect to the 2 sides of the rectangle.

$a \setminus b$	3	4	5	6	7	8	9	10	11	12	13
2	3	1	5	2	7	3	9	4	11	7	13
3		5	6	1	8	9	2	11	12	3	14
4			7	3	9	1	11	5	13	2	15
5				9	10	11	12	1	14	15	16
6					11	5	3	6	15	1	17
7						13	14	15	16	17	18
8							15	7	17	3	19
9								17	18	19	20

Table 1: Number of bouncing points at starting point $(0,0)$

If we look at the table where a and b are coprime numbers we can observe that the total amount of bouncing points is equal to $a + b - 2$.

In general we have :

Theorem 1. *The total amount of bouncing points in a rectangle with starting point $(0, 0)$ is*

$$\#bouncing\ points = \frac{a + b}{gcd(a, b)} - 2.$$

More precisely, we have

$$\#bouncing\ points = \frac{a}{gcd(a, b)} - 1$$

on the a side of the rectangle and we have

$$\#bouncing\ points = \frac{b}{gcd(a, b)} - 1$$

on the b side of the rectangle.

Proof. We start by drawing a square whose side is the $lcm(a, b)$, then we decompose the square into rectangles with sides a and b . Next, we draw the diagonal of the square. If we look at Figure 4 here below, we can see that the path of the billiard ball corresponds exactly to the diagonal of the square if we reflect the rectangle when a path segment hits a side.

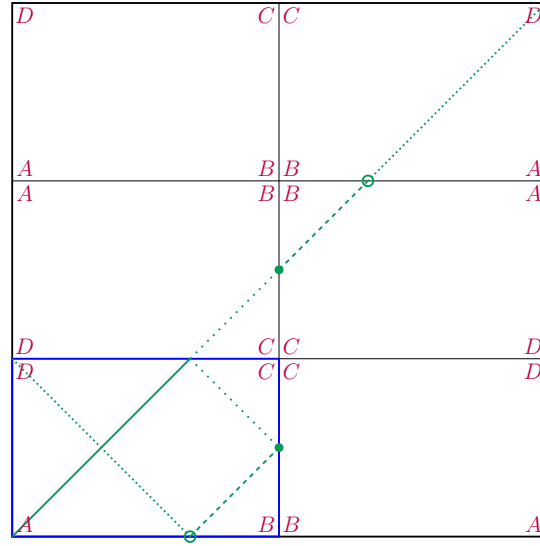


Figure 4: Reflecting the billiard, if the ratio between the two sides is $2/3$.

In Figure 4 we used a rectangle with sides 4 and 6, if we count the number of times the diagonal hits one side of the rectangle until it hits the other corner of the square, we have precisely the total amount of bouncing points. We can generalize this method for rectangles of different sizes.

We also know that $\frac{lcm(a,b)}{b} = \frac{a}{gcd(a,b)}$, so we have that the number of bouncing points at the side b is equal to $\frac{b}{gcd(a,b)} - 1$, similarly we have $\frac{a}{gcd(a,b)} - 1$ bouncing points on the side a . So if we want the total amount of bouncing points, we set

$$\# \text{bouncing points} = \frac{b}{gcd(a,b)} - 1 + \frac{a}{gcd(a,b)} - 1 = \frac{a+b}{gcd(a,b)} - 2.$$

□

2.2 Bouncing points starting at the point $(0, 1)$

We will start by writing a table where we write at the top the length of one side of the rectangle and on the left the length of the other side of the rectangle. In the middle we will put the number of the bouncing points with respect to the 2 sides of the rectangle. For simplicity we will one look at $(a \times b)$ rectangles for $a > b$.

$b \backslash a$	3	4	5	6	7	8	9	10	11	12	13
2	3	6	5	8	7	10	9	12	11	14	13
3		5	6	6	8	9	8	11	12	10	14
4			7	10	9	6	11	14	13	8	15
5				9	10	11	12	6	14	15	16
6					11	14	10	16	15	6	17
7						13	14	15	16	17	18
8							15	18	17	10	19
9								17	18	14	20

Table 2: Number of bouncing points at starting point $(0,1)$

If we look at the table for the a and b coprime numbers we can observe exactly the same numbers as for starting coordinates $(0, 0)$, so that the total amount of bouncing points for coprime numbers is equal to $a + b - 2$.

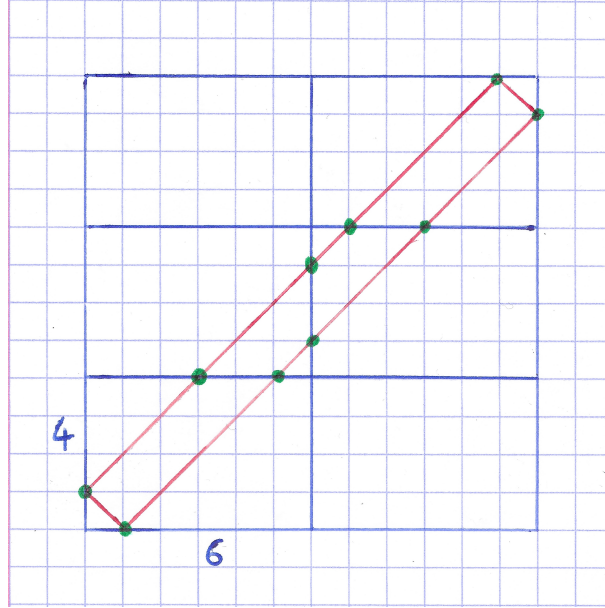
If we look now for all the numbers of a and b which have at least one common factor in their prime factorisation, we can observe that the total amount of bouncing points is equal to $2 \times (\frac{a}{\gcd(a,b)} + \frac{b}{\gcd(a,b)})$.

Theorem 2. *The total amount of bouncing points in a rectangle with starting coordinates $(0, 1)$ is*

$$\#bouncing\ points = \begin{cases} \frac{a+b}{\gcd(a,b)} - 2, & \text{for } a \text{ and } b \text{ coprime} \\ 2 \times (\frac{a}{\gcd(a,b)} + \frac{b}{\gcd(a,b)}), & \text{when } a \text{ and } b \text{ have a common factor.} \end{cases}$$

Proof. For the case where a and b are coprime numbers we can use the same proof as for starting coordinates $(0, 0)$ because up to reflecting the path, the path starts at $(0, 0)$.

For the case where a and b which have at least one common factor in their prime factorisation, we can use the same technique we used to prove Theorem 1. In the picture below we can see that the path has a shape.

Figure 5: Reflecting the billiard, for a 6×4 rectangle.

Let $L = \text{lcm}(a, b)$, then the ball bounces at $(L - 1, L)$, then immediately again at $(L, L - 1)$ and goes back all the way to $(1, 0)$, and finally bounces back to $(0, 1)$.

Now we can use this shape to count intersection points as we did in the first case, it's just slightly more complicated.

The only thing that we should pay attention to is when the ball goes through the corner. This could become a problem because the ball doesn't bounce in the corner, so it wouldn't continue the path. Luckily, one can show that this case only happens when $\gcd(a, b) = 1$. In fact, assume the ball crosses one of the corners in the big square. This means it has reached the point (ma, nb) for some positive integers m and n . If we take $a = 3$, $b = 2$, then $m = 1$, $n = 2$.

As we can see, the path of the ball until this point is the diagonal of the square with vertices $(0, 1)$, $(0, nb)$, $(ma, 1)$ and (ma, nb) . (This happens to be a square because we shoot the ball at a 45° angle.)

But since this is a square, its sides must be equal, in particular $nb - 1 = ma$ (here ma is the horizontal side and $nb - 1$ is the vertical side). But we can write this equality as $nb - ma = 1$, and by Bezout identity this implies that $\gcd(a, b) = 1$, as we wanted to show.

□

2.3 Bouncing points starting at the point $(0, 2)$

We will start by writing a table where we write at the top the length of one side of the rectangle and on the left the length of the other side of the rectangle. In the middle we will put the number of the bouncing points with respect to the two sides of the rectangle. For simplicity we will one look at $(a \times b)$ rectangles for $a > b$.

$b \backslash a$	4	5	6	7	8	9	10	11	12	13
3	5	6	6	8	9	8	11	12	10	14
4		7	3	9	6	11	5	13	8	15
5			9	10	5	12	6	14	6	16
6				11	5	10	6	15	12	17
7					13	14	15	16	17	18
8						15	7	17	10	19
9							17	18	14	20

Table 3: Number of bouncing points at starting point $(0,2)$

If we look at the table for the a and b coprime numbers we can observe exactly the same numbers as for starting coordinates $(0, 0)$ and $(0, 1)$, so that the total amount of bouncing points for coprime numbers is equal to $a + b - 2$ in these examples.

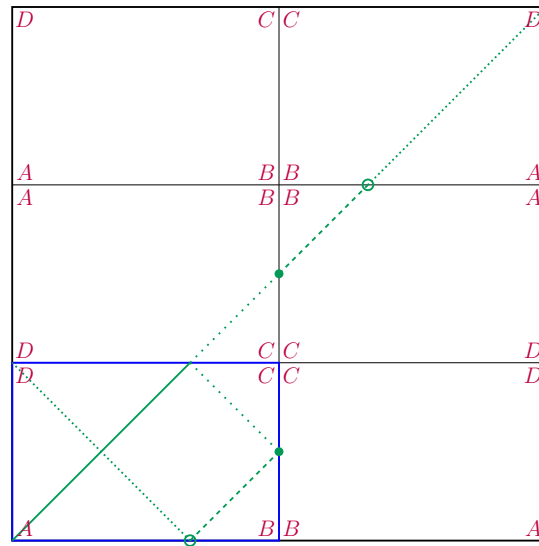
For b odd, we observe the same numbers as for starting coordinates $(0,1)$.

For b even, we observe for a a multiple of b that #bouncing points $= 2 \times (\frac{a}{gcd(a,b)} + \frac{b}{gcd(a,b)})$ and for a and b with a common factor we observe that #bouncing points $= \frac{a+b}{gcd(a,b)} - 2$.

