Perturbations of the Lorenz system

Experimental Mathematics
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Abstract

This article has been writen by a student at the University of Luxembourg and supervised by Dr. Andrew Bruce. This documents shows some computation of the Lorentz system and introduces the reader to the understanding of the behavior of the Lorentz system and which differences it can show for small changes.

At this point, I would like to thank Dr. Andrew Bruce for the great supervising and supporting, always when I had some problems with the programm. And I am really thankful that this topic has been proposed becauce I could, during this project, develop my interesting for the Lorentz system and the Chaos Theory.

Contents

1 Introduction

The Lorenz system is a system of ordinary equations first studied by Lorenz Edward Norton (1963) .
This system has chaotic solutions for certain parameter values and initial conditions. The set of chaotic solutions is called the Lorenz attractor. The real-world implications of the Lorenz attractor are known as the butterfly effect. The butterfly effect is the sensitive dependence on initial conditions in which a small change can result in large differences, which are well explained by the Lorenz system.
Two applications of the butterfly effect are the standard weather prediction and a number of cases in quantum physics.
For weather prediction, climate scientists explain that chaos is important in the development of weather prediction methods, because the weather prediction models are sensitive to initial conditions.
Actually every behaviour observed in the nature, which shows large differences caused by small changes on the initial conditions, can be visualized and explained by the Lorenz attractor, what made the Lorenz system interesting to study and it could help us for a better understanding of this phenomenon.
Let us give now the Lorentz system:

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x) \\ \frac{dx}{dt} = x(\rho - z) - y \\ \frac{dz}{dt} = xy - \beta z \end{cases}$$

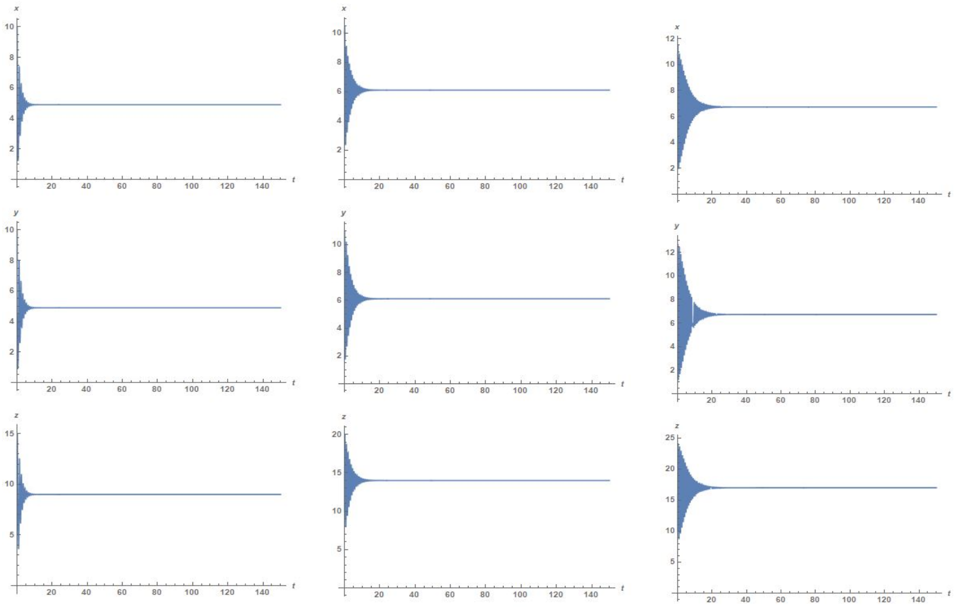
Here x,y and z are function with the variable t, in particular we have:

$$x : \mathbb{R} \to \mathbb{R},$$
$$y : \mathbb{R} \to \mathbb{R},$$
$$z : \mathbb{R} \to \mathbb{R}.$$

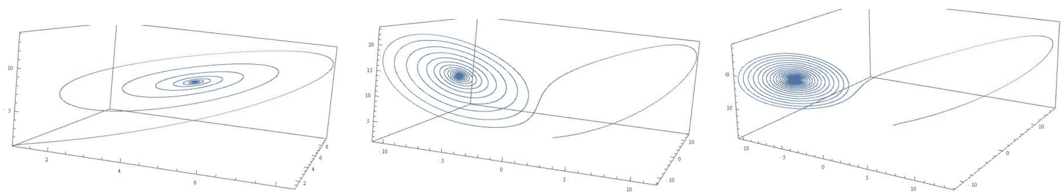
And σ, ρ and β are parameters for the Lorenz system, who are in $\mathbb{R}$.
Actually we will fixe σ and β for a better observation in the case when ρ changes.
In the following, we fix $\sigma = 10$ and $\beta = \frac{8}{3}$.

2 First step

As above mentioned, we fix σ and β. We will now compute the Lorentz system for different values of ρ to see if we can conclude something special in observing the behaviour of the system (the code for the following representation is given in the Appendix by $A1$).

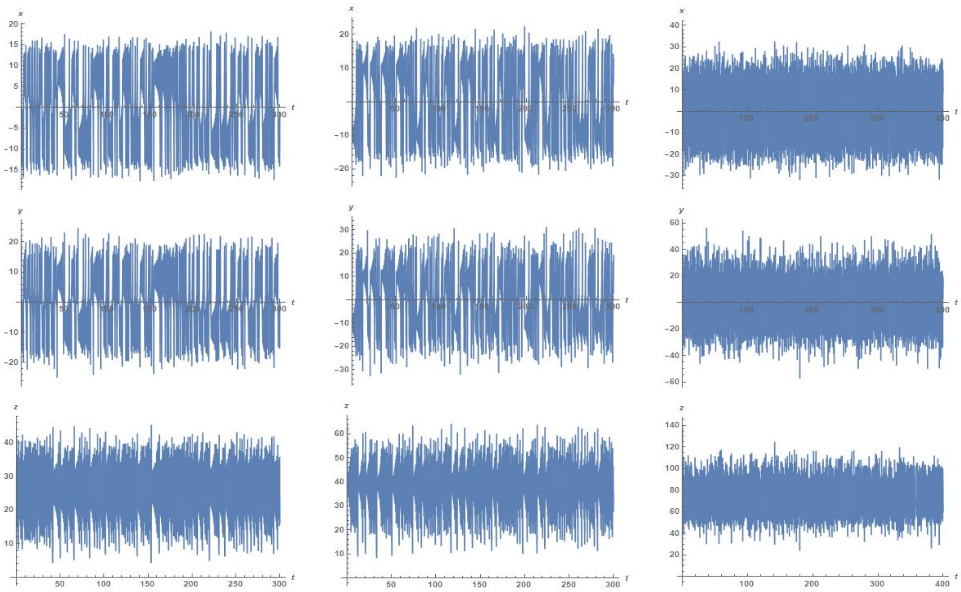


Here we have the computation of ρ for different values, from left to right, 10, 15 and 18. We can observe that for this values of ρ we have that the system converges to a certain value. In this cases we observe that it converges to the same value, but we should not forget that by observing the representation, the system oscillates at the beginning, more important is that for the different values of ρ the system gets stable after a different range. From the representation we have that the first system converges after t_1, the second one after t_2 and the third after t_3, actually we have that $t_1 < t_2 < t_3$, which is well shown by the representations, because for $\rho = 10$ the system oscillates a bit, for $\rho = 15$ the system oscillates a bit longer and for $\rho = 18$ the system oscillates longer before getting stable and starting to converge.
For the beginning, we can conclude that the Lorenz system converges for some small values of ρ, as for $\rho \leq 18$. We will now give the 3D computation for the values of ρ used in the above representation (the code for the following images are given in the Appendix by $A2$).

