Music in Fibonacci Series

Mathématiques Expérimentales
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Contents

Abstract

By choosing EXPERIMENTAL MATHEMATICS as an optional course during the 2^{nd} semester at the University of Luxembourg, we were given the opportunity to connect mathematics and music. The objective of this project is to establish a link between musical theory and the Fibonacci series, in order to create own melodies. These will be the final products of our work.

In the first section, we need to clarify some mathematical notions, in order to properly work with them. The next section concerns the theoretical basis of music, which is essential as well. The mathematical and musical theory has to be connected in the ensuing section, which is one of the most important ones, since it represents the main target of this project. To finalize this project, we will then give some musical examples. The presented melodies have been realized with MUSESCORE, a program that allowed us to retain the mathematically obtained melodies.

1 Mathematical Part

In this first section, we elucidate some important mathematical notions and theorems. Each explained term will be illustrated by an example. This section is purely mathematical, so no musical notions will be covered.

1.1 General Definitions and Propositions

Groups

Definition 1.1.1. Let G be a set, $e \in G$ an element and

$$\star : G \times G \to G$$

a map. The triple $(G, \star, e)$ is called a **group** if

- $\forall g_1, g_2, g_3 \in G : (g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3)$ (Associativity)
- $\forall g \in G : e \star g = g \star e = g$ (Neutral Element)
- $\forall g \in G \; \exists h \in G : h \star g = g \star h = e$ (Existence of the inverse)

Example 1.1.1. $(\mathbb{Z}, +, 0)$ is a group and the inverse element of $m \in \mathbb{Z}$ is $-m$ because $0 = (-m) + m = m + (-m)$.

$(\mathbb{N}, +, 0)$ is not a group as inverse elements do not exist.

Definition 1.1.2. Let $(G, \star, e)$ be a group. G is called **cyclic** if there exists an element $g \in G$ called a generator such that every element of G is of the form g^n for some $n \in \mathbb{Z}$ where

$$g^n = \begin{cases} e \quad \text{if } n = 0 \\ \underbrace{g \star g \star ... \star g}_{\text{n - times}} \quad \text{if } n > 0 \\ \underbrace{g^{-1} \star g^{-1} \star ... \star g^{-1}}_{|\text{n}| \text{ - times}} \quad \text{if } n < 0 \end{cases}$$

Example 1.1.2. The group $(\mathbb{Z}, +, 0)$ is cyclic. For example we have that $3 = 1 + 1 + 1$. Any element can be expressed in this way and therefore the generator of the group is 1.

For any $n \in \mathbb{N}$ the group $(\mathbb{Z}/n\mathbb{Z}, +, \overline{0})$ is cyclic.

Sequences

Definition 1.1.3. A **number sequence** is an ordered list of numbers. Therefore, unlike in a set, order matters and a particular term can appear multiple times at different positions in the sequence.

Each number in the sequence is called a term.
The notation $a_1, a_2, a_3, ..., a_{n-1}, a_n$ is used to denote the different terms in a sequence. The number of ordered elements (finite or infinite) is called the length of a sequence.

The terms of number sequences are often linked by a certain rule. This rule allows you to work out the next number of the sequence.

There is another way of defining a number sequence. We can associate it to a function from the natural numbers to the real numbers.

$$a : \mathbb{N} \longrightarrow \mathbb{R}, n \longmapsto a(n)$$

Example 1.1.3. The number sequence $3, 5, 7, 9, 11, ...$ goes on forever, it is thus an infinite sequence. We have a certain pattern: $a_{n+1} = a_n + 2$. If we suppose that $a_1 = 3$, $a_2 = 5$, $a_3 = 7$ and so on, we can even find another rule: $a(n) = n \times 2 + 1$.

Definition 1.1.4. A **periodic sequence** $a_1, a_2, a_3, a_4...$ is a sequence that is caracterized by its repetition of the same terms over and over again. It can therefore also be expressed in the following way :

$$a_1, a_2, ...a_{F-1}, a_F, a_1, a_2, ..., a_{F-1}, a_F, a_1, a_2, ...$$

with F being the number of repeated terms. F is also called a period of the periodic sequence.

A term of the above defined sequence satisfies the following formula:

$$a_n = a_{n+F} \quad \forall n \in \mathbb{N}_{>0}$$

Example 1.1.4. $0, 1, 0, 1, 0, 1, ...$ is a periodic sequence of (least) period 2.
$0, 1, 2, 3, 4, 0, 1, 2, 3, 4, ...$ is a periodic sequence of (least) period 5.

Division Algorithm

Definition 1.1.5. Let $a, b \in \mathbb{Z}$. One says that b **divides** a if there exists $q \in \mathbb{Z}$ such that $a = bq$. (Notation: b|a)

Example 1.1.5. 4 divides 12 because $12 = 4 \times 3$. $(4 \mid 12)$
5 does not divide 12 because $12 = 5 \times 2 + 2$ and $12 = 5 \times 3 - 3$. $(5 \nmid 12)$

Proposition 1.1.6. *Let $x, y \in \mathbb{Z}$ with $y \geq 1$. There exist unique $q, r \in \mathbb{Z}$ such that $x = qy + r$ and $0 \leq r \leq y - 1$*
(The Euclidean Division Algorithm)

Proof.