Theorem 3.

$$\#bouncing\ points = \begin{cases} \frac{a+b}{gcd(a,b)} - 2, & \text{for } gcd(a,b) = 1 \\ \frac{a+b}{gcd(a,b)} - 2, & \text{when } a \text{ and } b \text{ are both even, } gcd(a,b) \neq 1 \text{ and } b \nmid a \\ 2 \times (\frac{a}{gcd(a,b)} + \frac{b}{gcd(a,b)}), & \text{when one of } a \text{ and } b \text{ is even and the other one is odd, } gcd(a,b) \neq 1 \\ 2 \times (\frac{a}{gcd(a,b)} + \frac{b}{gcd(a,b)}), & \text{when } b|a. \end{cases}$$

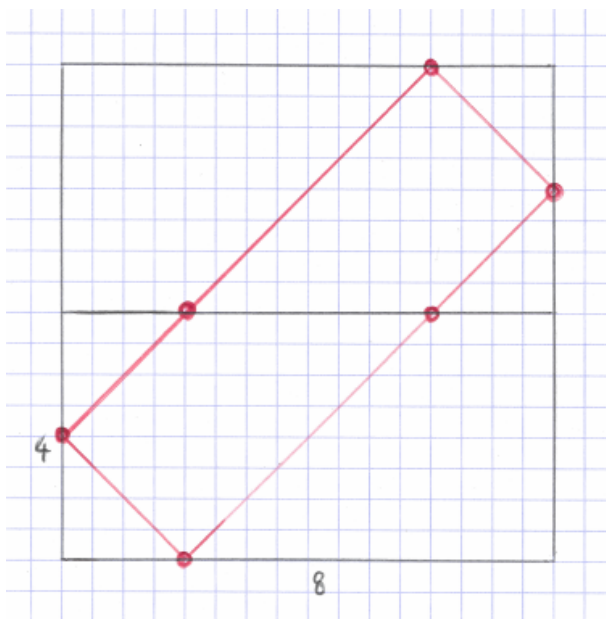
Proof. For the first part of the theorem, we start by drawing a square whose side is the $lcm(a, b)$, then we decompose the square into rectangles with sides a and b . Next, we draw the diagonal of the square. If we look at Figure 6 here below, we can see that the path of the billiard ball corresponds exactly to the diagonal of the square if we reflect the rectangle when a path segment hits a side.



In Figure 6 we used a rectangle with sides 4 and 6, if we count the number of times the diagonal hits one side of the rectangle until it hits the other corner of the square, we have precisely the total amount of bouncing points. We can generalize this method for rectangles of different sizes. We also know that $\frac{lcm(a,b)}{b} = \frac{a}{gcd(a,b)}$, so we have that the number of bouncing points at the side b is equal to $\frac{b}{gcd(a,b)} - 1$, similarly we have $\frac{a}{gcd(a,b)} - 1$ bouncing points on the side a . So if we want the total amount of bouncing points, we set

$$\# \text{bouncing points} = \frac{b}{\gcd(a, b)} - 1 + \frac{a}{\gcd(a, b)} - 1 = \frac{a + b}{\gcd(a, b)} - 2.$$

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Figure 7: Reflecting the billiard, for a 8×4 rectangle.

Let $L = \text{lcm}(a, b)$, then the ball bounces at $(L - 1, L)$, then immediately again at $(L, L - 1)$ and goes back all the way to $(1, 0)$, and finally bounces back to $(0, 1)$.

Now we can use this shape to count intersection points as we did in the first case, it's just slightly more complicated.

The only thing that we should pay attention to is when the ball goes through the corner. This could become a problem because the ball doesn't bounce in the corner, so it wouldn't continue the path. Luckily, one can show that this case only happens when $\gcd(a, b) = 1$. In fact, assume the ball crosses one of the corners in the big square. This means it has reached the point (ma, nb) for some positive integers m and n . If we take $a = 3$, $b = 2$, then $m = 1$, $n = 2$.

As we can see, the path of the ball until this point is the diagonal of the square with vertices $(0, 1)$, $(0, nb)$, $(ma, 1)$ and (ma, nb) . (This happens to be a square because we shoot the ball at a 45° angle.)

But since this is a square, its sides must be equal, in particular $nb - 1 = ma$ (here ma is the horizontal side and $nb - 1$ is the vertical side). But we can write this equality as $nb - ma = 1$, and by Bezout identity this implies that $\gcd(a, b) = 1$, as we wanted to show.

□

2.4 Bouncing points starting at the point $(0, 3)$

We will start by writing a table where we write at the top the length of one side of the rectangle and on the left the length of the other side of the rectangle. In the middle we will put the number of the bouncing points with respect to the two sides of the rectangle. For simplicity we will one look at $(a \times b)$ rectangles for $a > b$.

$b \backslash a$	5	6	7	8	9	10	11	12	13
4	7	10	9	6	11	14	13	8	15
5		9	10	11	12	6	14	15	16
6			11	14	10	16	15	6	17
7				13	14	15	16	17	18
8					15	18	17	10	19
9						17	18	14	20

Table 4: Number of bouncing points at starting point $(0,3)$

If we look now at the table we see exactly the same values as for starting point $(0, 1)$. We can explain this because up to reflecting we can use the same statement as we used for the case where the path starts at the point $(0, 1)$.

2.5 Conclusion for the total amount of bouncing points for any starting point

If we start at the coordinates (x_0, y_0) such that $x_0 + y_0 = 0$ we have the following formula:

$$\# \text{bouncing points} = \frac{a+b}{\gcd(a,b)} - 2.$$

If we start at the coordinates (x_0, y_0) such that $x_0 + y_0$ is odd we have the following formulas:

$$\# \text{bouncing points} = \begin{cases} \frac{a+b}{\gcd(a,b)} - 2, \text{for } a \text{ and } b \text{ coprime} \\ 2 \times \left(\frac{a}{\gcd(a,b)} + \frac{b}{\gcd(a,b)} \right), \text{when } a \text{ and } b \text{ have a common factor.} \end{cases}$$

If we start at the coordinates (x_0, y_0) such that $x_0 + y_0$ is even we have the following formulas:

$$\# \text{bouncing points} = \begin{cases} \frac{a+b}{\gcd(a,b)} - 2, \text{for } \gcd(a,b) = 1 \\ \frac{a+b}{\gcd(a,b)} - 2, \text{when } a \text{ and } b \text{ are both even, } \gcd(a,b) \neq 1 \text{ and } b \nmid a \\ 2 \times \left(\frac{a}{\gcd(a,b)} + \frac{b}{\gcd(a,b)} \right), \text{when one of } a \text{ and } b \text{ is even and the other one is odd, } \gcd(a,b) \neq 1 \\ 2 \times \left(\frac{a}{\gcd(a,b)} + \frac{b}{\gcd(a,b)} \right), \text{when } b|a. \end{cases}$$

3 On which corner will the ball land if we start at the point (0,0)

In this section we want to know in which corner the ball will land if we start shooting the ball at the point (0, 0).

We will begin to define what it means when we say that a path is symmetric.

Definition 3. The path is called *pointsymmetric* if it is symmetric with respect to the center of the rectangle, in other words, the starting and the ending corner are opposite.

Else, the path is symmetric with respect to the bisector of the side connecting the starting and the ending corner.

We will start to compute how many times the ball bounces on the side b of the rectangle according to the formula $\frac{b}{\gcd(a,b)}$. Then we will compute how many times the ball bounces on the side a of the rectangle according to the formula $\frac{a}{\gcd(a,b)}$.

Now we have 4 different cases:

- $\frac{b}{\gcd(a,b)}$ is even and $\frac{a}{\gcd(a,b)}$ is even
- $\frac{b}{\gcd(a,b)}$ is even and $\frac{a}{\gcd(a,b)}$ is odd
- $\frac{b}{\gcd(a,b)}$ is odd and $\frac{a}{\gcd(a,b)}$ is even
- $\frac{b}{\gcd(a,b)}$ is odd and $\frac{a}{\gcd(a,b)}$ is odd

3.1 If a and b are both even, in which pocket will the ball land on an $a \times b$ table?

If a and b are both even, it's not possible to know in which pocket the ball will land without more information. Here below in figure 8, we see that on a 8×20 rectangle, the ball lands on the top left corner. On a 6×10 , the ball lands in the top right corner. On a 10×12 , the ball lands on the bottom right corner.

Thus, we can't be certain in which pocket the ball will land without further information. The problem is that the quotient of two even numbers can be either even or odd. For example $20 \div 4 = 5$ and $8 \div 4 = 2$.

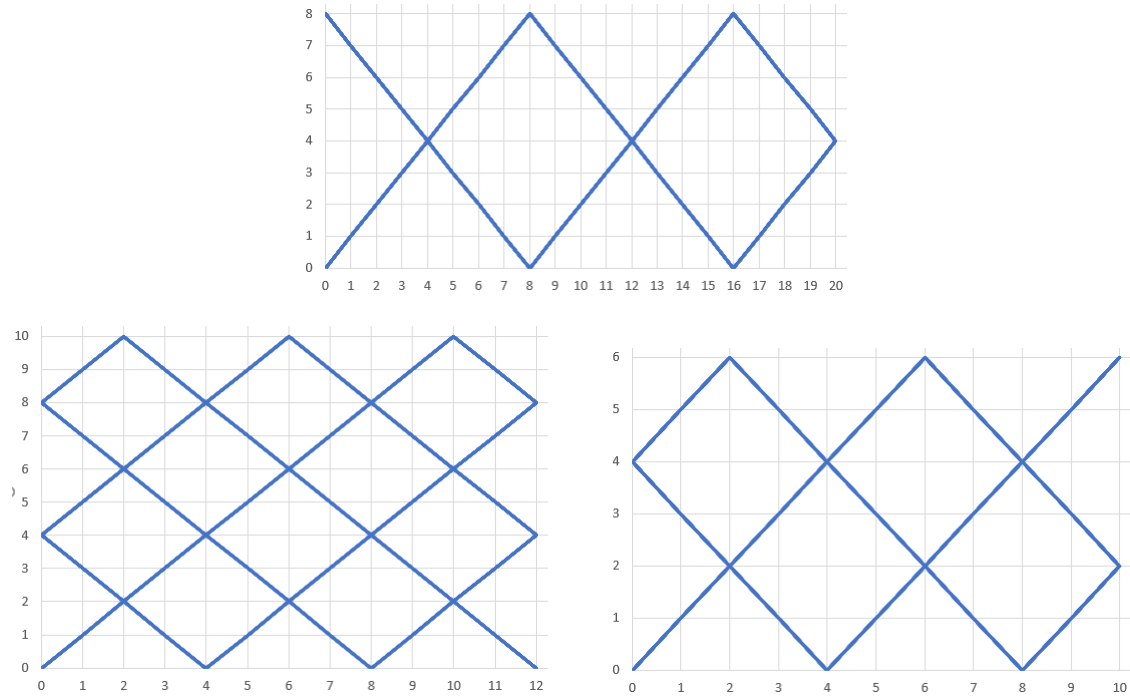


Figure 8: An example of 20×8 , 12×10 and 10×6 rectangles

3.2 If a and b are coprime, in which pocket will the ball land on an $a \times b$ table?

We know that if a (resp. b) is even (resp. odd) then there is an odd (resp. even) number of bouncing points on the two b sides (resp. a sides).