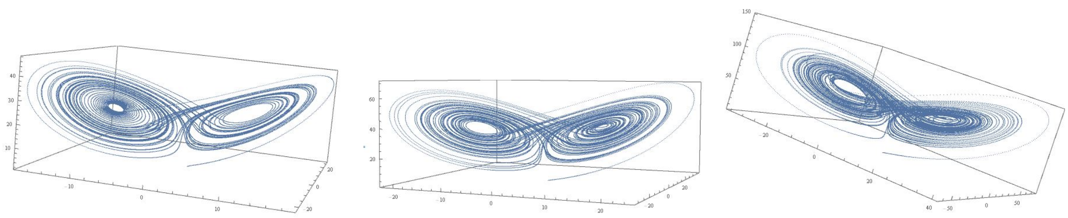


Here one can see how the system oscillates and later converges and get stable.

After observing the Lorenz system for a small ρ, it is intuitive and interesting to ask us what will happen if take a bigger ρ, will it show the same behaviour or not ? Let us try for ρ equal to 28, 40 and 80.



By observing the representation, we can say that we have totally different results as for the values 10, 15 and 18. The x and y axes shows a chaotic oscillation, but the z axes shows although the chaotic behaviour of x and y, that they are bounded in an area of z, which means that x and y do not converge to a value, but oscillate around an area. In particular, they oscillate around a sphere, which will be more clear by computating the Lorenz system in 3D for the same values.

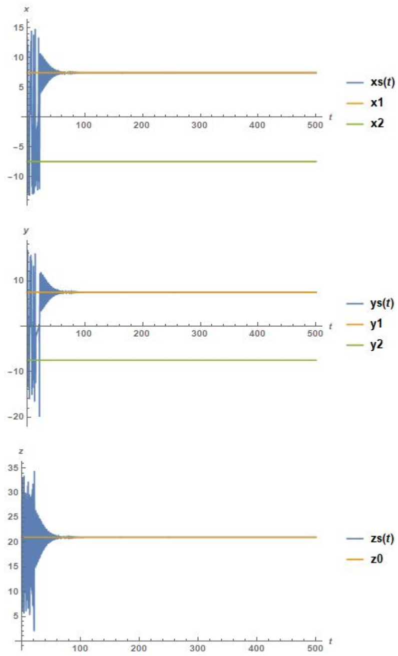


As we can see, it seems that x and y oscillate around two spheres with different centers and radius, for the values 28, 40 and 80 of ρ. This is actually the representation of the chaotic solutions of the Lorenz system, i.e. the Lorenz Attractor.
At this point it would by interesting to know how big ρ can become without that the solutions of the Lorenz systems shows chaotic behaviour, for this the above representation of the Lorenz Attractor helps, because by observing it we find two spheres with two centers. We can conclude that there are maybe two, instead of only one point, to which the solution of the Lorenz system converges. This points we call *equilibrium points*.

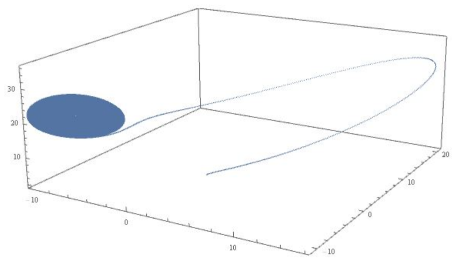
3 Second step

In this section, we will have a look at the solution of the Lorenz system and see if it approaches to one of the following equilibrium points $(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1)$.

Let us observe it for $\rho = 22$. We expect that at the beginning the solution will oscillate and show after a ceratin range $t \in \mathbb{R}$ it will start to converge at one of the two equilibrium points. Let us observe the representation of this Lorenz system. To help us for a better visualization we will add two lines, which represent the equilibrium points (the code for the following representation is given in the Appendix by $A3$).



As expected the representation shows us that at the beginning the solutions oscillate and show chaotic behaviour, but after a certain time they start to get more stable and finally converge to the equilibrium point $(\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1)$.

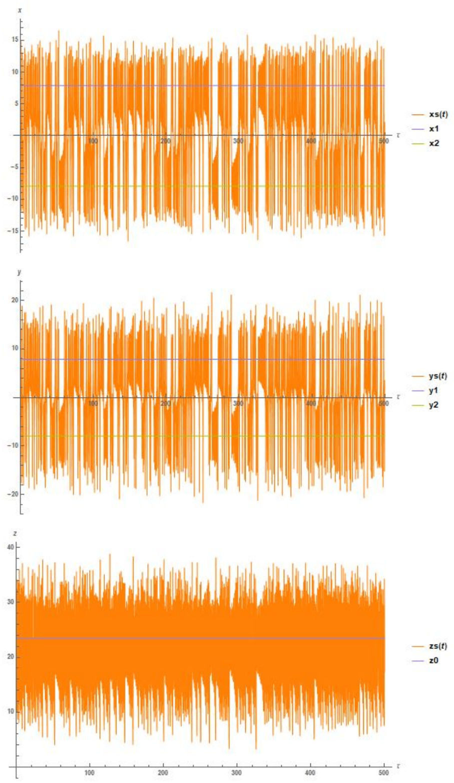


Here we can see that the solution converges to a point, more precise to the equilibrium point. By doing some more computation, we can assume that for $\rho < 24$ the solutions of the system will converge and for $\rho \geq 24$ the solutions shows chaotic behaviour.

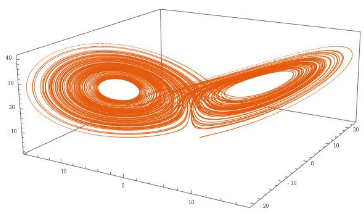
3.1 $\rho = 24.5$

In this subsection, we will observe the behaviour of the solution of the Lorentz system for $\rho = 24.5$. From this point we will try to reduce ρ to obtain the last value for which the solution of the Lorentz system do not have chaotic behaviour, in other words the last value for which the solution of the Lorentz system converges.