Existence:

Let $M = \{x - zy | z \in \mathbb{Z}\} \cap \mathbb{N}$. This is a non-empty subset of $\mathbb{N}$. As $\mathbb{N}$ is well-ordered, there exists the least element $r \in M$ which is automatically of the form $r = x - qy$ for $q \in \mathbb{Z}$. If $r \geq y$, then $r - y = x - (q+1)y \in M$ gives an element which is even smaller than the least element. Thus $r < y$.

Uniqueness:

Suppose that $x = qy + r = q'y + r'$.
$\Rightarrow (q - q')y = r' - r$
$\Rightarrow y|(r' - r)$
One also has $-y < r' - r < y$.

0 ist the only multiple of y which is strictly larger than $-y$ and strictly less than y.
$\Rightarrow r' - r = 0$
$\Rightarrow r = r'$ and $y = y'$

□

Example 1.1.6. $12 = 4 \times 3 + 0$ and $12 = 5 \times 2 + 2$

Note that if $r = 0$, we know that $y \mid x$.

Definition 1.1.7. Let $n \in \mathbb{N}_{>0}$. Two integers $x, y \in \mathbb{Z}$ are called **congruent modulo** n if $n|(x - y)$. (Notation: $x \equiv y \pmod{n}$)

Example 1.1.7. $98 \equiv 18 \pmod 5$ because $5 \mid (98 - 18)$.
$98 \not\equiv 16 \pmod 5$ because $5 \nmid (98 - 16)$.
$17 \equiv 1 \pmod 4$ because $4 \mid (17 - 1)$.
$17 \equiv 2 \pmod 5$ because $5 \mid (17 - 2)$.
$17 \equiv 3 \pmod 2$ because $2 \mid (17 - 3)$.

Lemma 1.1.8. *Let $n \in \mathbb{N}$ and $x_1, x_2, y_1, y_2 \in \mathbb{Z}$ such that $x_1 \equiv y_1 \pmod n$ and $x_2 \equiv y_2 \pmod n$. Then*

$$x_1 + x_2 \equiv y_1 + y_2 \pmod n \quad \text{and} \quad x_1 \cdot x_2 \equiv y_1 \cdot y_2 \pmod n$$

Proof.

$n|(x_1 - y_1)$ and $n|(x_2 - y_2)$
$\Rightarrow n|(x_1 - y_1) + (x_2 - y_2)$
$\Rightarrow n|(x_1 + x_1) + (y_1 + y_2)$
$\Rightarrow (x_1 + x_2) \equiv (y_1 + y_2) \pmod n$

$n|(x_1 - y_1)$ and $n|(x_2 - y_2)$
$\Rightarrow n|(x_1 - y_1)x_2$ and $n|(x_2 - y_2)y_1$
$\Rightarrow n|(x_1 - y_1)x_2 + (x_2 - y_2)y_1$

$\Rightarrow n|(x_1x_2 - y_1x_2 + x_2y_1 - y_2y_1)$
$\Rightarrow n|(x_1x_2 - y_2y_1)$
$\Rightarrow (x_1x_2) \equiv (y_1y_2) \pmod n$

$\square$

Example 1.1.8. $19 \equiv 13 \pmod 3$ because $3 \mid (19 - 13)$ and
$29 \equiv 2 \pmod 3$ because $3 \mid (29 - 2)$. Therefore we also have
$(19 + 29) \equiv (13 + 2) \pmod 3 \Rightarrow 48 \equiv 15 \pmod 3$.
We can also conclude that $(19 \times 29) \equiv (13 \times 2) \pmod 3$.

Lemma 1.1.9. *Let $n \in \mathbb{N}_{>0}$ and $x, y \in \mathbb{Z}$. $x \equiv y \pmod n$ if and only if the remainder of the Euclidean division of x by n is the same as the remainder of the division of y by n.*

Proof.

Let $x = q_1n+r_1$ and $y = q_2n+r_2$ with $0 \leq r_1 \leq n-1$ and $0 \leq r_2 \leq n-1$.

Suppose that $x \equiv y \pmod n$. Then $n \mid (x-y)$. As $n \mid (q_1-q_2)n$, it follows that $n \mid (x-y)-(q_1-q_2)n$. We also know that $(x-y)-(q_1-q_2)n = r_1-r_2$. Therefore we have that $n|(r_1-r_2)$ and $-n < r_1-r_2 < n$, which implies that $r_1 = r_2$.

Suppose that $r_1 = r_2$. This implies that $x - y = (q_1 - q_2)n$ so that we also know that $n \mid (x - y)$. Hence $x \equiv y \pmod n$. $\square$

Example 1.1.9. $19 \equiv 7 \pmod 6$ and we have
$19 = 6 \times 3 + 1$ and $7 = 6 \times 1 + 1$

Definition 1.1.10. Let $n \in \mathbb{N}$. The **congruence modulo** n defines an equivalence relation $\mathcal{R}_n$:

$$\forall (x, y) \in \mathbb{Z}^2,\ x\mathcal{R}_ny \Leftrightarrow x \equiv y \pmod n$$

The quotient set $\mathbb{Z}/\mathcal{R}_n$ is denoted by $\mathbb{Z}/n\mathbb{Z}$. One has:

$$\mathbb{Z}/n\mathbb{Z} = \{\overline{0}, \overline{1}, ..., \overline{n-1}\}$$

And:
$\overline{0} = \{..., -2n, -n, 0, n, 2n, ...\}$, $\overline{k} = \{..., -2n+k, -n+k, k, n+k, 2n+k, ...\}$
The class of an integer k in the range between 0 and n-1 is the subset of $\mathbb{Z}$ formed by integers whose remainder in the Euclidean division by n is equal to k.