3.2.1 The case where a and b are odd

If a and b are both odd then the starting point and the ending point are on opposite corners.

As we see in Figure 9 for a 7×5 rectangle the starting and ending point are on opposite corners.

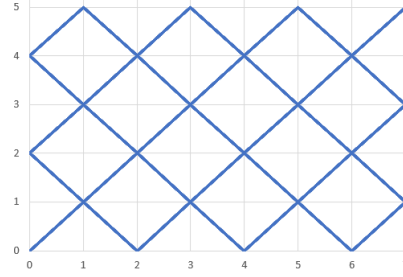


Figure 9: An example of 7×5 rectangle

3.2.2 The case where a is odd and b is even

If a is odd and b is even then the starting point and the ending point are connected by the y -axis.

As we see on Figure 10 for a 7×4 rectangle the starting and ending point are connected by the y -axis (in other words the starting and ending point are connected by the left side of the rectangle).

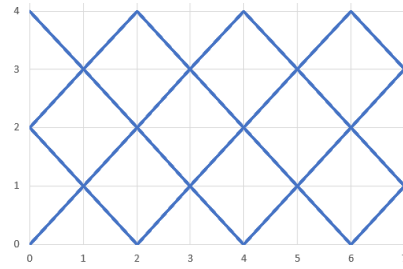


Figure 10: An example of 7×4 rectangle

3.2.3 The case where a is even and b is odd

If a is even and b is odd then the starting point and the ending point are connected by the x-axis.

As we see in Figure 11 for a 8×5 rectangle the starting and ending point are connected by the x-axis (in other words the starting and ending point are connected by the bottom side of the rectangle).

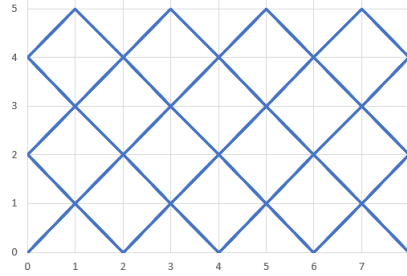


Figure 11: An example of 8×5 rectangle

3.3 If a and b are not necessarily coprime, in which pocket will the ball land on an $a \times b$ table?

We know that if a (resp. b) is even (resp. odd) then there is an odd (resp. even) number of bouncing points on the two b sides (resp. a sides).

If a and b are not necessarily coprime we also have three different cases, as follows.

3.3.1 The case where a and b have the same number of factors 2 in their prime factorisation

If a and b have the same number of factors 2 in their prime factorisation then the starting point and the ending point are on opposite corners.

As we see on figure 12 for a 12×4 rectangle the starting and ending point are on opposite corners.

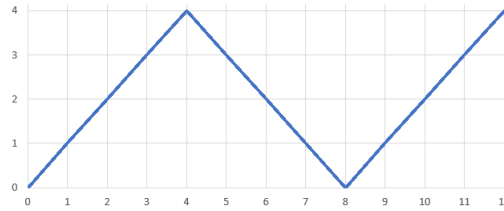


Figure 12: An example of 12×4 rectangle

3.3.2 The case where a has more factors 2 in its prime factorisation than b

If a has more factors 2 in its prime factorisation than b then the starting point and the ending point are connected by the x-axis.

As we see in Figure 13 for a 8×6 rectangle the starting and ending point are connected by the x-axis (in other words the starting and ending point are connected by the bottom side of the rectangle).

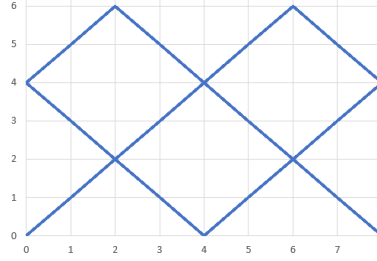


Figure 13: An example of 8×6 rectangle

3.3.3 The case where a has less factors 2 in its prime factorisation than b

If a has less factors 2 in its prime factorisation than b then the starting point and the ending point are connected by the y-axis.

As we see in Figure 14 for a 6×4 rectangle the starting and ending point are connected by the y-axis (in other words the starting and ending point are connected by the left side of the rectangle).

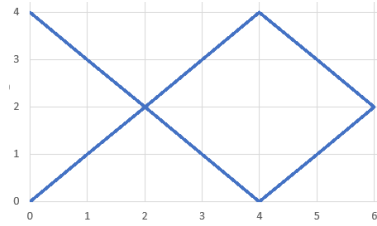


Figure 14: An example of 6×4 rectangle

4 How many intersection points does the path have?

In this section we are interested in finding the exact amount of intersection points on a $a \times b$ rectangle. First we will define what we mean by an intersection point or even better a intersection point.

Definition 4. (Intersection point) An intersection point is a point which the arithmetic billiard path crosses two or more times.

Example 2. In the three pictures below, you can see three different rectangles of different sizes. On the 10×6 rectangle we can see that the path has exactly 3 intersection points, on the 12×10 rectangle we can see that the path has exactly 10 intersection points and on the 20×8 rectangle we can see that the path has exactly 4 intersection points.

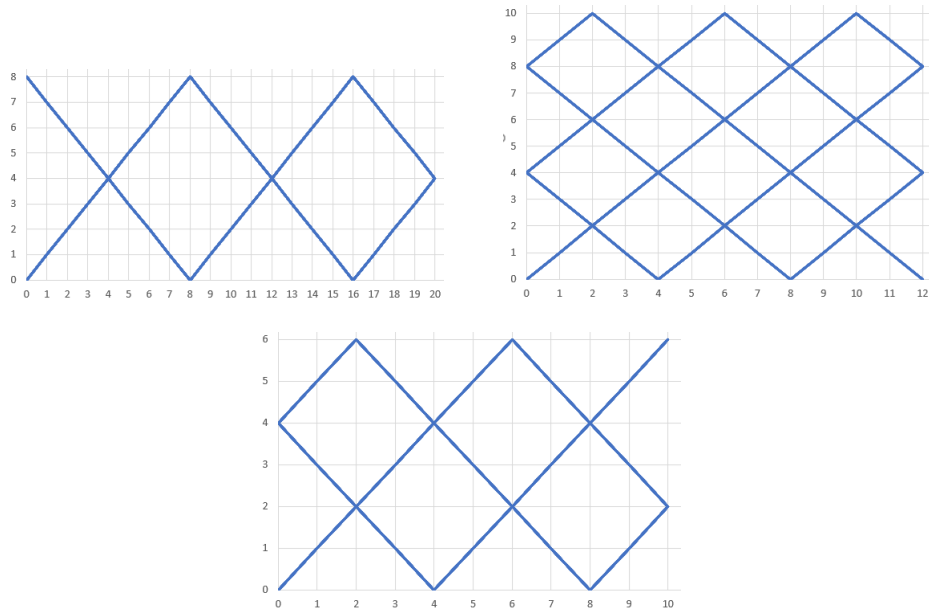


Figure 15: An example of 20×8 , 12×10 and 10×6 rectangles

4.1 How many intersection points does the path have if we start at (0,0)?

We will begin by writing a table with some precise values for a and b and look if we can find something similar for all rectangle sizes. For simplicity, we will suppose that $a > b$ and we will just write the amount of intersection points for this case.

$b \backslash a$	3	4	5	6	7	8	9	10	11
2	1	0	2	0	3	0	4	0	5
3		3	4	0	6	7	0	9	10
4			6	1	9	0	12	2	15
5				10	12	14	16	0	20
6					15	3	1	4	25
7						21	24	27	30
8							28	6	35
9								36	40

Table 5: Number of intersection points at starting coordinates (0,0)

If we look at the Table 5 we immediately see that if a is a multiple of b then there are no intersection points. We also see that the 6×4 rectangle has as many intersection points as the 3×2 rectangle. This last observation is also the case for other $a \times b$ rectangles which have 2 as a factor in their prime factorisation. After some observation we can say that the number of intersection points at starting coordinates (0,0) has something to do with the number of bouncing points on the a and b side of the rectangle.

Theorem 4. *The total amount of intersection points for a path starting at (0,0) is*

$$\#intersection\ points = \frac{(\frac{a}{gcd(a,b)} - 1) \times (\frac{b}{gcd(a,b)} - 1)}{2}.$$

Proof. For simplicity and by rescaling the rectangle, we can prove this theorem for two coprime numbers. So without loss of generality let b be odd, then there are exactly $a + 1$ lines of the unit grid in the rectangle and parallel to the sides of length b and on each of them we have $\frac{b+1}{2}$ points of the path. If we remove now the bouncing points and the two points at the corners from the rectangle sides, we have:

$$\#intersection\ points = \frac{(a+1) \times (b+1)}{2} - (a + b) = \frac{(a-1)(b-1)}{2}.$$

We can now rescale the above formula and we get:

$$\#intersection\ points = \frac{(\frac{a}{gcd(a,b)} - 1) \times (\frac{b}{gcd(a,b)} - 1)}{2}.$$

□

4.2 How many intersection points does the path have if we start at the point (0,1)?

We will begin by writing a table with some precise values for a and b and look if we can find something similar for all rectangle sizes. For simplicity, we will suppose that $a > b$ and we will just write the amount of intersection points for this case.