We will visualize the solution for $\rho = 24,5$ in two different ways to avoid errors and misunderstanding.

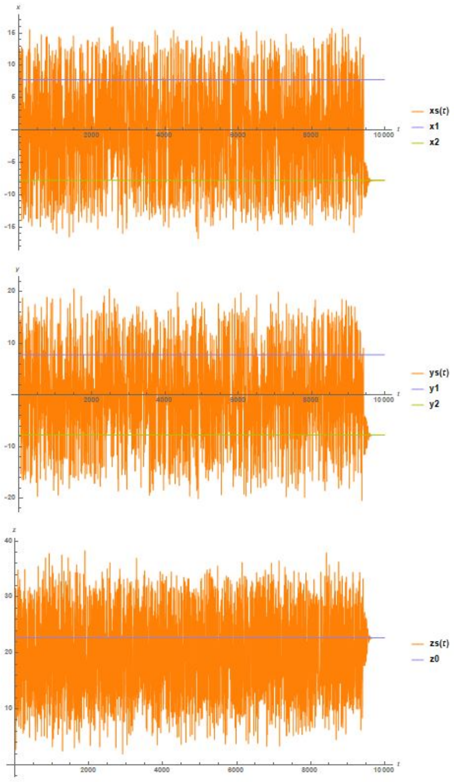


From this visualization of the solution set $\{x, y, z\}$, we can observe that the values x,y and z oscillate around the equilibrium points. It is a classical example for the Lorenz Attractor.



This $3D$ representation of the solution shows us that the values oscillate around two spheres with different centers and different radius, hence this graphical representation supports the above visualization and the fact that the solution of the Lorentz system for $\rho = 24.5$ is a Lorentz Attractor. Furthermore, for this values the Lorentz system shows bistability which we can be seen too from the representation, because the solution oscillates at most around the two equilibrium points, which made the solution stable around them.

Now we will try to bracket the values of ρ for which we did not have done a simulation of the solution to observe the behaviour. Until now, we know that for $\rho = 22$ the solution converges to one of the two equilibrium points and for $\rho = 24.5$ we have seen a chaotic behaviour. Let δ be the value of ρ which we did not have observed until now, i.e. $22 < \delta < 24,5$. Our goal is to reduce δ until it is equal to zero.
After a lot of computation in our programm we can say that for $\rho = 23,7$ the solution of the Lorenz syszem converges to one of our equilibrium point, hence we have now $23,7 < \delta < 24,5$. The following representation shows how many computation we needed to find out that for $\rho = 23,7$ the solution converges after a certain range.



If we continue this process, we found out that for $\rho \leqslant 24,06$ we have that the solution converges, that means the last value for which the solution show this behaviour is $24,06$. Note that it is remarkable that if we choose some $\rho > 24,74$, then the equilibrium points become unstable.

4 Periodic perturbation

In the previous section, we have observed the Lorenz attractor, in the case where we fix $\sigma = 10$ and $\beta = \frac{8}{3}$, and let varied ρ. We found out that for $\rho \leqslant 24,06$ the Lorenz attractor converges to one of the two equilibrium points. And for $\rho \geqslant 24,06$ we could observe that the Lorenz attractor shows chaotic behavior.
Now we will have a closer look at the Lorenz system for $\rho = \rho(t) = 24,4 + A \times sin(\omega \times t)$, we will choose $\omega = \{0,3; 0,05\}$ and vary the amplitude A for each of these two values between $1 \leqslant A \leqslant 10$.
In particular for $\omega = 0,3$ the range $4 \leqslant A \leqslant 5$ is probably sufficient to analyze the main features of the dynamics.
For fixed values of (ω, A), we show two time series, e.g. $x(t)$ and $z(t)$, as well as $\rho(t)$ over a relatively long time interval of, say, length 2000, we will plot the simulations for $0 \leqslant t \leqslant 2000$.
Before we start plotting and observing the behaviour of the Lorenz Attractor by simulation, we will study $\rho(t) = 24,4 + A \times sin(\omega \times t)$ to have an idea what we could expect.

<u>Observing "$A \times sin(\omega \times t)$":</u>

Let us have a closer look at $sin(x)$ $\forall x \in \mathbb{R}$, we know that the sinus function is a bounded function, we have

$$\forall x \in \mathbb{R} : -1 \leqslant sin(x) \leqslant 1$$

In our case, we have that $\omega = 0,3$ or $\omega = 0,05$ and $t \geqslant 0$, hence

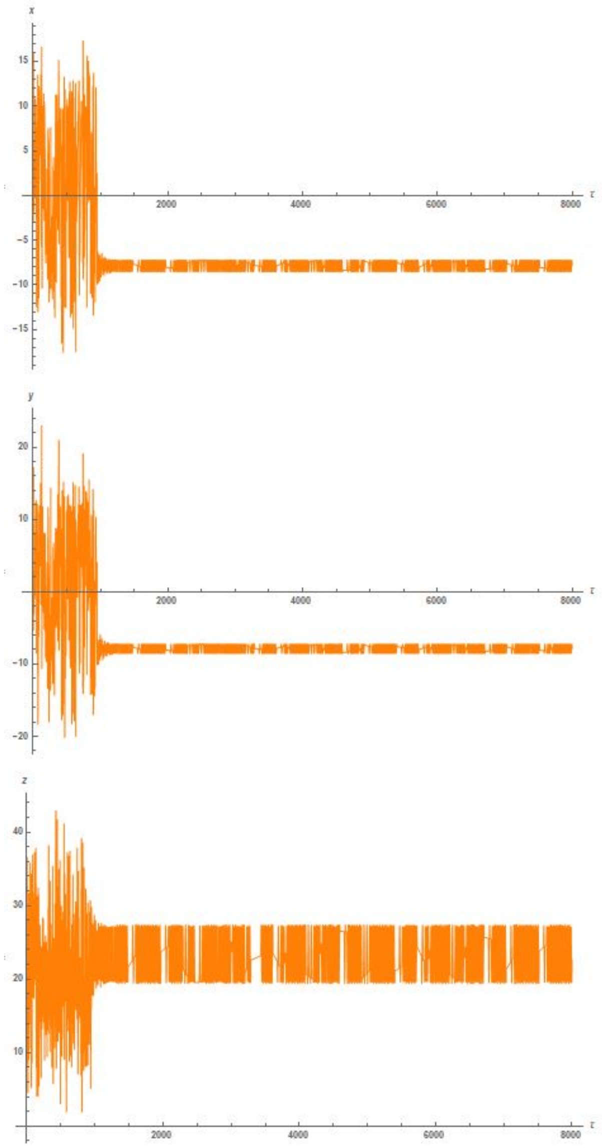
$$-1 \leqslant sin(\omega \times t) \leqslant 1$$