Example 1.1.10. $\mathbb{Z}/7\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$ with:
$\overline{0} = \{..., -14, -7, 0, 7, 14, ...\}$
$\overline{1} = \{..., -13, -6, 1, 8, 15, ...\}$

$\overline{2} = \{..., -12, -5, 2, 9, 16, ...\}$
$\overline{3} = \{..., -11, -4, 3, 10, 17, ...\}$
$\overline{4} = \{..., -10, -3, 4, 11, 18, ...\}$
$\overline{5} = \{..., -9, -2, 5, 12, 19, ...\}$
$\overline{6} = \{..., -8, -1, 6, 13, 20, ...\}$

1.2 The Fibonacci Sequence

Definition 1.2.1. The **Fibonacci sequence** is a linear recursion defined by

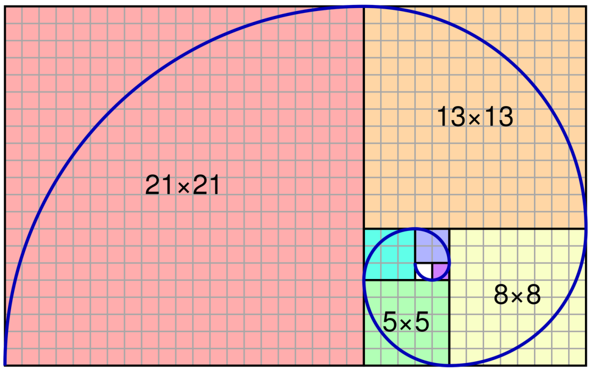
$$f_{n+1} = f_{n-1} + f_n \quad \text{for } n \in \mathbb{N}_{\geq 1}$$

where f_n is the n-th Fibonacci number with $f_0 = 0$ and $f_1 = f_2 = 1$

Remark 1.2.1. This means that each number in the sequence is the sum of the two preceding ones. Starting with 0 and 1 as the first two terms of the sequence, the Fibonacci sequence looks like this for the first few terms :

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

When we make squares with those widths, we get a nice spiral.



Fibonacci Spiral

Definition 1.2.2. Let $n \in \mathbb{N}_{>0}$. By applying the modulo n to an integer $m \in \mathbb{Z}$, we understand that we take the remainder of the Euclidean Division by n. Every result will belong to the set $\{0, 1, ..., n-1\}$.

Remark 1.2.2. If we apply $\pmod 4$ to 9, we obtain 1, because $9 = 2 \times 4 + 1$. If we apply $\pmod 4$ to 30, we obtain 2, because $30 = 7 \times 4 + 2$.

Definition – Proposition 1.2.3. *Let $n \in \mathbb{N}_{>0}$. By applying the modulo n to every element of the Fibonacci Sequence, we obtain a periodical sequence of least period F_n, where $F_n \in \mathbb{N}_{>0}$ is finite. F_n represents the number of elements in such a cycle.*

Proof.

We know that $(\mathbb{Z}, +, 0)$ is a group.
Let m be the function from $\mathbb{Z}$ to $\{0, 1, ..., n-1\}$ which applies to every element of $\mathbb{Z}$ the modulo of n. m is a group homomorphism because $\forall a, b \in \mathbb{Z}$ we can write $a = q_1 \times n + r_1$ and $b = q_2 \times n + r_2$. We then know that $a + b = (q_1 + q_2) \times n + (r_1 + r_2)$. Therefore,

$$m(a) + m(b) = r_1 + r_2 = m(a + b)$$

Thus, the Fibonacci Sequence is not random at all and is defined by the rule

$$f_{n+1} = f_{n-1} + f_n \quad \text{for } n \in \mathbb{N}_{\geq 1}$$

where f_n is the n-th Fibonacci number with $f_0 = 0$ and $f_1 = f_2 = 1$.
The function m being a group homomorphism, we have

$$m(f_{n+1}) = m(f_{n-1} + f_n) = m(f_{n-1}) + m(f_n) \quad \text{for } n \in \mathbb{N}_{\geq 1}$$

The obtained sequence takes values from the set $\{0, 1, ..., n - 1\}$. It is sufficient that two new sequent numbers coincide with two subsequent at an earlier point of the sequence to get a period. This occurrence is a matter of probability and ergodics. We can therefore assume that there exists $p, n \in \mathbb{N}$ such that for f_p, f_n, f_{p+1} and f_{n+1} we have :

$$m(f_p) = m(f_n) \text{ and } m(f_{p+1}) = m(f_{n+1})$$

$$\begin{aligned} \Rightarrow m(f_{p+2}) &= m(f_p + f_{p+1}) \\ &= m(f_p) + m(f_{p+1}) \\ &= m(f_n) + m(f_{n+1}) \\ &= m(f_n + f_{n+1}) \\ &= m(f_{n+2}) \end{aligned}$$

This results in a periodic sequence. □

Example 1.2.3. If we apply the (mod 2) to the Fibonacci Sequence, we obtain the following periodic sequence: $0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, ...$ We can conclude that in this case $F_2 = 3$. There are 3 elements $(0, 1, 1)$ that repeat over and over again in this exact same formation.