$b \setminus a$	3	4	5	6	7	8	9	10	11	12	13
2	1	1	2	2	3	3	4	4	5	5	6
3		3	4	1	6	7	2	9	10	3	12
4			6	7	9	1	12	13	15	2	18
5				10	12	14	16	1	20	22	24
6					15	17	7	22	25	1	30
7						21	24	27	30	33	36
8							28	31	35	7	42
9								36	40	17	48

Table 6: Number of intersection points at starting coordinates (0,1)

On this table we immediately see that we have the same amount of intersection points for a and b coprime as for starting point $(0,0)$.

This is because up to reflection we may suppose without loss of generality that the starting point is $(0,0)$. This is because if a and b are coprime the path goes through every point which is not a corner point and starts and ends in a corner

For a a multiple of b we see that the number of intersection points is equal to $\frac{a}{\gcd(a,b)} - 1$.

4.3 How many intersection points does the path have if we start at the point (0,2)?

We will begin by writing a table with some precise values for a and b and look if we can find something similar for all rectangle sizes. For simplicity, we will suppose that $a > b$ and we will just write the amount of intersection points for this case.

$b \backslash a$	4	5	6	7	8	9	10	11	12	13
3	3	4	1	6	7	2	9	10	3	12
4		6	1	9	1	12	2	15	2	18
5			10	12	14	16	1	20	22	24
6				15	3	7	4	25	1	30
7					21	24	27	30	33	36
8						28	6	35	7	42
9							36	40	17	48

Table 7: Number of intersection points at starting coordinates (0,2)

On this table we immediately see that we have the same amount of intersection points for a and b coprime as for starting point (0,0).

This is because up to reflection we may suppose without loss of generality that the starting point is (0,0). This is because if a and b are coprime the path goes through every point which is not a corner point and starts and ends in a corner.

For a a multiple of b we see that the number of intersection points is equal to $\frac{a}{\gcd(a,b)} - 1$, which is the same as for starting point (0,1)

The only number of intersection points that changes, compared to the table of starting point (0,0) is at $a = 9$ and $b = 6$.

4.4 How many intersection points does the path have if we start at the point (0,3)?

We will begin by writing a table with some precise values for a and b and look if we can find something similar for all rectangle sizes. For simplicity, we will suppose that $a > b$ and we will just write the amount of intersection points for this case.

$b \backslash a$	5	6	7	8	9	10	11	12	13
4	6	7	9	1	12	13	15	2	18
5		10	12	14	16	1	20	22	24
6			15	17	1	22	25	1	30
7				21	24	27	30	33	36
8					28	31	35	7	42
9						36	40	3	48

Table 8: Number of intersection points at starting coordinates (0,3)

On this table we immediately see that we have the same amount of intersection points for a and b coprime as for starting point $(0, 0)$.

This is because up to reflection we may suppose without loss of generality that the starting point is $(0, 0)$. This is because if a and b are coprime the path goes through every point which is not a corner point and starts and ends in a corner.

For a a multiple of b we see that the number of intersection points is equal to $\frac{a}{\gcd(a,b)} - 1$, which is the same as for starting point $(0, 1)$

The only number of intersection points that changes, compared to the table of starting point $(0, 1)$ is at $a = 9$ and $b = 6$, $a = 12$ and $b = 9$ and $a = 15$ and $b = 12$.

5 Determine the closest distance to a corner point for a point on a side

Remark 1. Without loss of generality we can start at the point $(0, y_0)$, where y_0 is the parameter that determines the path. Without loss of generality by reversing time or symmetry we can choose if the ball is shot upwards or downwards.

Without loss of generality we can choose $y_0 \leq \frac{b}{2}$ by reversing time and by symmetry.

Without loss of generality we can say that the ball does not land in a corner (that case is already known). So we will exclude $y_0 \in \gcd(a, b) \times \mathbb{Z}$

Without loss of generality y_0 is the closest point to $(0, 0)$ on the left side of the rectangle

Remark 2. (Injectivity of the path) A period or more periods have passed. Along the same segments, the ball goes on the right direction and later back on the left direction. We can call past the time period where the ball goes on the right direction and future where the ball goes back on the left direction. So at some point the past becomes the future. We will call $t - 1$ the past, t the present and $t + 1$ the future. The variation of t (Δt) equals 1, we can decrease Δt by 2 and it can not become 0, so it has to get to 1.

We will call l_1 the closest point to the lower left corner, l_2 the closest point to the lower right corner, l_3 the closest point to the upper right corner and l_4 the closest point to the upper left corner.

The first guess is that

$$\{l_1, l_2, l_3, l_4\} = \{L_1, L_2\}.$$

So the $\#set \leq 2$ and $L_1 + L_2 = \gcd(a, b)$. This also gives a way of computing the l_i , for $i \in \{1, 2, 3, 4\}$, because $l_1 = y_0$

The second guess is that

$$l_i = y_0,$$

for $i \in \{2, 3, 4\}$ the one value beyond $i = 1$ because $l_1 = y_0$ is true by definition.

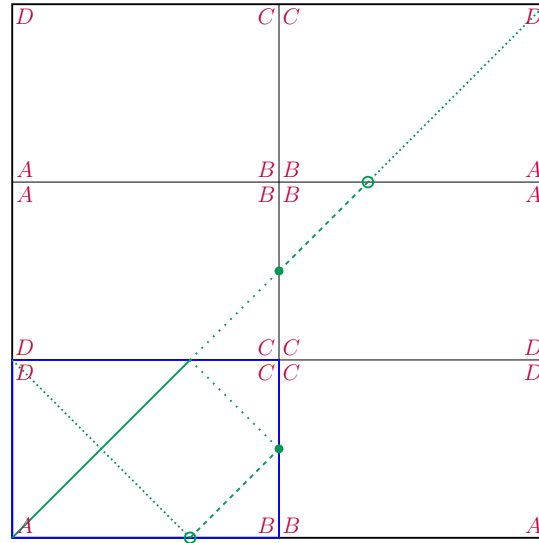


Figure 16: Reflecting the billiard

In Figure 16, we are now interested in the period where the ball touches again the left side and goes up. We can see that this point is in the period of $[0, 2a]$ and it repeats itself again on the periods $[2a, 4a], \dots, [na, (n+2)a]$.

$$\left\lfloor \frac{b}{2} \right\rfloor \times 2a = \begin{cases} b \times a, & \text{for } b \text{ even} \\ (b-1) \times a, & \text{for } b \text{ odd.} \end{cases}$$

Example 3. We will look how many times the ball goes on the upper right direction on a interval for $a = 3$ and $b = 8$. In the interval $(0, 6)$ it goes 1 time on the upper right direction, in $(1, 7)$ it goes 0 times, in $(2, 8)$ it goes 1 times, in $(2, 9)$ it goes 1 times, in $(3, 10)$ it goes 1 times, in $(9, 15)$ it goes 0 times. For the intervals $(9, 15)$ and $(1, 7)$ we get the same value, this is because $9 \equiv 1 \pmod{b}$

At $2ka$ we have:

$$\begin{cases} y_0 + 2ka \pmod{b}, & \text{if } \# \text{ Flip is even} \\ b - (y_0 + 2ka) \pmod{b}, & \text{if } \# \text{ Flip is odd} \end{cases}$$

So $\#Flip = (y_0, y_0 + 2ka) \cap (b \times \mathbb{Z})$

We will call q the quotient of $\frac{2ka}{b}$ and r the remainder of $\frac{2ka}{b}$.

$y_0 + r \geq b$, for $y_0 \in \{0, \dots, b\}$ and $r \in 0, \dots, b-1$. So, $0 \leq y_0 + r < 2b$ and if we have that $y_0 + r \geq b$ we get then $+1$.

$$\#Flip = \left\lfloor \frac{2ka}{b} \right\rfloor + \left\lfloor \frac{y_0 + r}{b} \right\rfloor$$

, with $\left\lfloor \frac{2ka}{b} \right\rfloor = q$ and $\left\lfloor \frac{y_0 + r}{b} \right\rfloor$ is fixed so it doesn't depend on k and equals to 0 or 1.

Let $P_k = (0, y_k)$, where $y_k = \begin{cases} y_0 + 2ka \\ b - (y_0 + 2ka) \end{cases}$

It doesn't need to alternate because $\left\lfloor \frac{2a}{b} \right\rfloor$ could be even.

Maybe this method allows us now to count the P_k distinct because of the period.

The length of the periodic path until it hits the right side of the rectangle is a . We also have that $b \leq b - y_0 + y_1 < 2b$ for a periodic path and to be more precise it is at most $b - 2$ because it is not a corner path.

$$z = b - y_0 + y_1 = \begin{cases} r \\ r + b \end{cases}$$

$$a - z \in (b \times \mathbb{Z})$$

$$z \in [0, 2b - z]$$

$$a - r \in (b \times \mathbb{Z})$$

$$r \in [0, b)$$

$$a - (r + b) \in (b \times \mathbb{Z})$$

$$r + b \in [b, 2b)$$

$$y_1 = \begin{cases} r - (b - y_0) \\ r + b - (b - y_0) \end{cases} = \begin{cases} r + y_0 \\ r - b + y_0 \end{cases}$$

So $y_1 \in \{0, \dots, b\}$ and $y_1 = r + y_0 \pmod{b}$

Suppose that $y_1 \leq \frac{b}{2}$, now if we double the path we can say that all $y_i \leq \frac{b}{2}$, for $i \in \mathbb{Z}$.

We need to prove that y_0 is in the set. We can also say that $r^k \equiv 0 \pmod{b}$, for some k . Equivalently, we can prove that $b - y_0$ is in the set.

I think this will lead us nowhere, so I will start looking at other approaches.

6 Different shooting angles

In this section we want to look at different angles and see if we have similar or even the same properties as for a 45° degree angle. In particular we are going to look at the bouncing points and the intersection points and we hope to see similarities. We will look at two different angles, the first one with a 2 to 1 ratio and the second one a 1 to 2 ratio. In the pictures here below we can see which angles we are going to look at.