Furthermore,

$$-A \leqslant A \times sin(\omega \times t) \leqslant A$$

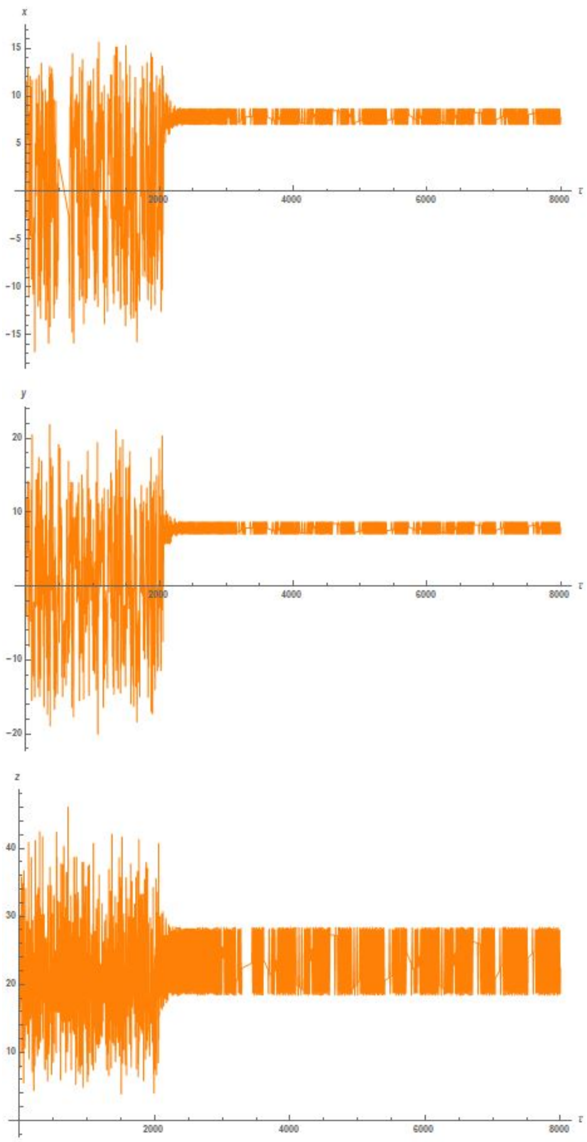
As we will observe $\rho(t) = 24,4 + A \times sin(\omega \times t)$ we have actually that
$\rho(t) \in [24,4 - A, 24,4 + A]$.
This implies that for some t_0 we have that $\rho(t_0)$ shows the same behavior as we have seen in the previous sections for some specific A.

We will now compute the Lorenz system for the couple $(0,3;4)$ and observe the behavior of the Lorenz Attractor (the code for the following representation is given in the Appendix by $A4$).



In the above visualization, we can notice that $x(t), y(t)$ and $z(t)$ emerge around some values around, this means that the solutions of this Lorenz equation oscillate. We know that for $\omega = 0,3$ it suffices to study the Lorenz equation for $A \in [4,5]$.
We will in the following see the case where $A = 5$, so we could compare the two cases to observe better the behavior of the Lorenz Attractor for $\omega = 0,3$ and $A \in [4,5]$.

Here you can see the visualization of the solution of the Lorenz equation for the couple $(0,3;5)$.

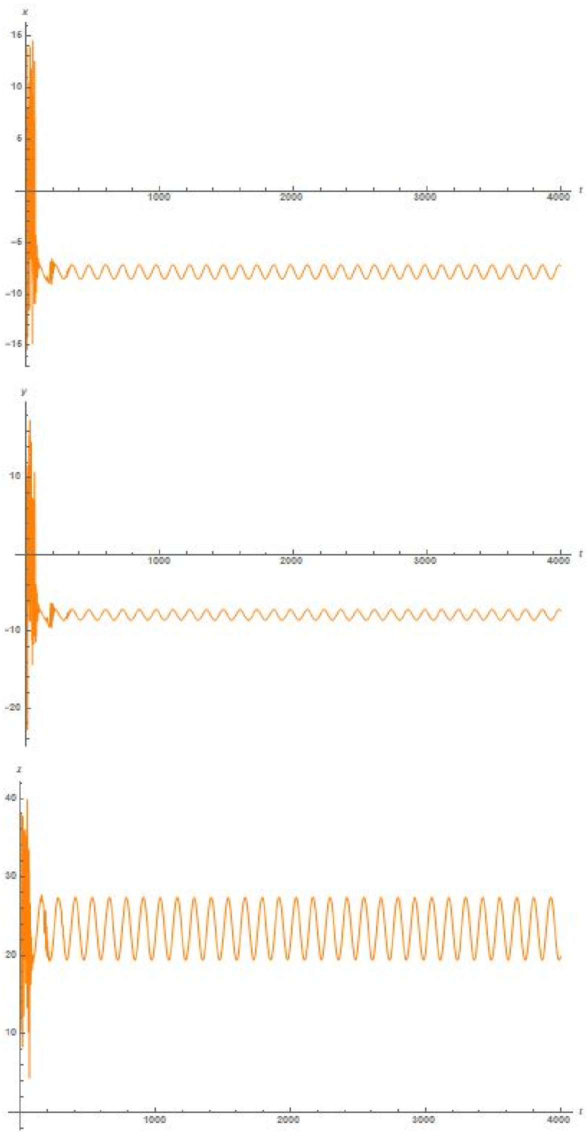


From this representation, we can clearly recognize the same behavior as in the last representation, we have that $x(t), y(t)$ and $z(t)$ osciallte around some values. Let us compare theses two representation with those of the section 3, by paying more attention on the values where $x(t), y(t)$ and $z(t)$ oscillate. We can conclude that they oscillate around the equilibrium points. As well by a closer look, we can observe that for $A = 4$ the solution oscillates around $(-\sqrt{\beta(\rho(t)-1)}, -\sqrt{\beta(\rho(t)-1)}, \rho(t)-1)$, what would too explain the behavior of $z(t)$. For $A = 5$ we have that the solution oscillates around $(\sqrt{\beta(\rho(t)-1)}, \sqrt{\beta(\rho(t)-1)}, \rho(t)-1)$.
From this observation, we find out that for this Lorenz equation the change of the value A has an impact on the solution in such a way that the solution oscillates around the other equilibrium point.