Remark 1.2.4. Let $n \in \mathbb{N}_{>0}$. We can observe that if n belongs to one of the elements of the original Fibonacci Sequence, F_n is a smaller number than if n doesn't belong to the sequence.

2 Musical Part

In this part, we specifically define some musical terms. This section should not be neglected, as it concerns some important musical notions that are necessary for the coming steps. It is the last required step to finally be able to combine music with mathematics.

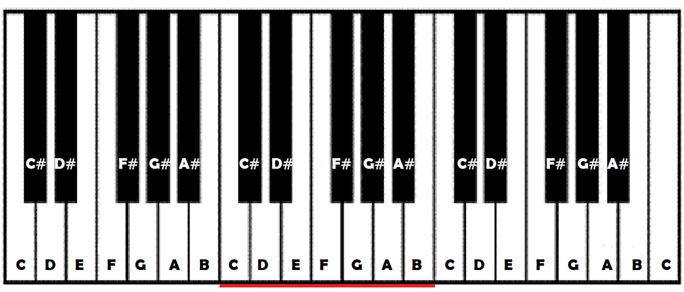
2.1 General Definitions

Definition 2.1.1. Notes are the building blocks from which all chords and melodies are created. It's the raw material of music. In Western music, one can count a total of twelve notes per octave:

C, C#, D, D#, E, F, F#, G, G#, A, A#, B

Notes

Remark 2.1.1. It is often simpler to have in mind the keys of a keyboard. One can observe that the sharp notes, also called accidentals, fall on the black keys and the regular notes fall on the white keys. After an octave, the cycle repeats.



Piano Keys With Corresponding Notes

One Cycle equals One Octave

Definition 2.1.2. An **interval** in music is defined as a distance in pitch between any two notes. The larger the interval between two notes, then the greater the difference in pitch between the notes and vice versa.

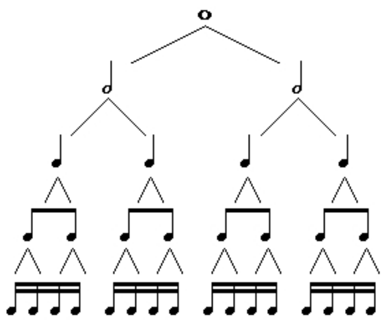
Remark 2.1.2. Semitones and tones (also called half steps and whole steps) are the building blocks of intervals. A semitone is the smallest possible interval, meaning that it is the next higher or lower note. A tone is an interval of two semitones.

Semitone between E and F

Whole Tone between A and B

Definition 2.1.3. In music, the **rhythm** can be defined as the placements of sounds in time and is thus one of the fundamental aspects of music theory. The rhythm relies on the beat, which is a regular pulse. The speed of this pulsation is called the tempo.

Remark 2.1.3. There exists the whole note, the half note, the quarter note and so on. A full note corresponds to 4 beats, a half note to 2 beats and a quarter note to 1 beat. The eighth note corresponds to half a beat and the sixteenth note to a quarter beat.



Rhythm Diagram

Definition 2.1.4. A **melody** is a rhythmic succession of single tones organized as an aesthetic whole.

Melody of Happy Birthday

2.2 Scales

Definition 2.2.1. A musical **scale** consists of a set of notes, which starts with a certain note and goes following a defined sequence of intervals until it returns to the initial note again. The intervals between the notes follow a set pattern that depends on the type of the scale.

Remark 2.2.1. In this section, we chose some scales that later on will play an important role. Note that there are many more scales we could have chosen.

Definition 2.2.2. The **Major Scale** is obtained by giving a certain starting note and following the logic:

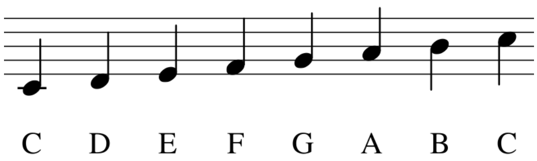
Tone, Tone, Semitone, Tone, Tone, Tone, Semitone.

Example 2.2.2. The C Major Scale is obtained by taking the note C and applying the above sequence for major scales to this starting note.

C + Whole Tone = D
D + Whole Tone = E
E + Semitone = F
F + Whole Tone = G
G + Whole Tone = A
A + Whole Tone = B
B + Semitone = C

We then return to the initial note C. That means that our cycle is complete and that we have obtained the C Major Scale:

C – D – E – F – G – A – B



C Major Scale

Definition 2.2.3. The **Minor Scale** is formed by the following sequence:
Tone, Semitone, Tone, Tone, Semitone, Tone, Tone.

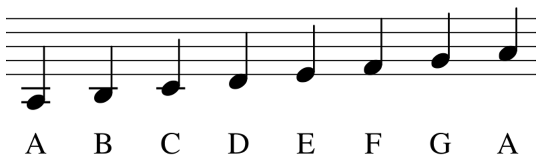
One starts with a certain note and applies this sequence in order to obtain the minor scale.

Example 2.2.3. The A Minor Scale is obtained by taking the note A and applying the above sequence for minor scales to this starting note.