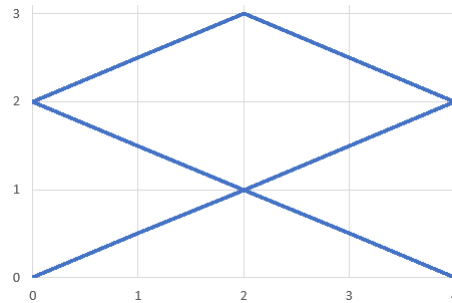


Figure 17: angle of ratio 2 to 1

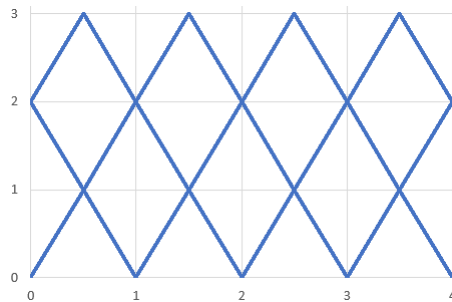


Figure 18: angle of ratio 1 to 2

6.1 Bouncing points

In this section we want to find an easy way to calculate how many bouncing points we have if we start shooting the ball at the bottom left corner with respect to the sides of the rectangle and we hope to find some formula which holds for every point. We will start with an angle of ratio 2 to 1 and then look at the angle of ratio 1 to 2.

6.1.1 Bouncing points starting at the bottom left corner with an angle of ratio 2 to 1

We will start by writing a table where we write at the top the length of one side of the rectangle and on the left the length of the other side of the rectangle. In the middle we will put the number of the bouncing points with respect to the 2 sides of the rectangle.

$b \backslash a$	3	4	5	6	7	8	9	10	11	12	13
2	5	0	7	3	9	1	11	5	13	2	15
3		3	9	0	11	5	3	6	15	1	17
4			11	5	13	0	15	7	17	3	19
5				6	15	7	17	0	19	9	21
6					17	3	5	9	21	0	23
7						9	21	10	23	11	25
8							23	11	25	5	27
9								12	27	3	29

Table 9: Number of bouncing points at starting point (0,0) for a angle of ratio 2 to 1

What we can observe in this table is that if a is a even number and we call a_1 the division of a by 2, then we obtain the same number of bouncing points for b and a_1 as we did for b and a starting at the point (0,0) and shoot at an angle of 45° .

Another thing we can observe is that for $b = 2$ and for a an odd number, we have that the number of bouncing points equals to $a + 2$.

A similar observation we get for $b = 3$ and for a an odd number such that $\gcd(a, b) = 1$, we have that the total number of bouncing points is equal to $a + b + 1$, which is the same as $a + 2b - 2$.

We will also look for $b = 5$ and $b = 7$ with the same hypothesis that a is an odd number such that $\gcd(a, b) = 1$. So for $b = 5$, we have that the total number of bouncing points is equal to $a + b + 3$, which is the same as $a + 2b - 2$.

For $b = 7$, we have that the total number of bouncing points is equal to $a + b + 3$, which is the same as $a + 2b - 2$.

So $\forall b$ and $\forall a$ an odd number such that $\gcd(a, b) = 1$, we have that the total amount of bouncing points is equal to $a + 2b - 2$.

If we have that a is a multiple of b and a is an odd number then the total number of bouncing points is $\frac{a}{b}$, as we can see on the table here above.

If we now have that a is a multiple of b and a is an even number then the total number of bouncing points is $\frac{a}{2b} - 1$, as we can see on the table here above.

Theorem 5. *The total amount of bouncing points in a rectangle with starting coordinates $(0, 0)$ and an angle of ratio 2 to 1 is:*

$$\# \text{bouncing points} = \begin{cases} \frac{\frac{a}{2} + b}{\gcd(\frac{a}{2}, b)} - 2, & \forall b \text{ and } a \text{ even} \\ a + 2b - 2, & \forall b, a \text{ odd and } \gcd(a, b) = 1 \\ \frac{a+b}{\gcd(a, b)} - 1, & \forall b, a \text{ odd and } \gcd(a, b) \neq 1. \end{cases}$$

Proof. The first part of this theorem we can prove in the exact same way as we did for Theorem 1 in section 2.1.

We shot the ball at an angle of ratio 2 to 1, so we have to divide a by 2 to get somehow the angle of 45° . We start by drawing a square whose side is the $\text{lcm}(\frac{a}{2}, b)$, then we decompose the square into rectangles with sides $\frac{a}{2}$ and b . Next, we draw a line at an angle of ratio 2 to 1.

If we count the number of times the line at an angle of ratio 2 to 1 hits one side of the rectangle until it hits the other corner of the square, we have precisely the total amount of bouncing points.

We also know that $\frac{\text{lcm}(\frac{a}{2}, b)}{b} = \frac{\frac{a}{2}}{\gcd(\frac{a}{2}, b)}$, so we have that the number of bouncing points at the side b is equal to $\frac{b}{\gcd(\frac{a}{2}, b)} - 1$, similarly we have $\frac{\frac{a}{2}}{\gcd(\frac{a}{2}, b)} - 1$ bouncing points on the side $\frac{a}{2}$. So if we want the total amount of bouncing points, we set

$$\# \text{bouncing points} = \frac{b}{\gcd(\frac{a}{2}, b)} - 1 + \frac{\frac{a}{2}}{\gcd(\frac{a}{2}, b)} - 1 = \frac{\frac{a}{2} + b}{\gcd(\frac{a}{2}, b)} - 2.$$

For the second part of the theorem we can't divide a by 2 because then we wouldn't get a natural number, so instead we multiply b by 2. Now we can prove this in the exact same way as we did for the first part.

We know that $\frac{\text{lcm}(a, 2b)}{2b} = \frac{a}{\gcd(a, 2b)}$, so we have that the number of bouncing points at the side $2b$ is equal to $\frac{2b}{\gcd(a, 2b)} - 1$, similarly we have $\frac{a}{\gcd(a, 2b)} - 1$ bouncing points on the side a . So if we want the total amount of bouncing points, we set

$$\# \text{bouncing points} = \frac{2b}{\gcd(a, 2b)} - 1 + \frac{a}{\gcd(a, 2b)} - 1 = \frac{a + 2b}{\gcd(a, 2b)} - 2.$$

We also know that $\gcd(a, 2b) = 1$, so we have:

$$\# \text{bouncing points} = a + 2b - 2.$$

For the third part we can do it as we did before.

We start by drawing a square whose side is the $\text{lcm}(a, b)$, then we decompose the square into rectangles with sides a and b . Next, we draw a line at an angle of ratio 2 to 1. If we look at Figure 19 here below, we can see that the path of the billiard ball corresponds exactly to the line at an angle of ratio 2 to 1 of the square if we reflect the rectangle when a path segment hits a side.

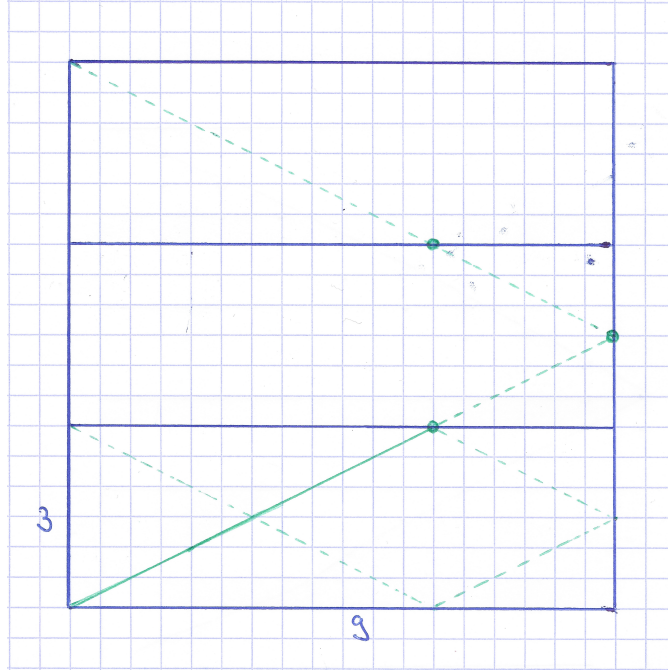


Figure 19: Reflecting the billiard

In Figure 19 we used a rectangle with sides 3 and 9, if we count the number of times the line at an angle of ratio 2 to 1 hits one side of the rectangle until it hits the other corner of the square, we have precisely the total amount of bouncing points.

We also know that $\frac{lcm(a,b)}{b} = \frac{a}{gcd(a,b)}$, so we have that the number of bouncing points at the side b is equal to $\frac{b}{gcd(a,b)}$, similarly we have $\frac{a}{gcd(a,b)} - 1$ bouncing points on the side a . So if we want the total amount of bouncing points, we set

$$\# \text{bouncing points} = \frac{b}{gcd(a,b)} + \frac{a}{gcd(a,b)} - 1 = \frac{a+b}{gcd(a,b)} - 1.$$

□

6.1.2 Bouncing points starting at the bottom left corner with an angle of ratio 1 to 2

We will start by writing a table where we write at the top the length of one side of the rectangle and on the left the length of the other side of the rectangle. In the middle we will put the number of the bouncing points with respect to the 2 sides of the rectangle.

$b \backslash a$	3	4	5	6	7	8	9	10	11	12	13
2	2	3	4	5	6	7	8	9	10	11	12
3		9	11	3	15	17	5	21	23	7	27
4			5	2	7	3	9	4	11	5	13
5				15	17	19	21	3	25	27	29
6					8	9	2	11	12	3	14
7						21	23	25	27	29	31
8							15	5	13	2	15
9								27	29	9	33

Table 10: Number of bouncing points at starting point (0,0) for a angle of ratio 1 to 2

What we can observe in this table is that if b is a even number and we call b_1 the division of b by 2, then we obtain the same number of bouncing points for a and b_1 as we did for a and b starting at the point (0,0) and shoot at an angle of 45° .

Another thing we can observe is that $\forall a$ and $\forall b$ an odd number such that $\gcd(a, b) = 1$, we have that the total amount of bouncing points is equal to $2a + b - 2$.

If we have that a is a multiple of b and b is an odd number then the total number of bouncing points is $\frac{2a}{b} - 1$, as we can see on the table here above.

Theorem 6. *The total amount of bouncing points in a rectangle with starting coordinates (0,0) and an angle of ratio 2 to 1 is:*

$$\#bouncing\ points = \begin{cases} \frac{\frac{b}{2} + a}{\gcd(\frac{b}{2}, a)} - 2, & \forall a \text{ and } b \text{ even} \\ 2a + b - 2, & \forall a, b \text{ odd and } \gcd(a, b) = 1 \\ \frac{2a}{b} - 1, & \forall a, b \text{ odd and } \gcd(a, b) \neq 1. \end{cases}$$

Proof. The first part of this theorem we can prove in the exact same way as we did for Theorem 1 in section 2.1.