As mentionned at the beginning of this section we will study the following Lorenz equation:

$$\begin{cases} \frac{dx}{dt} = \sigma(y - x) \\ \frac{dx}{dt} = x(\rho(t) - z) - y \\ \frac{dz}{dt} = xy - \beta z \end{cases}$$

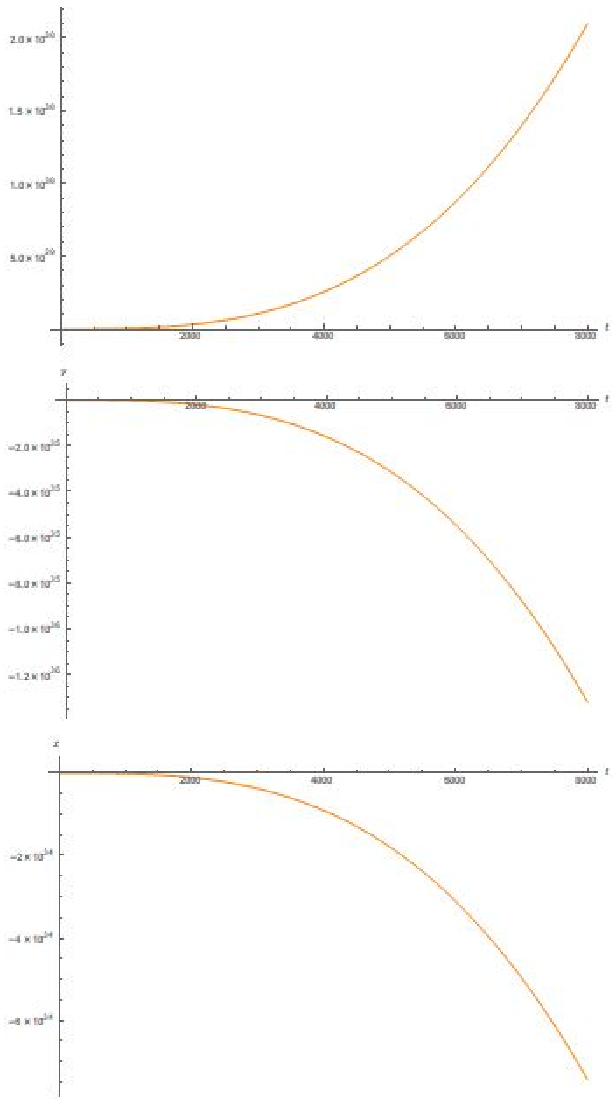
where $\rho(t) = 24,4 + A \times sin(\omega \times t)$ with $\omega = \{0,05; 0,3\}$ and $1 \leqslant A \leqslant 10$, as we have already studied the case for $\omega = 0,3$. It is intuitive to have a closer look for $\omega = 0,05$, but before starting the simulation, let us have a small idea what could happen if we replace $\omega = 0,3$ by $\omega = 0,05$.
As we know that the sinus function is a periodic function and that ω affects the speed how fast the function increases and decreases, then we can conclude that by the fact that $0,05 < 0,3$, we have that the function will have slower movement than for $0,3$. We will now plot the Lorenz equations for $(\omega; A) = (0,05; 4)$ to have a better view on the change which is caused by ω. As we can see on the above representation, the



solution shows the same behavior as for $(0,3;4)$ only that in this case we got a more smooth representation, which is actually the confirmation of our previous estimation.

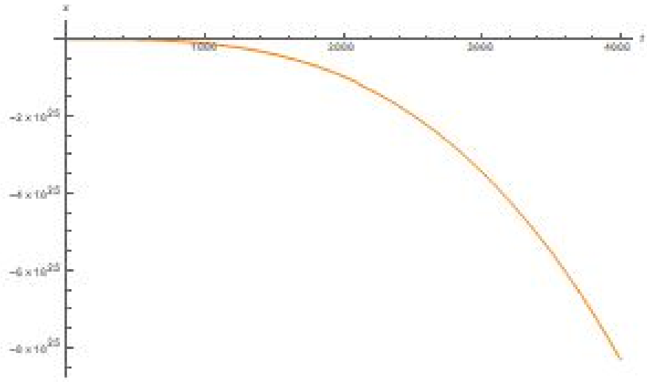
5 Exponential perturbation

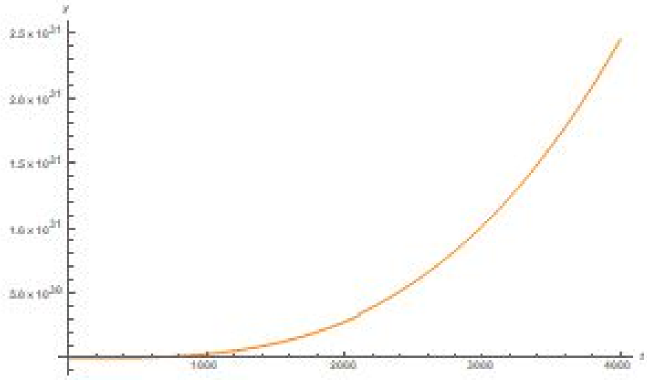
We have studied the case where ρ is equal to a sinus function in the previous section. We will pass now to the observation of an exponential function, this means we will take $\rho = a + b \exp ct$ with $a, b, c \in \mathbb{R}$, where a, b, c are choosen arbitrary .
We still need to have in mind that the exponential function has the property to increase really fast. By the above observations, we know that the solution converges for $\rho \leqslant 24,06$ as we will study the behavior of $\rho = a + b \exp ct$, we can image that in this case ρ will pass $24,06$ really fast. This would imply that the solution does not converge.
Let us observe some computation (the code is given in Appendix $A5$).

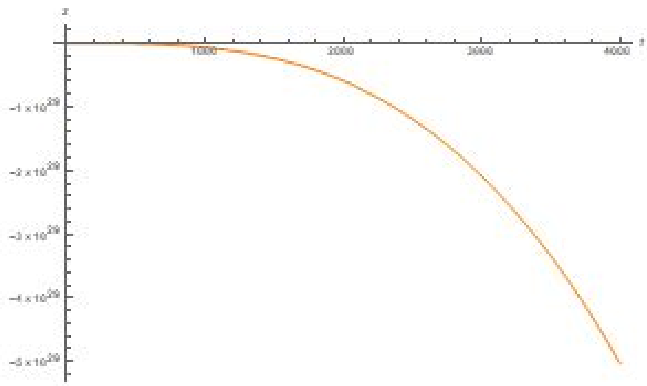


We used in this case $a = 0, b = 1$ and $c = 0,3$.
As we can observe here the Lorentz solution diverges, i.e. we have that $x(t) \to +\infty$, $y(t) \to -\infty$ and $z(t) \to -\infty$ for $t \to +\infty$. This behavior was actually expected because of the exponential function. We will now compute the Lorentz system for different values of a, b and c and compare the results.

Let us take $a = 5, b = 3$ and $c = 0, 3$.