A + Whole Tone = B
B + Semitone = C
C + Whole Tone = D
D + Whole Tone = E
E + Semitone = F
F + Whole Tone = G
G + Whole Tone = A

We then return to the initial note A. That means that our cycle is complete and that we have obtained the A Minor Scale:

A – B – C – D – E – F – G



A Minor Scale

Definition 2.2.4. The **Pentatonic scale**, also called a five-note scale, is a musical scale containing only five different tones. It is appears in many cultures and dates back to the ancient Greek lyre. It's a great way to start writing a song and to improvise. In fact, major pentatonic scales can be expressed by the following series of intervals where the numbers represent the number of half steps between the notes:

$$2 - 2 - 3 - 2 - 3$$

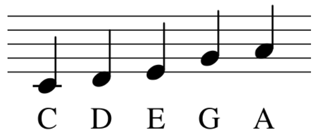
A similar relation exists for minor pentatonic scales.

$$3 - 2 - 2 - 3 - 2$$

is the sequence of intervals for minor pentatonic scales, where the numbers represent the number of half steps between two consecutive notes.

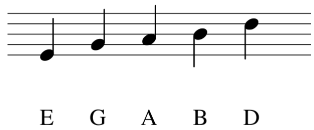
Example 2.2.4. One very simple pentatonic scale is the C Major Pentatonic Scale. It's notes are C – D – E – G – A.

C + 2 Semitones = D
D + 2 Semitones = E
E + 3 Semitones = G
G + 2 Semitones = A
A + 3 Semitones = C



C Major Pentatonic Scale

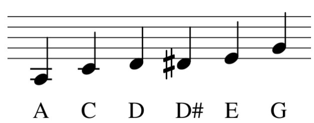
Thus E – G – A – B – D forms the E Minor Pentatonic Scale.
E + 3 Semitones = G
G + 2 Semitones = A
A + 2 Semitones = B
B + 3 Semitones = D
D + 2 Semitones = E



E Minor Pentatonic Scale

Definition 2.2.5. The **Blues Scale** is a common hexatonic scale, meaning it contains six different tunes. The Blues scale is a minor pentatonic scale to which one adds the diminished fifth. In the case of a major pentatonic scale, one adds the flattened third. This additional note is known as the "Blue Note".

Example 2.2.5. For the A Minor Blues Scale, one takes the A Minor Pentatonic Scale (A – C – D – E – G) and adds the flattened fifth (D#).



A Minor Blues Scale

3 Relation between Music and the Fibonacci Sequence

This section is the most important one, as it unites the two preceding parts. The connection between music and mathematics becomes self-evident the moment one clarifies the approach. It thus requires a certain knowledge in both domains. The final section, meaning the section in which will be represented the created melodies, fully relies on this link between musical theory and the Fibonacci sequence.

3.1 Notes - Fibonacci Sequence

The following section explains the procedure of associating musical notes to the Fibonacci Series by an iteration of different choices.

The very first step is the choice of our scale. We first need to choose the set of notes we want to use for this melody. Thus, let n be the number of notes of the chosen scale. Based on this, we know that we have to apply the modulo n to each term of the Fibonacci Sequence.

As seen before in the mathematical section, we then obtain a periodical sequence of least period F_n. The following step consists in associating the numbers from 0 to $n-1$ to the notes. Each note obtains an unique number, starting with the first note of the scale corresponding to 0. The second note then corresponds to 1 and so on.

Once these relations are established, the periodic sequence we obtained in one of the first steps has to be written as a sequence of notes. We therefore replace the number values with the corresponding notes. This finally gives us a sequence of notes, which can be transformed in a complete melody by adding some rhythm.

Since this procedure has to be done for different choices of scales and note sets in order to obtain a variety of melodies, we created a Python program which simplifies the whole process. The program allows us to retrieve the Fibonacci Sequence itself, but also the sequence obtained by applying a certain modulo to it. We then are asked to associate to each value from 0 to n a certain note. The program finally returns us the sequence of notes that is necessary for creating a melody. We could even say that the resulting note sequence is the melody.

```
>>>
= RESTART: C:\Users\beatr\Documents\Semestre 2\Music in Fibonacci Numbers\Fibonacci.py
The Fibonacci Sequence

Enter the number of elements of the Fibonacci Sequence you want to see: 10

[0, 1, 1, 2, 3, 5, 8, 13, 21, 34]

Enter the number of the modulo you want to treat: 4

By applying mod 4  to every element of the above extrait of the Fibonacci Sequence, we obtain:

[0, 1, 1, 2, 3, 1, 0, 1, 1, 2]

This is the periodical sequence we are looking for:

[0, 1, 1, 2, 3, 1]

To convert the numbers into sheet music,you need to associate a note to every value.
To  0  we associate the note: C
To  1  we associate the note: D
To  2  we associate the note: E
To  3  we associate the note: F

['C', 'D', 'D', 'E', 'F', 'D']

>>> |
```

Output of our Python Program

3.2 Rhythm - Fibonacci Sequence

There exist rhythmic patterns that can be derived from the Fibonacci sequence. Of course, one does not have to restrict the rhythm of an entire melody on the following scheme. This section simply underlines the fact, that entire melodies, meaning the notes and the rhythm, can be extracted from the Fibonacci Sequence.