We shot the ball at an angle of ratio 1 to 2, so we have to divide b by 2 to get somehow the angle of 45° .

We start by drawing a square whose side is the $lcm(a, \frac{b}{2})$, then we decompose the square into rectangles with sides a and $\frac{b}{2}$. Next, we draw a line at an angle of ratio 1 to 2.

If we count the number of times the line at of angle of ratio 1 to 2 hits one side of the rectangle until it hits the other corner of the square, we have precisely the total amount of bouncing points.

We also know that $\frac{lcm(a, \frac{b}{2})}{\frac{b}{2}} = \frac{a}{gcd(a, \frac{b}{2})}$, so we have that the number of bouncing points at the side $\frac{b}{2}$ is equal to $\frac{\frac{b}{2}}{gcd(a, \frac{b}{2})} - 1$, similarly we have $\frac{a}{gcd(a, \frac{b}{2})} - 1$ bouncing points on the side a . So if we want the total amount of bouncing points, we set

$$\# \text{bouncing points} = \frac{\frac{b}{2}}{gcd(a, \frac{b}{2})} - 1 + \frac{a}{gcd(a, \frac{b}{2})} - 1 = \frac{a + \frac{b}{2}}{gcd(a, \frac{b}{2})} - 2.$$

For the second part of the theorem we can't divide b by 2 because then we wouldn't get a natural number, so instead we multiply a by 2. Now we can prove this in the exact same way as we did for the first part.

We know that $\frac{lcm(2a, b)}{b} = \frac{2a}{gcd(2a, b)}$, so we have that the number of bouncing points at the side b is equal to $\frac{b}{gcd(2a, b)} - 1$, similarly we have $\frac{2a}{gcd(2a, b)} - 1$ bouncing points on the side a . So if we want the total amount of bouncing points, we set

$$\# \text{bouncing points} = \frac{b}{gcd(2a, b)} - 1 + \frac{2a}{gcd(2a, b)} - 1 = \frac{2a + b}{gcd(2a, b)} - 2.$$

We also know that $gcd(2a, b) = 1$, so we have:

$$\# \text{bouncing points} = 2a + b - 2.$$

For the third part we can do it as we did before.

We start by drawing a square whose side is the $lcm(a, b)$, then we decompose the square into rectangles with sides a and b . Next, we draw a line at an angle of ratio 1 to 2. If we look at Figure 20 here below, we can see that the path of the billiard ball corresponds exactly to the line at an angle of ratio 1 to 2 of the square if we reflect the rectangle when a path segment hits a side.

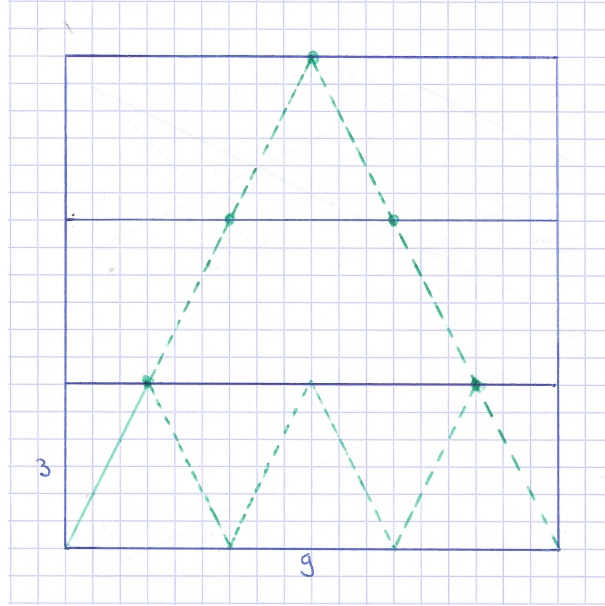


Figure 20: Reflecting the billiard

In Figure 20 we used a rectangle with sides 3 and 9, if we count the number of times the line at an angle of ratio 1 to 2 hits one side of the rectangle until it hits the other corner of the square, we have precisely the total amount of bouncing points.

We also know that $\frac{lcm(a,b)}{b} = \frac{a}{gcd(a,b)}$, so we have that the number of bouncing points at the side b is equal to $\frac{b}{gcd(a,b)} - 1$, similarly we have $\frac{a}{gcd(a,b)} + 2$ bouncing points on the side a . So if we want the total amount of bouncing points, we set

$$\# \text{bouncing points} = \frac{b}{gcd(a,b)} - 1 + \frac{a}{gcd(a,b)} + 2 = \frac{a+b}{gcd(a,b)} + 1.$$

□

6.2 How many intersection points does the path have if we start at the point (0,0)?

We will begin by writing a table with some precise values for a and b and look if we can find something similar for all rectangle sizes. For simplicity, we will suppose that $a > b$ and we will just write the amount of intersection points for this case. We will start with an angle of ratio 2 to 1 and then look at the angle of ratio 1 to 2.

6.2.1 Intersection points starting at the bottom left corner with an angle of ratio 2 to 1

We will start by writing a table where we write at the top the length of one side of the rectangle and on the left the length of the other side of the rectangle. In the middle we will put the number of the bouncing points with respect to the 2 sides of the rectangle.

$b \setminus a$	3	4	5	6	7	8	9	10	11	12	13
2	3	0	6	1	9	0	12	2	15	0	18
3		1	10	0	15	3	1	4	25	0	30
4			14	3	21	0	28	6	35	1	42
5				4	27	6	36	0	45	10	54
6					33	1	3	10	55	0	66
7						9	52	12	65	15	72
8							60	14	75	3	84
9								16	85	1	96

Table 11: Number of intersection points at starting point (0,0) for a angle of ratio 2 to 1

One thing we can observe in this table is that if a is a even number and we call a_1 the division of a by 2, then we obtain the same number of intersection points for b and a_1 as we did for b and a starting at the point (0, 0) and shoot at an angle of 45° .

For example for $b = 2$ and $a = 6$, we have that $a_1 = \frac{a}{2} = 3$. If we put this values in the formula we used for intersection points at starting coordinates (0, 0) at an angle of 45° we get the result we wanted. So, $\frac{(\frac{a_1}{gcd(a_1, b)} - 1) \times (\frac{b}{gcd(a_1, b)} - 1)}{2} = 1$.

Another thing we can observe is that for $b = 2$ and for a an odd number, we have that the number of intersection points equals to $(\frac{a}{gcd(a, b)} - 1) \times (\frac{b}{gcd(a, b)} - 1) + (\frac{\frac{b}{gcd(a, b)} - 1}{2})$.

A similar observation we get for $b = 3$ and for a an odd number such that $gcd(a, b) = 1$, we have that the total number of intersection points is equal to $(\frac{a}{gcd(a, b)} - 1) \times (\frac{b}{gcd(a, b)} - 1) + (\frac{\frac{b}{gcd(a, b)} - 1}{2})$.

We will also look for $b = 5$ and $b = 7$ with the same hypothesis that a is an odd number such that $gcd(a, b) = 1$. So for $b = 5$, we have that the total number of intersection points is again equal to

$$\left(\frac{a}{\gcd(a,b)} - 1\right) \times \left(\frac{b}{\gcd(a,b)} - 1\right) + \left(\frac{\frac{b}{\gcd(a,b)} - 1}{2}\right).$$

For $b = 7$, we have that the total number of intersection points is equal to $\left(\frac{a}{\gcd(a,b)} - 1\right) \times \left(\frac{b}{\gcd(a,b)} - 1\right) + \left(\frac{\frac{b}{\gcd(a,b)} - 1}{2}\right)$.

So $\forall b$ and $\forall a$ an odd number such that $\gcd(a, b) = 1$, we have that the total amount of intersection points is equal to $\left(\frac{a}{\gcd(a,b)} - 1\right) \times \left(\frac{b}{\gcd(a,b)} - 1\right) + \left(\frac{\frac{b}{\gcd(a,b)} - 1}{2}\right)$.

If we have that a is a multiple of b and a is an odd number then the total number of bouncing points is equal to $\left\lfloor \frac{a}{2b} \right\rfloor$.

Theorem 7. *The total amount of intersection points in a rectangle with starting coordinates $(0,0)$ and an angle of ratio 2 to 1 is:*

$$\#intersection\ points = \begin{cases} \frac{\left(\frac{\frac{a}{2}}{\gcd(\frac{a}{2}, b)} - 1\right) \times \left(\frac{b}{\gcd(\frac{a}{2}, b)} - 1\right)}{2}, & \forall b \text{ and } \forall a \text{ even} \\ \left(\frac{a}{\gcd(a,b)} - 1\right) \times \left(\frac{b}{\gcd(a,b)} - 1\right) + \left(\frac{\frac{b}{\gcd(a,b)} - 1}{2}\right), & \forall b, \forall a \text{ odd such that } \gcd(a, b) = 1 \\ \left\lfloor \frac{a}{2b} \right\rfloor, & \forall b, \forall a \text{ odd such that } \gcd(a, b) \neq 1. \end{cases}$$

Proof. For simplicity and by rescaling the rectangle, we can prove this theorem for two coprime numbers. We shot the ball at an angle of ratio 2 to 1, so we have to divide a by 2 to get somehow the angle of 45° .

So without loss of generality let b be odd and a be even, then there are exactly $\frac{a}{2} + 1$ lines of the unit grid in the rectangle and parallel to the sides of length b and on each of them we have $\frac{b+1}{2}$ points of the path. If we remove now the bouncing points and the two points at the corners from the rectangle sides, we have:

$$\#intersection\ points = \frac{\left(\frac{a}{2} + 1\right) \times (b + 1)}{2} - \left(\frac{a}{2} + b\right) = \frac{\left(\frac{a}{2} - 1\right)(b - 1)}{2}.$$

We can now rescale the above formula and we get:

$$\#intersection\ points = \frac{\left(\frac{\frac{a}{2}}{\gcd(\frac{a}{2}, b)} - 1\right) \times \left(\frac{b}{\gcd(\frac{a}{2}, b)} - 1\right)}{2}.$$

□

6.2.2 Intersection points starting at the bottom left corner with an angle of ratio 1 to 2

We will start by writing a table where we write at the top the length of one side of the rectangle and on the left the length of the other side of the rectangle. In the middle we will put the number of the bouncing points with respect to the 2 sides of the rectangle.