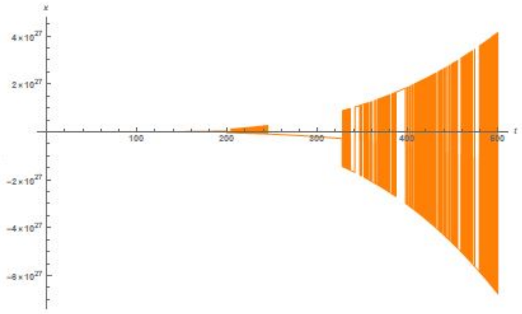


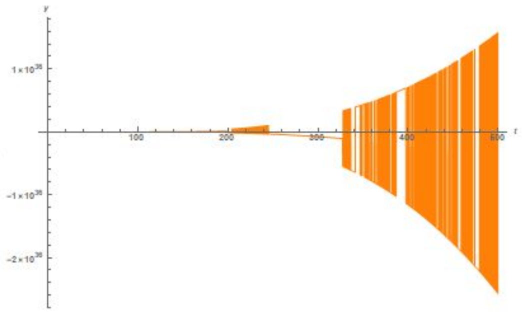


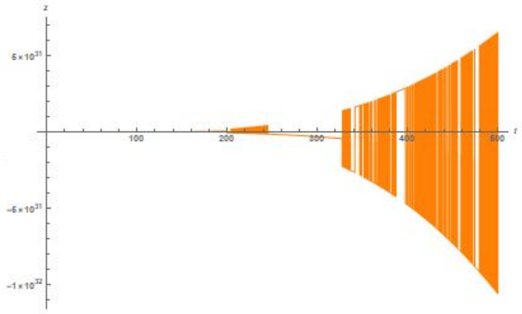


We have the same behavior as in the previous computation, which means both Lorentz solution diverges. But we can see that $x(t)$ and $y(t)$ diverge in the different direction, as in the first example. This difference needs more attention, this is why we will compute the code for a smaller tend.

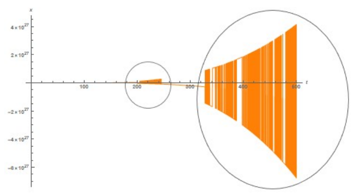
Let's take $tend = 500$:





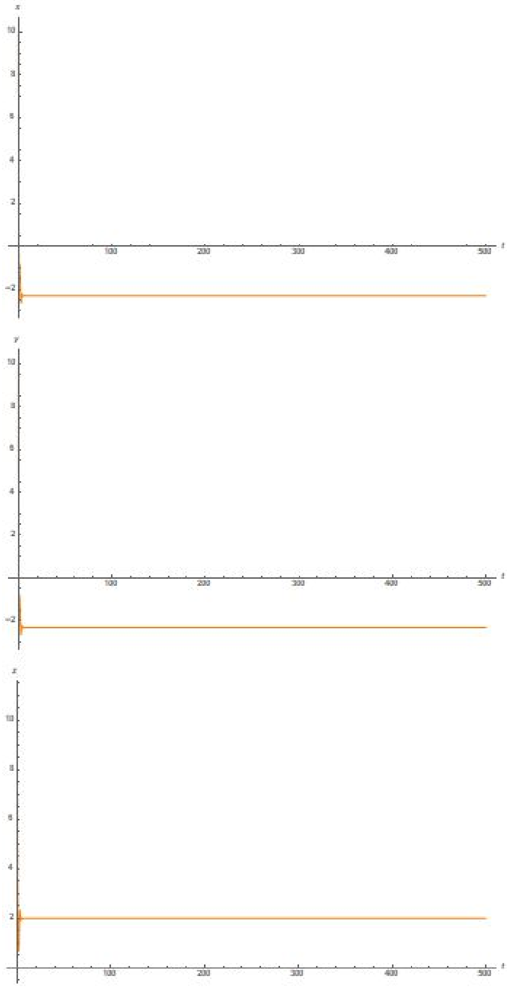


From this representation, we can analyze what happens before the system start to diverge. It seems that the system oscillate between some values which starts to grow and after a certain t, the function diverges.



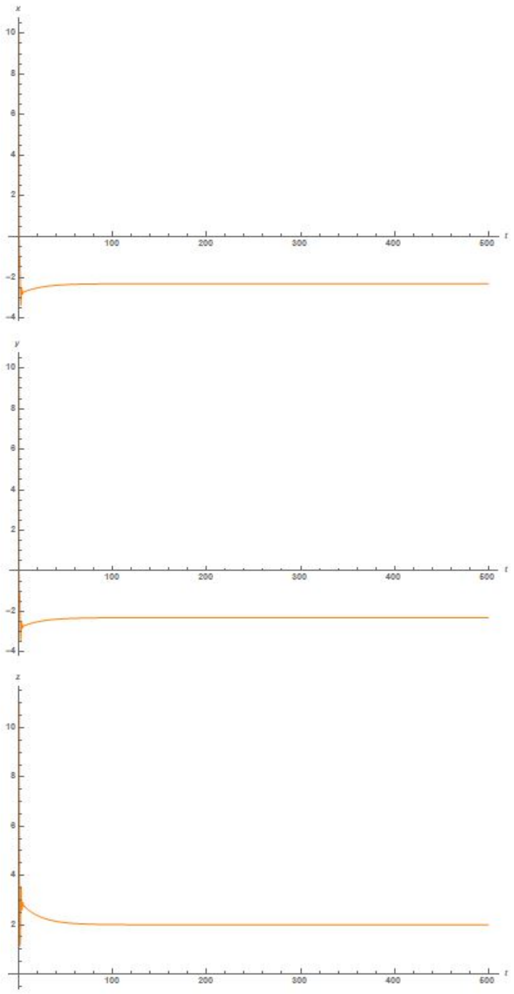
One can clearly see that in both cercles the function oscillates between some values which grow and as this is the representation for $\rho(t) = 5 + 3\exp(0, 3t)$, we know from the illustration for $tend = 8000$ that the Lorentz system diverges.
At this point, we can say that the solution shows some oscillation at the beginning and after a certain $t \in \mathbb{R}$ it diverges.
As a next step maybe it will be possible to find two functions which bound the values of oscillation, because at this point we do not know for sure what determines if the solutions diverge to $+\infty$ or $-\infty$. This will not be treated in this article, we hope that interested readers will continue where we needed to stop.

Let us see what happens if take a negativ c. We did the following computation for the value $(3, 1, -3)$.