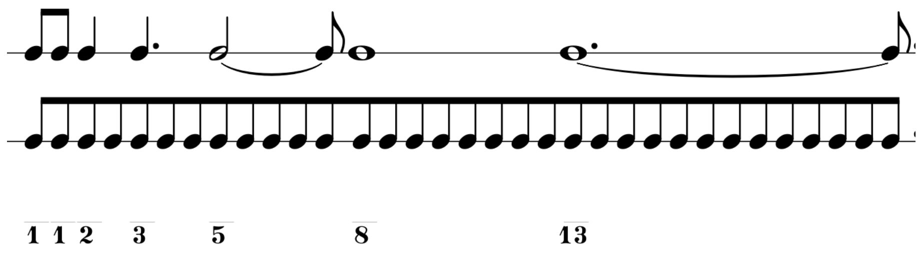
One way of obtaining a rhythmic pattern is by taking the numbers of the Fibonacci sequence and to each number associate the rhythm that corresponds to n times an eighth note.

One Eighth Note

A Quarter Note Corresponding to 2 Eighth Notes

Rhythm Corresponding to 3 Eighth Notes

This leads us to the following rhythmic sequence:



Play

A video that gives further explanation and inspired this section can be found via this link:

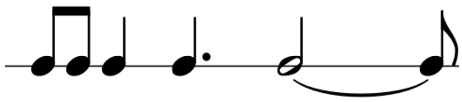
https://www.youtube.com/watch?v=mOMLRMfIYf0

4 Results

This final part concludes all the work that has been done before with some musical examples. The melodies have been created withMuseScore and are based on the relations established in the previous section. The examples differ essentially by the choice of the scale, that is the choice of the modulo.

4.1 Rhythm

For our project, we decided that the rhythm itself plays a secondary role. In the following examples we will therefore at first focus on the harmony of the note sequence and later on add some rhythm. We will work with the rhythms obtained in the last sections and most likely will repeat the following rhythmic pattern:



Play

Note that this pattern is simply a matter of choice, relying on the taste of the authors.

4.2 Modulo 5

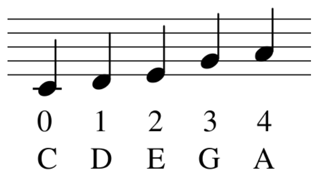
4.2.1 C Major Pentatonic Scale

The C Major Pentatonic Scale consists of 5 notes:
C – D – E – G – A
This means that we have to work with modulo 5.
Now we have to apply the modulo 5 to each element of the Fibonacci Sequence, which leads us to a cycle:

$$(0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1)$$

The least period F_5 equals to 20.

To each note we now associate a number:
C corresponds to 0,
D corresponds to 1,
E corresponds to 2,
G corresponds to 3,
A corresponds to 4.

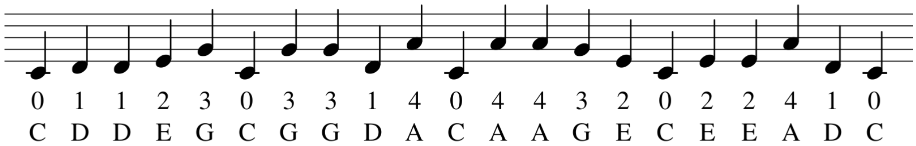


C Major Pentatonic Scale

By associating the corresponding note to every number of our newly obtained periodic sequence, we get that

$$(0,1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1)$$
$$\Leftrightarrow$$
$$(\mathrm{C,D,D,E,G,C,G,G,D,A,C,A,A,G,E,C,E,E,A,D})$$

Note	Fibonacci Sequence	Result
C	0	0
D	1	1
D	1	1
E	2	2
G	3	3
C	5	0
G	8	3
G	13	3
D	21	1
A	34	4
C	55	0
A	89	4
A	144	4
G	233	3
E	377	2
C	610	0
E	987	2
E	1597	2
A	2584	4
D	4181	1



C Major Pentatonic - Modulo 5

Play

By now also applying our rhythmic pattern we get:

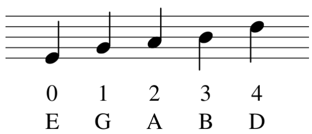
C Major Pentatonic with Rhythm - Modulo 5

Play

4.2.2 E Minor Pentatonic Scale

The E Minor Pentatonic Scale consists of 5 notes:
E – G – A – B – D
This means that we have to work with modulo 5.
Now we have to apply the modulo 5 to each element of the Fibonacci Sequence, which leads us to a periodic sequence of least period 20:

$$(0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1)$$



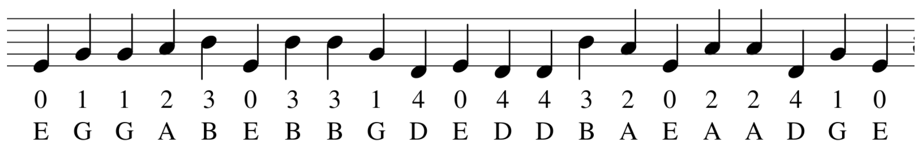
E Minor Pentatonic Scale

To each note we now associate a number:
E corresponds to 0,
G corresponds to 1,
A corresponds to 2,
B corresponds to 3,
D corresponds to 4.