$b \backslash a$	3	4	5	6	7	8	9	10	11	12	13
2	0	0	0	0	0	0	0	0	0	0	0
3		7	9	0	13	15	0	19	21	0	25
4			2	0	3	0	4	0	5	0	6
5				22	26	30	34	0	42	46	50
6					6	7	0	9	10	0	12
7						45	51	57	63	69	75
8							12	2	15	0	18
9								76	84	7	100

Table 12: Number of intersection at starting point (0,0) for an angle of ratio 1 to 2

One thing we can observe in this table is that if $b = 2$ we always have 0 intersection points, this seems plausible because we have chosen an angle of ratio 1 to 2.

We will look now all b such that b is even but is not equal to 2. So for b an even number, we call b_1 the division of b by 2, then we obtain the same number of intersection points for a and b_1 as we did for a and b starting at the point $(0, 0)$ and shoot at an angle of 45° .

For example for $b = 6$ and $a = 8$, we have that $b_1 = \frac{b}{2} = 3$. If we put this values in the formula we used for intersection points at starting coordinates $(0, 0)$ at an angle of 45° we get the result we wanted. So,
$$\frac{(\frac{b_1}{\gcd(b_1, a)} - 1) \times (\frac{a}{\gcd(b_1, a)} - 1)}{2} = 7.$$

Another thing we can observe is that for b an odd number and $\forall a$, we have that the number of intersection points equals to $(b - 1) \times (a - \frac{1}{2})$.

For example, if we take $b = 3$ and $a = 7$ we have: $(3 - 1) \times (7 - \frac{1}{2}) = 2 \times (7 - \frac{1}{2}) = 13$.

Let's look at another example, $b = 5$ and $a = 9$, then we have: $(5 - 1) \times (9 - \frac{1}{2}) = 4 \times (9 - \frac{1}{2}) = 34$. So, this formula seems to work.

If we look now at b an odd number and a a multiple of b , in other words $\gcd(a, b) = b$, we have that the number of intersection points is always equal to 0. To explain this, we can use the same argument we did for an angle of 45° .

Theorem 8. *The total amount of intersection points in a rectangle with starting coordinates $(0, 0)$ and an angle of ratio 2 to 1 is:*

$$\# \text{intersection points} = \begin{cases} \frac{(\frac{b}{\gcd(\frac{b}{2}, a)} - 1) \times (\frac{a}{\gcd(\frac{b}{2}, a)} - 1)}{2}, & \forall a \text{ and } \forall b \text{ even } (b \neq 2) \\ (b-1) \times (a - \frac{1}{2}), & \forall a, \forall b \text{ odd such that } \gcd(a, b) = 1 \\ 0, & \forall b \text{ odd and } \forall a \text{ such that } \gcd(a, b) \neq 1. \end{cases}$$

Proof. For simplicity and by rescaling the rectangle, we can prove this theorem for two coprime numbers. We shot the ball at an angle of ratio 1 to 2, so we have to divide b by 2 to get somehow the angle of 45° .

So without loss of generality let b be even, then there are exactly $a + 1$ lines of the unit grid in the rectangle and parallel to the sides of length b and on each of them we have $\frac{b+1}{2}$ points of the path. If we remove now the bouncing points and the two points at the corners from the rectangle sides, we have:

$$\# \text{intersection points} = \frac{(a+1) \times (\frac{b}{2} + 1)}{2} - (a + \frac{b}{2}) = \frac{(a-1)(\frac{b}{2} - 1)}{2}.$$

We can now rescale the above formula and we get:

$$\# \text{intersection points} = \frac{(\frac{a}{\gcd(a, \frac{b}{2})} - 1) \times (\frac{\frac{b}{2}}{\gcd(a, \frac{b}{2})} - 1)}{2}.$$

□

7 Arithmetic billiards in dimension 3

In this section we will do exactly the same we did before but instead of the usual arithmetic billiards we are looking at an $3D$ arithmetic billiards. So we have exactly three coordinates, which we will call a , b and c . For simplicity, we will fix c and just vary a and b . We will try to look at the bouncing points and at the intersection points and we hope to find similar formulas as we did in the sections before. We will just look at the case where we shoot a ball at an 45° angle to all the coordinates. For simplicity we will suppose that $a > b$.

7.1 Bouncing points

In this section we want to find an easy way to calculate how many bouncing points we have if we start shooting the ball at the bottom left corner with respect to the sides of the cuboid and we hope to find some formula which holds for every point.

7.1.1 Bouncing points for $c = 2, 3, 4$

We will start by writing some tables where we write at the top the length of one side of the cuboid and on the left the length of the other side of the cuboid. In the middle we will put the number of the bouncing points with respect to the 3 sides of the cuboid.

$b \setminus a$	3	4	5	6	7	8	9	10	11	12	13
2	3	1	5	2	7	3	9	4	11	5	13
3		7	21	3	29	15	11	19	45	7	53
4			11	5	15	3	19	9	23	5	27
5				17	45	23	57	5	69	35	81
6					23	11	9	14	35	5	41
7						31	77	39	93	47	109
8							39	19	47	11	55
9								49	117	19	137

Table 13: Number of bouncing points for $c = 2$

b \ a	3	4	5	6	7	8	9	10	11	12	13
2	3	7	21	3	29	15	11	19	45	7	53
3		5	6	1	8	9	2	11	12	3	14
4			35	5	47	11	17	31	71	5	83
5				13	56	63	20	13	84	27	98
6					17	9	5	11	25	3	29
7						83	26	101	110	35	128
8							29	55	123	9	143
9								35	38	11	44

Table 14: Number of bouncing points for $c = 3$

b \ a	3	4	5	6	7	8	9	10	11	12	13
2	7	1	11	5	15	3	19	9	23	5	27
3		5	35	5	47	11	17	31	71	5	83
4			7	3	9	1	11	5	13	2	15
5				27	67	15	83	7	99	23	115
6					35	7	13	21	51	3	59
7						19	107	55	127	29	147
8							23	11	27	5	31
9								67	155	11	179

Table 15: Number of bouncing points for $c = 4$

In this three tables we immediately see that the formula has to be somehow $\frac{lcm(a,b,c)}{a} + \frac{lcm(a,b,c)}{b} + \frac{lcm(a,b,c)}{c} - \frac{lcm(a,b,c)}{lcm(a,b)} - \frac{lcm(a,b,c)}{lcm(a,c)} - \frac{lcm(a,b,c)}{lcm(b,c)}$. If we try this formula and compare it to the values we got, it seems to work.

7.1.2 Bouncing points for any a, b and c

Theorem 9. *The total amount of bouncing points in a cuboid starting at $(0, 0, 0)$ is:*

$$\#bouncing\ points = \frac{lcm(a, b, c)}{a} + \frac{lcm(a, b, c)}{b} + \frac{lcm(a, b, c)}{c} - \frac{lcm(a, b, c)}{lcm(a, b)} - \frac{lcm(a, b, c)}{lcm(a, c)} - \frac{lcm(a, b, c)}{lcm(b, c)}$$

Proof. We know that the path has a length of $lcm(a, b, c)$ for a cuboid. So it hits a corner at the earliest at the step $lcm(a, b, c)$. We know that the number of bouncing points at the side a is equal to $\frac{lcm(a, b, c)}{a}$, similarly we have $\frac{lcm(a, b, c)}{b}$ bouncing points on the b side of the cuboid and we also have that the number of bouncing points at the side c is equal to $\frac{lcm(a, b, c)}{c}$.

Now we know how many times the ball bounces on each side, but we have to subtract the number of times the ball bounces on two sides at the same time. So, the ball bounces $\frac{lcm(a, b, c)}{lcm(a, b)}$ times on the a and b side at the same time, $\frac{lcm(a, b, c)}{lcm(a, c)}$ times on the a and c side at the same time and $\frac{lcm(a, b, c)}{lcm(b, c)}$ times on the b and c side at the same time. (We don't add the point where the ball touches all the three sides because this is a corner.) If we take now everything together we get:

$$\#bouncing\ points = \frac{lcm(a, b, c)}{a} + \frac{lcm(a, b, c)}{b} + \frac{lcm(a, b, c)}{c} - \frac{lcm(a, b, c)}{lcm(a, b)} - \frac{lcm(a, b, c)}{lcm(a, c)} - \frac{lcm(a, b, c)}{lcm(b, c)}$$

□

7.2 How many intersection points does the path have on a cuboid?

In this section we will look at the intersection points for a fixed point c and we will variate the points a and b . We want to find an easy way to calculate how many intersection points we have if we start shooting the ball at the bottom left corner with respect to the sides of the cuboid and we hope to find some formula which holds for every point.

We will start by writing three tables where we write at the top the length of one side of the cuboid and on the left the length of the other side of the cuboid. In the middle we will put the number of the bouncing

points with respect to the 3 sides of the cuboid. We will look at the cases where $c = \begin{cases} 2 \\ 3 \\ 4 \end{cases}$.

$b \setminus a$	3	4	5	6	7	8	9	10	11	12
2	1	0	2	0	3	0	4	0	5	0
3		3	9	0	13	7	4	9	21	0
4			6	1	9	0	12	2	15	0
5				10	23	14	30	0	37	22
6					15	3	1	4	25	0

Table 16: Number of intersection points for $c = 2$

$b \setminus a$	3	4	5	6	7	8	9	10	11	12
2	1	3	9	0	12	7	4	9	21	0
3		3	4	0	6	7	0	9	10	0
4			19	1	27	7	12	19	43	0
5				10	34	39	16	9	54	22
6					15	3	1	4	25	0

Table 17: Number of intersection points for $c = 3$

$b \setminus a$	3	4	5	6	7	8	9	10	11	12
2	3	0	6	1	9	0	12	2	15	0
3		3	19	1	27	7	12	19	43	0
4			6	1	9	0	12	2	15	0
5				21	45	14	58	2	71	22
6					31	3	6	9	50	0

Table 18: Number of intersection points for $c = 4$

If we look at the table $c = 2$ we immediately see that if a is a multiple of b and c , then we have no intersection points. This observation we also have for the other cases.