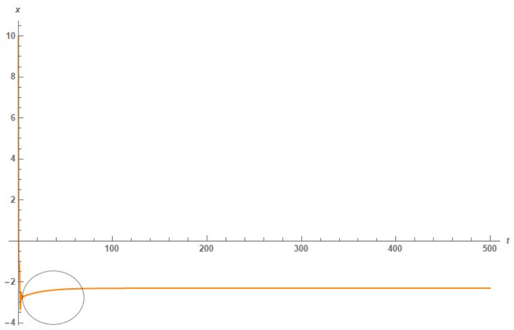


By taking a closer look, we see that for this values the lorenz system oscillates in the beginning but this behavior is really weak. If we had done the computation for a higher a bigger t, the oscillation would vanish. In particular if t approaches ∞ then it will be negligible. After this oscillation the system converges really fast to some value. It will be interesting to analize if the system for a smaller c.

Let us briefly have a look at the system for a c smaller than the previous one. We will take the value $(3, 1, -0.05)$ for the following computation.



By comparing this figure with the previous one, we see that the Lorenz system for $c = -0.05$ oscillate at the beginning as for $c = -3$. The only difference is that in this case the Lorenz system starts converging really slow which can be observed by the area bounded by the circle.



We actually do not know why this happens, we can only say that maybe the exponential function is the reason for the behavior, because we know that $exp(x) \to 0$ if $x \to -\infty$, i.e. it converges for the negative values, but it diverges for postive values, as $exp(x) \to +\infty$ if $x \to +\infty$.

6 Conclusion

At the end, we can say that the Lorenz system is a interesting topic, which we could observe in a more simple illustrated way, but it is important to keep in mind that this topic contains a complex theorie behind in Chaos Theory. We hope that we could inspire some readers to start their interest for the Chaos Theory.

7 Appendix

Here we will give the code for the numerically visualization and the code for the representation of the solution of the Lorentz system. Every reader is welcome to try the codes in Wolfram Mathematica and to compute the solution of the Lorentz system with parameters and initial conditions, which seem interesting for themselves.

<u>*A*1</u>

```
Clear[μ, σ, ρ, β]
|lösche
tend = 500;
eq = {x'[t] == σ (y[t] - x[t]) , y'[t] == x[t] (ρ - z[t]) - y[t], z'[t] == x[t] × y[t] - β z[t]};
init = {x[0] == 10, y[0] == 10, z[0] == 10};
pars = {σ → 10, ρ → 24.00, β → 8 / 3};


{xs, ys, zs} = NDSolveValue[{eq /. pars, init}, {x, y, z}, {t, 0, tend}, AccuracyGoal → 10];
               |löse Wert der Differentialgleichung numerisch          |Genauigkeitsziel
(* ParametricPlot3D[{xs[t],ys[t],zs[t]},{t,100,1000}, Axes → False, Boxed → False,
 ColorFunction→Function[{x,y,z,t},ColorData["SolarColors"][√(x²+ y² + z²)]],PlotStyle→{AbsoluteThickness[0], Opacity[0.2]}
 Background → Black, ImageSize→{1000,1000},PlotPoints → 100,PerformanceGoal->"Quality",MaxRecursion→12, WorkingPrecision→6,
 Exclusions → None]*)


StringForm["x*=`1`,y*=`2`,z*=`3`." , ToString[xs[tend]], ToString[ys[tend]], ToString[zs[tend]]]
|in Zeichenkettenform                |als Zeichenkette     |als Zeichenkette     |als Zeichenkette
Plot[xs[t], {t, 0, tend}, PlotRange → All, AxesLabel → {t, x}, LabelStyle → Directive[Bold], ImageSize → Large]
|stelle Funktion graphisch dar |Koordinate··· |alle |Achsenbeschriftung |Beschriftun··· |Anweisung |fett |Bildgröße |groß
Plot[ys[t], {t, 0, tend}, PlotRange → All, AxesLabel → {t, y}, LabelStyle → Directive[Bold], ImageSize → Large]
|stelle Funktion graphisch dar |Koordinate··· |alle |Achsenbeschriftung |Beschriftun··· |Anweisung |fett |Bildgröße |groß
Plot[zs[t], {t, 0, tend}, PlotRange → All, AxesLabel → {t, z}, LabelStyle → Directive[Bold], ImageSize → Large]
|stelle Funktion graphisch dar |Koordinate··· |alle |Achsenbeschriftung |Beschriftun··· |Anweisung |fett |Bildgröße |groß
```

Here above is the code for the numerical illustration for the solution and it will follow the code, which gives us the representation of the solution for the same parameters and initial conditions.

A2

```
Clear[LorenzSystemPoints, p, s, pts]
lösche

LorenzSystemPoints[parameters_List, steps_Integer] :=
  Module[{σ, ρ, β, updates, δ = 1.*^-3, pt0 = {1., 1., 1.}}, {σ, ρ, β} = parameters;
  Modul
   updates = x ↦ {{1 - δ σ, δ σ, 0}, {δ ρ, 1 - δ, -δ #}, {0, δ #, 1 - δ β}}.{##} & @@ x;
   NestList[updates, pt0, steps]];
   Liste verschachtelter Ergebnisse

p = {10., 23., 8 / 3.};
s = 100000;
pts = LorenzSystemPoints[p, s];

ListPointPlot3D[pts, PlotRange → All, PlotTheme → "Scientific", ImageSize → Large]
listenbezogenes 3D-Streu···  Koordinatenb··  alle   Thema der graphischen Darstell···  Bildgröße   groß
```

Here is the code with the added lines, which was done in Wolfram Mathematica.