By associating the corresponding note to every number of our newly obtained periodic sequence, we get that

$$(0,1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1)$$
$$\Leftrightarrow$$
$$(\mathrm{E,G,G,A,B,E,B,B,G,D,E,D,D,B,A,E,A,A,D,G})$$

Note	Fibonacci Sequence	Result
E	0	0
G	1	1
G	1	1
A	2	2
B	3	3
E	5	0
B	8	3
B	13	3
G	21	1
D	34	4
E	55	0
D	89	4
D	144	4
B	233	3
A	377	2
E	610	0
A	987	2
A	1597	2
D	2584	4
G	4181	1



E Minor Pentatonic - Modulo 5

Play

By now also applying our rhythmic pattern we get:

E Minor Pentatonic with Rhythm - Modulo 5

Play

4.3 Modulo 6

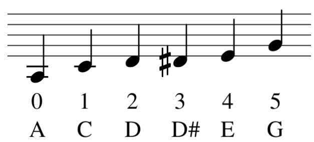
4.3.1 A Minor Blues Scale

The A Minor Blues Scale consists of 6 notes:
A – C – D – D# – E – G
This means that we have to work with modulo 6.
Now we have to apply the modulo 6 to each element of the Fibonacci Sequence, which leads us to a cycle:

$$(0,1,1,2,3,5,2,1,3,4,1,5,0,5,5,4,3,1,4,5,3,2,5,1)$$

The least period F_6 equals to 24.

To each note we now associate a number:
A corresponds to 0,
C corresponds to 1,
D corresponds to 2,
D# corresponds to 3,
E corresponds to 4,
G corresponds to 5.



A Minor Blues Scale

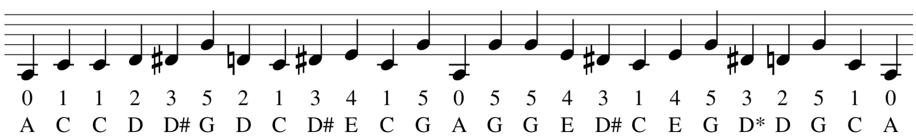
By associating the corresponding note to every number of our newly obtained periodic sequence, we get that

$$(0,1,1,2,3,5,2,1,3,4,1,5,0,5,5,4,3,1,4,5,3,2,5,1)$$

$$\Leftrightarrow$$

(A,C,C,D,D#,G,D,C,D#,E,C,G,A,G,G,E,D#,C,E,G,D#,D,G,C)

Note	Fibonacci Sequence	Result
A	0	0
C	1	1
C	1	1
D	2	2
D#	3	3
G	5	5
D	8	2
C	13	1
D#	21	3
E	34	4
C	55	1
G	89	5
A	144	0
G	233	5
G	377	5
E	610	4
D#	987	3
C	1597	1
E	2584	4
G	4181	5
D#	6765	3
D	10946	2
G	17711	5
C	28657	1



A Minor Blues Scale - Modulo 5

Play

By now also applying our rhythmic pattern we get:

A Minor Blues Scale with Rhythm - Modulo 5

Play

4.4 Modulo 7

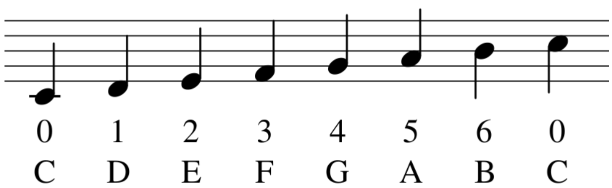
4.4.1 C Major Scale

The C Major Scale consists of 7 notes:
C – D – E – F – G – A – B
This means that we have to work with modulo 7.
Now we have to apply the modulo 7 to each element of the Fibonacci Sequence, which leads us to a cycle:

$$(0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1)$$

The least period F_7 equals to 16.

To each note we now associate a number:
C corresponds to 0,
D corresponds to 1,
E corresponds to 2,
F corresponds to 3,
G corresponds to 4,
A corresponds to 5,
B corresponds to 6.



C Major

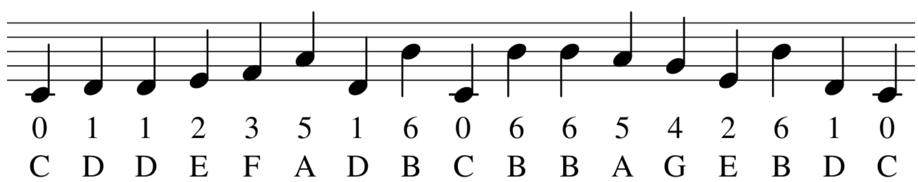
By associating the corresponding note to every number of our newly obtained periodic sequence, we get that

$$(0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1)$$

$$\Leftrightarrow$$

(C,D,D,E,F,A,D,B,C,B,B,A,G,E,B,D)

Note	Fibonacci Sequence	Result
C	0	0
D	1	1
D	1	1
E	2	2
F	3	3
A	5	5
D	8	1
B	13	6
C	21	0
B	34	6
B	55	6
A	89	5
G	144	4
E	233	2
B	377	6
D	610	1



C Major Scale - Modulo 5

Play

By now also applying our rhythmic pattern we get:

C Major Scale with Rhythm - Modulo 5

Play

4.5 Modulo 8

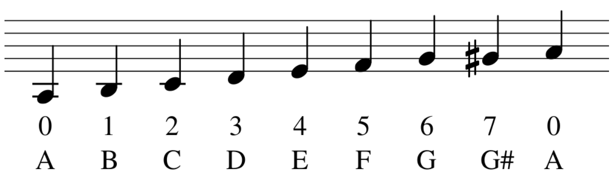
4.5.1 Harmonic A Minor Scale

The A Minor Blues Scale consists of 8 notes:
A – B – C – D – E – F – G – G#
This means that we have to work with modulo 8.
Now we have to apply the modulo 8 to each element of the Fibonacci Sequence, which leads us to a cycle:

$$(0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1)$$

The least period F_8 equals to 12.