If we have that a , b and c are prime numbers such that $a \neq b \neq c$, or even better when $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$, then we have that the number of intersection points is equal to $\text{lcm}(a, b, c) - \#\text{bouncing points}$. This observation only worked for the case where a , b or c is equal to 2. If we take $a = 7$, $b = 5$ and $c = 3$, $\text{lcm}(3, 5, 7) = 105$, bouncing points for $a = 7$, $b = 5$ and $c = 3$ is 56 and $105 - 56 = 49 \neq 34$.

If a is an odd number, a is a multiple of b and $\gcd(a, c) = \gcd(b, c) = 1$, then the number of intersection points is equal to $(a - 1)\left(\frac{c-1}{2}\right)$. This seems to work for all points a , b and c .

We think it must have something to do with the $\gcd(a, b, c)$ and the $\text{lcm}(a, b, c)$ but we couldn't find a formula which works.

8 Arithmetic billiards in dimension n

We assume that the reader is familiar with the setting of the two-dimensional arithmetic billiards. First of all, we know that a square has 4 corners, 4 edges and 1 square face. It is also easy to see that a cube has 8 corners, 12 edges and 6 squares.

Every time we move a cube to generate a cube in the next higher dimension, the number of corners doubles. That is easy to see since we have an initial position and a final position, each with the same number of corners. Using this we can find a formula for the number of corners of a cube in any dimension, namely 2 raised to the power of n for the n^{th} dimension.

Theorem 10. *Let us call $x_1, x_2, \dots, x_n$, the (positive and integer) variables, and let $a_1, a_2, \dots, a_n$, their maximal value. Then the total length of the path in a n -dimensional cuboid is equal to $\text{lcm}(a_1, a_2, \dots, a_n)$.*

Proof. First of all, when the path hits one corner, the coordinates of that corner must be:

$$\begin{cases} x_1 = 0 \text{ or } x_1 = a_1 \\ x_2 = 0 \text{ or } x_2 = a_2 \\ \cdot \\ \cdot \\ x_n = 0 \text{ or } x_n = a_n \end{cases}$$

Let us now call $c_1, c_2, \dots, c_n$, the coordinates of the hit corner. Then we know that $c_1, c_2, \dots, c_n$ have to be a positive common multiple of $a_1, a_2, \dots, a_n$, so at least $\text{lcm}(a_1, a_2, \dots, a_n)$. On the other hand, after that amount of unitary steps in the path (independently of the reflections) each coordinate x_i is a multiple of a_i and hence we are in a corner. $\square$

Let $I = \{1, \dots, n\}$ and for every non-empty subset $J \subseteq I$ write $\text{lcm}(a_J)$ for the least common multiple of the numbers a_i with $i \in J$.

Theorem 11. *Let us call $x_1, x_2, \dots, x_n$, the (positive and integer) variables, and let $a_1, a_2, \dots, a_n$, their maximal value. The total amount of bouncing points in the n -dimensional arithmetic billiard is:*

$$\# \text{bouncing points} = \text{lcm}(a_I) \cdot \sum_{J \subseteq I, J \neq \emptyset} (-1)^{\#J+1} \frac{1}{\text{lcm}(a_J)}$$

Proof. The number of bouncing points on the faces $x_i = 0$ or $x_i = a_i$ equals $\frac{\text{lcm}(a_1, a_2, \dots, a_n)}{a_i}$. Similarly, on an edge $x_i = 0$ or $x_i = a_i$ for all $i \in J$, where J is a non-empty subset of $\{1, \dots, n\}$, equals $\frac{\text{lcm}(a_I)}{\text{lcm}(a_J)}$. This formula is then obtained by the inclusion-exclusion principle while counting the bouncing points on the faces (we have to keep track of points which are bouncing points for several faces simultaneously). $\square$

9 Conclusion

As a conclusion, we can say that if we start at a corner and we go on a 45° angle we find an easy way to calculate the bouncing points and the intersection points. We can even predict on which corner we will land at the end of the path. We also found a way to calculate the bouncing points and the intersection points whenever we change the starting coordinates.

If we change the angle, it is difficult to calculate the bouncing points and the intersection points because the formula we found, just holds for this angle and not for every angle.

Looking at Arithmetic billiards on a cuboid (3^{rd} dimension) is difficult to represent but we manage to discover a formula to calculate the bouncing points. Unfortunately, we couldn't find a formula to compute the intersection points.

We even managed to calculate the length of the path in the n^{th} dimension and we found a formula to count the bouncing points in the n^{th} dimension.

Arithmetic Billiards is an interesting problem, we could even use at school to introduce the greatest common divisor and the lowest common multiple. It catches the attention of the students because most students have played billiard once in their lives.

Overall, I appreciated the project, as it is oftentimes straightforward, while still being challenging.

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Appendices

A Arithmetic billiards in dimension n

We assume that the reader is familiar with the setting of the two-dimensional arithmetic billiards.

Theorem 12. *Let us call $x_1, x_2, \dots, x_n$, the (positive and integer) variables, and let $a_1, a_2, \dots, a_n$, their maximal value. Then the total length of the path in a n -dimensional cuboid is equal to $\text{lcm}(a_1, a_2, \dots, a_n)$.*

Proof. First of all, when the path hits one corner, the coordinates of that corner must be:

$$\begin{cases} x_1 = 0 \text{ or } x_1 = a_1 \\ x_2 = 0 \text{ or } x_2 = a_2 \\ \vdots \\ x_n = 0 \text{ or } x_n = a_n \end{cases}$$

Let us now call $c_1, c_2, \dots, c_n$, the coordinates of the hit corner. Then we know that $c_1, c_2, \dots, c_n$ have to be a positive common multiple of $a_1, a_2, \dots, a_n$, so at least $\text{lcm}(a_1, a_2, \dots, a_n)$. On the other hand, after that amount of unitary steps in the path (independently of the reflections) each coordinate x_i is a multiple of a_i and hence we are in a corner. $\square$

Let $I = \{1, \dots, n\}$ and for every non-empty subset $J \subseteq I$ write $\text{lcm}(a_J)$ for the least common multiple of the numbers a_i with $i \in J$.

Theorem 13. *Let us call $x_1, x_2, \dots, x_n$, the (positive and integer) variables, and let $a_1, a_2, \dots, a_n$, their maximal value. The total amount of bouncing points in the n -dimensional arithmetic billiard is:*

$$\text{lcm}(a_I) \cdot \sum_{J \subseteq I, J \neq \emptyset} (-1)^{\#J+1} \frac{1}{\text{lcm}(a_J)}$$

Proof. The number of bouncing points on the faces $x_i = 0$ or $x_i = a_i$ equals $\frac{\text{lcm}(a_1, a_2, \dots, a_n)}{a_i}$. Similarly, on an edge $x_i = 0$ or $x_i = a_i$ for all $i \in J$, where J is a non-empty subset of $\{1, \dots, n\}$, equals $\frac{\text{lcm}(a_I)}{\text{lcm}(a_J)}$. This formula is then obtained by the inclusion-exclusion principle while counting the bouncing points on the faces (we have to keep track of points which are bouncing points for several faces simultaneously). $\square$

B Open questions

For an n -th dimensional billiard with the path starting at the point $(0, \dots, 0)$ and each coordinate increasing or decreasing by one at each step:

- In which corner does the ball land?
This question is probably clarified by considering the highest power of 2 dividing the numbers a_i .
- How many intersection points does the path have?
Many examples need to be investigated to formulate a conjecture.

In general, what happens if the starting point is not $(0, \dots, 0)$ but another point in the arithmetic billiard?

C Codes

At the beginning of this year, I didn't know how to write a code, so I tried to use **Excel** to do it. I will put here what I did with some explanations.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	x	y	z	horizontal	vertical	diagonal					max of x	max of y	max of z
2	0	0	0	1	1	1					7	3	2
3	1	1	1	1	1	1							
4	2	2	2	1	1	-1							
5	3	3	1	1	-1	-1							
6	4	2	0	1	-1	1							
7	5	1	1	1	-1	1							
8	6	0	2	1	1	-1							
9	7	1	1	-1	1	-1							
10	6	2	0	-1	1	1							
11	5	3	1	-1	-1	1							
12	4	2	2	-1	-1	-1							
13	3	1	1	-1	-1	-1							
14	2	0	0	-1	1	1							
15	1	1	1	-1	1	1							
16	0	2	2	1	1	-1							

x , y and z are the coordinates of the ball. **horizontal**, **vertical** and **diagonal** are going to give us the next step, so in other words, the next coordinates the ball will reach. **max of x**, **max of y** and **max of z** are the dimensions of the cuboid. If we just want to look at it on a 2 dimensional way, we can erase z , **diagonal** and **max of z**. We can even draw a graph if we just look at it on a 2 dimensional way.

On the second row, we write for x , y and z the starting coordinates. (In the case here above, the starting coordinates are $(0, 0, 0)$).

For **horizontal**, **vertical** and **diagonal** we put 1 because we will continue the path with an angle of 45° (If we want to change the angle we put instead of 1 a number between 0 and 1 at x , y or z).

On the third row, we write for x : $= A2 + D2$, for y : $= B2 + E2$ and for z : $= C2 + E2$. (We do this to go to the next coordinate.)

We write for **horizontal**: $= IF(OR(A3 = 0; A3 = K2); -1D2; D2)$, for **vertical**: $= IF(OR(B3 = 0; B3 = L2); -1E2; E2)$ and for **diagonal**: $= IF(OR(C3 = 0; C3 = M2); -1F2; F2)$. This is to change directions, when we hit **max of x**, **max of y** or **max of z**. In other words when a coordinate is at its maximal value (max of x , max of y , max of z), then we have to subtract $-1 \times \text{horizontal}$ or $-1 \times \text{vertical}$ or $-1 \times \text{diagonal}$, depending which coordinate is at its maximum.

On the fourth row, we write for x : $= A3 + D3$, for y : $= B3 + E3$ and for z : $= C3 + E3$.

We write for **horizontal**: $= IF(OR(A4 = 0; A4 = K2); -1D3; D3)$, for **vertical**: $= IF(OR(B4 = 0; B4 = L2); -1E3; E3)$ and for **diagonal**: $= IF(OR(C4 = 0; C4 = M2); -1F3; F3)$.

For the next rows, we continue doing this.