<u>A3</u>

```
Clear[μ, σ, ρ, β]
|lösche
tend = 500;
eq = {x'[t] == σ (y[t] - x[t])  , y'[t] == x[t] (ρ - z[t]) - y[t], z'[t] == x[t] × y[t] - β z[t]};
init = {x[0] == 10, y[0] == 10, z[0] == 10};
a = 10; b = 23; c = 8 / 3;
pars = {σ → a, ρ → b, β → c};


{xs, ys, zs} = NDSolveValue[{eq /. pars, init}, {x, y, z}, {t, 0, tend}, AccuracyGoal → 10];
               |löse Wert der Differentialgleichung numerisch          |Genauigkeitsziel
(* ParametricPlot3D[{xs[t],ys[t],zs[t]},{t,100,1000}, Axes → False, Boxed → False,
 ColorFunction→Function[{x,y,z,t},ColorData["SolarColors"][√(x²+ y² + z²)]],
 PlotStyle→{AbsoluteThickness[0], Opacity[0.2]}, Background → Black, ImageSize→{1000,1000},
 PlotPoints → 100,PerformanceGoal->"Quality",MaxRecursion→12, WorkingPrecision→6,
 Exclusions → None] *)

x1 = (c (b - 1)) ^ (1 / 2);
x2 = - (c (b - 1)) ^ (1 / 2);
y1 = (c (b - 1)) ^ (1 / 2);
y2 = - (c (b - 1)) ^ (1 / 2);
z0 = b - 1;
StringForm["x*=`1`,y*=`2`,z*=`3`." , ToString[xs[tend]], ToString[ys[tend]], ToString[zs[tend]]]
|in Zeichenkettenform                 |als Zeichenkette     |als Zeichenkette    |als Zeichenkette
Plot[{xs[t], x1, x2}, {t, 0, tend}, PlotRange → All, AxesLabel → {t, x}, LabelStyle → Directive[Bold],
|stelle Funktion graphisch dar         |Koordinatenb··· |alle |Achsenbeschriftung |Beschriftungssti |Anweisung |fett
 ImageSize → Large]
 |Bildgröße  |groß
Plot[{ys[t], y1, y2}, {t, 0, tend}, PlotRange → All, AxesLabel → {t, y}, LabelStyle → Directive[Bold],
|stelle Funktion graphisch dar         |Koordinatenb··· |alle |Achsenbeschriftung |Beschriftungssti |Anweisung |fett
 ImageSize → Large]
 |Bildgröße  |groß
Plot[{zs[t], z0}, {t, 0, tend}, PlotRange → All, AxesLabel → {t, z}, LabelStyle → Directive[Bold],
|stelle Funktion graphisch dar     |Koordinatenb··· |alle |Achsenbeschriftung |Beschriftungssti |Anweisung |fett
 ImageSize → Large]
 |Bildgröße  |groß
```

We will give you here the code where ρ is replaced by $\rho(t)$

A4

```
Clear[μ, σ, ρ, β]
lösche
tend = 4000;
A = 4
ω = 0.3
eq = {x'[t] == σ (y[t] - x[t])  , y'[t] == x[t] (24.4 + A * Sin[ω * t] - z[t]) - y[t],
                                                      Sinus
   z'[t] == x[t] × y[t] - β z[t]};
init = {x[0] == 10, y[0] == 10, z[0] == 10};
pars = {σ → 10, β → 8 / 3};


{xs, ys, zs} = NDSolveValue[{eq /. pars, init}, {x, y, z}, {t, 0, tend}, AccuracyGoal → 10];
               löse Wert der Differentialgleichung numerisch                Genauigkeitsziel
(* ParametricPlot3D[{xs[t],ys[t],zs[t]},{t,100,1000}, Axes → False, Boxed → False,
 ColorFunction→Function[{x,y,z,t},ColorData["SolarColors"][√(x² + y² + z²)]],
 PlotStyle→{AbsoluteThickness[0], Opacity[0.2]}, Background → Black, ImageSize→{1000,1000},
 PlotPoints → 100,PerformanceGoal->"Quality",MaxRecursion→12, WorkingPrecision→6,
 Exclusions → None]*)


StringForm["x*=`1`,y*=`2`,z*=`3`." , ToString[xs[tend]], ToString[ys[tend]], ToString[zs[tend]]]
in Zeichenkettenform                  als Zeichenkette     als Zeichenkette     als Zeichenkette
Plot[xs[t], {t, 0, tend}, PlotRange → All, AxesLabel → {t, x}, LabelStyle → Directive[Bold],
stelle Funktion graphisch dar  Koordinatenb··· alle  Achsenbeschriftung  Beschriftungssti Anweisung  fett
 ImageSize → Large]
 Bildgröße   groß
Plot[ys[t], {t, 0, tend}, PlotRange → All, AxesLabel → {t, y}, LabelStyle → Directive[Bold],
stelle Funktion graphisch dar  Koordinatenb··· alle  Achsenbeschriftung  Beschriftungssti Anweisung  fett
 ImageSize → Large]
 Bildgröße   groß
Plot[zs[t], {t, 0, tend}, PlotRange → All, AxesLabel → {t, z}, LabelStyle → Directive[Bold],
stelle Funktion graphisch dar  Koordinatenb··· alle  Achsenbeschriftung  Beschriftungssti Anweisung  fett
 ImageSize → Large]
 Bildgröße   groß
```

A5

```
Clear[μ, σ, ρ, β]
|lösche
tend = 8000;
a = 5;
b = 3;
c = 0.3;
eq = {x'[t] == σ (y[t] - x[t])  , y'[t] == x[t] (a + b * Exp[c * t] - z[t]) - y[t], z'[t] == x[t] × y[t] - β z[t]};
                                              |Exponentialfunktion
init = {x[0] == 10, y[0] == 10, z[0] == 10};
pars = {σ → 10, β → 8 / 3};


{xs, ys, zs} = NDSolveValue[{eq /. pars, init}, {x, y, z}, {t, 0, tend}, AccuracyGoal → 10];
               |löse Wert der Differentialgleichung numerisch                |Genauigkeitsziel
(* ParametricPlot3D[{xs[t],ys[t],zs[t]},{t,100,1000}, Axes → False, Boxed → False,
 ColorFunction→Function[{x,y,z,t},ColorData["SolarColors"][√(x² + y² + z²)]],
 PlotStyle→{AbsoluteThickness[0], Opacity[0.2]}, Background → Black, ImageSize→{1000,1000},PlotPoints → 100,
 PerformanceGoal->"Quality",MaxRecursion→12, WorkingPrecision→6, Exclusions → None] *)


StringForm["x*=`1`,y*=`2`,z*=`3`." , ToString[xs[tend]], ToString[ys[tend]], ToString[zs[tend]]]
|in Zeichenkettenform                 |als Zeichenkette      |als Zeichenkette      |als Zeichenkette
Plot[xs[t], {t, 0, tend}, PlotRange → All, AxesLabel → {t, x}, LabelStyle → Directive[Bold], ImageSize → Large]
|stelle Funktion graphisch dar |Koordinatenb··· |alle |Achsenbeschriftung |Beschriftungssti |Anweisung |fett |Bildgröße |groß
Plot[ys[t], {t, 0, tend}, PlotRange → All, AxesLabel → {t, y}, LabelStyle → Directive[Bold], ImageSize → Large]
|stelle Funktion graphisch dar |Koordinatenb··· |alle |Achsenbeschriftung |Beschriftungssti |Anweisung |fett |Bildgröße |groß
Plot[zs[t], {t, 0, tend}, PlotRange → All, AxesLabel → {t, z}, LabelStyle → Directive[Bold],  ImageSize → Large]
|stelle Funktion graphisch dar |Koordinatenb··· |alle |Achsenbeschriftung |Beschriftungssti |Anweisung |fett |Bildgröße |groß
```

References:

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2. Wikipedia Butterfly effect

3. Youtube video: "What is the Lorenz system? Chaotic Attractor?" from Jonathan Mitchell

4. Article: Perturbation of the Lorentz system, January 31. 2020, UNI LU