To each note we now associate a number:
A corresponds to 0,
B corresponds to 1,
C corresponds to 2,
D corresponds to 3,
E corresponds to 4,
F corresponds to 5,
G corresponds to 6,
G# corresponds to 7.



Harmonic A Minor Scale

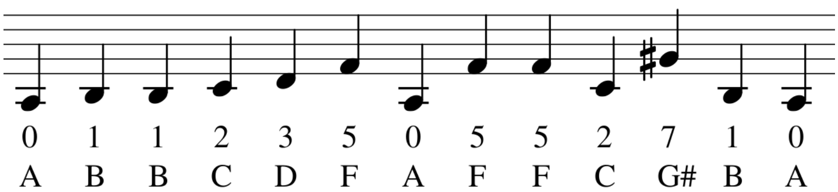
By associating the corresponding note to every number of our newly obtained periodic sequence, we get that

$$(0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1)$$

$$\Leftrightarrow$$

$$(\text{A,B,B,C,D,F,A,F,F,C,G\#,B})$$

Note	Fibonacci Sequence	Result
A	0	0
B	1	1
B	1	1
C	2	2
D	3	3
F	5	5
A	8	0
F	13	5
F	21	5
C	34	2
G#	55	7
B	89	1



Harmonic A Minor Scale - Modulo 5

Play

By now also applying our rhythmic pattern we get:

Harmonic A Minor Scale with Rhythm- Modulo 5

Play

4.6 Modulo 12

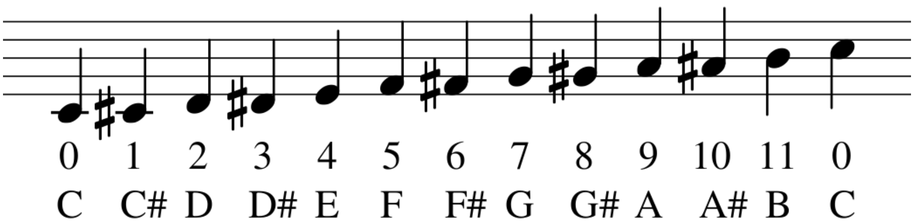
The A Minor Blues Scale consists of 8 notes:
C – C# – D – D# – E – F – F# – G – G# – A – A# – B
This means that we have to work with modulo 12.

Now we have to apply the modulo 12 to each element of the Fibonacci Sequence, which leads us to a cycle:

$$(0,1,1,2,3,5,8,1,9,10,7,5,0,5,5,10,3,1,4,5,9,2,11,1)$$

The least period F_{12} equals to 24.

To each note we now associate a number:
C corresponds to 0,
C# corresponds to 1,
D corresponds to 2,
D# corresponds to 3,
E corresponds to 4,
F corresponds to 5,
F# corresponds to 6,
G corresponds to 7,
G# corresponds to 8,
A corresponds to 9,
A# corresponds to 10,
B corresponds to 11.



Full Scale

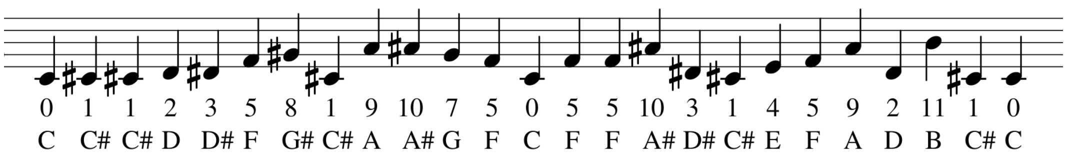
By associating the corresponding note to every number of our newly obtained periodic sequence, we get that

$$(0,1,1,2,3,5,8,1,9,10,7,5,0,5,5,10,3,1,4,5,9,2,11,1)$$

$$\Leftrightarrow$$

(C,C#,C#,D,D#,F,G#,C#,A,A#,G,F,C,F,F,A#,D#,C#,E,F,A,D,B,C#)

Note	Fibonacci Sequence	Result
C	0	0
C#	1	1
C#	1	1
D	2	2
D#	3	3
F	5	5
G#	8	8
C#	13	1
A	21	9
A#	34	10
G	55	7
F	89	5
C	144	0
F	233	5
F	377	5
A#	610	10
D#	987	3
C#	1597	1
E	2584	4
F	4181	5
A	6765	9
D	10946	2
B	17711	11
C#	28657	1



Full Scale - Modulo 12

Play

By now also applying our rhythmic pattern we get:

Full Scale with Rhythm- Modulo 12

Play

4.7 Others

In this section, we want to show the reader some more creative results of this work. The all have the Fibonacci Sequence as their origin, but we added our personal touch.

Modulo 8

Play

Modulo 5

Play

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