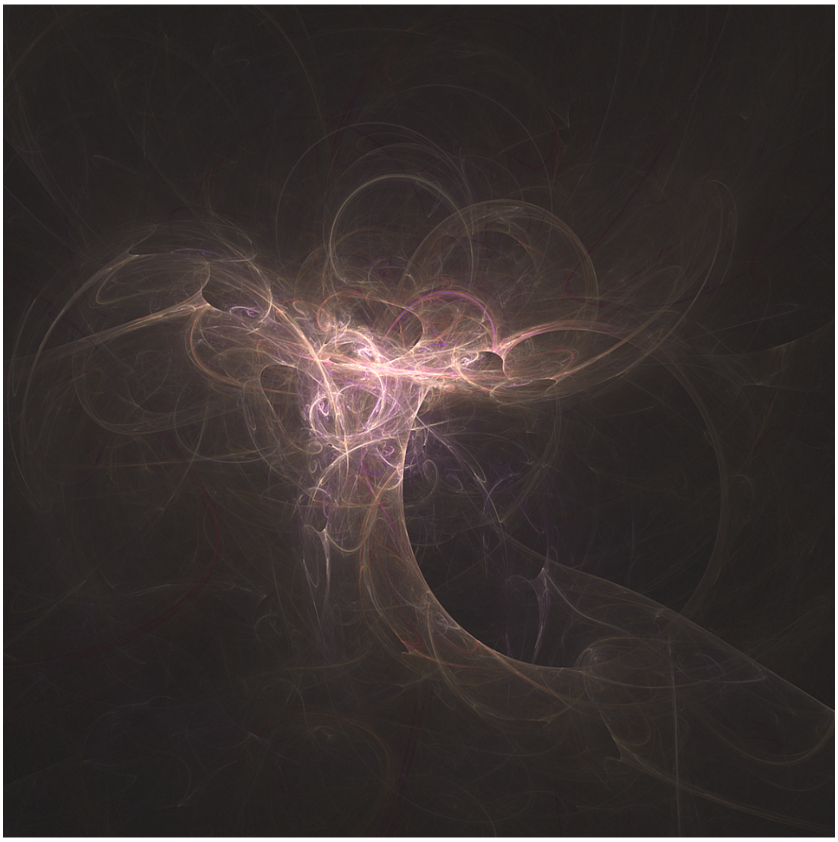
Université du Luxembourg

Bachelor en Sciences et Ingénierie

Mathématiques expérimentales 3

2019-2020

Fractal limit sets and Schottky groups



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1 Introduction

1.1 Möbius maps

A transformation of the form $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by $f(z) = \frac{az+b}{cz+d}$ where $z \in \mathbb{C}$, $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ is called a Möbius transformation (or homography) of the complex plane.
Geometrically speaking, a Möbius transformation can be obtained by first applying a stereographic projection[1] from the plane to the unit sphere, then rotating the sphere to a new position and orientation in space and finally performing another stereographic projection from that new position of the sphere to the plane.

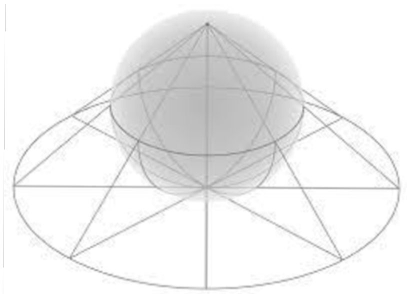

Figure 1: Stereographic projection from the plane to the unit sphere [2]

The stereographic projections identify $\hat{\mathbb{C}}$ with a sphere, which is the Riemann sphere, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e. the complex plane with the additional point at infinity.

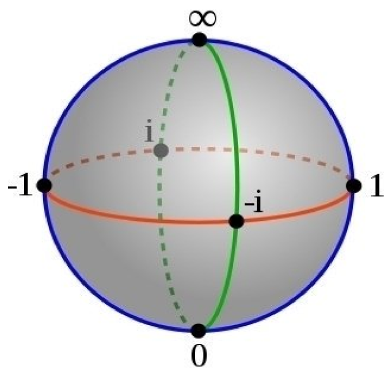

Figure 2: Riemann sphere [3]

For example, one can perform simple translations, dilations, rotations and inversions to a rectangular plane of dimension 2. When we take now the same plane in the third dimension, we can put a Riemann sphere on top of that plane. The plane is then projected onto the sphere. Now by raising the sphere for instance, the plane dilates, by spinning the sphere, the plane rotates and by rotating the sphere about a horizontal axis, we obtain an inversion of the plane (the plane is turned inside out). Shortly, by rotating the sphere and/or repositioning it, the plane that was projected onto the sphere is then reprojected back to the plane.
Hence, these transformations map the object to itself while preserving the essential structure of the object. Möbius transformations map straight lines either to a line or circle or map a circle to a line or a circle and preserve angles, because the essential structure is kept the same.

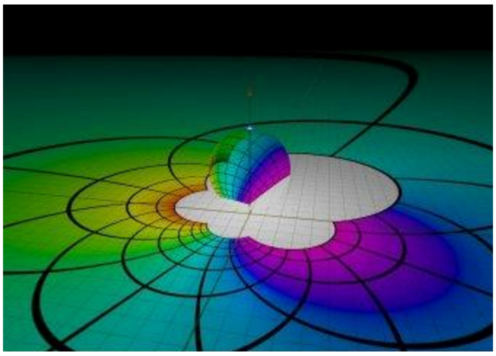

Figure 3: Example of a Möbius transformation with stereographic projection [4]

The set of Möbius transformations form the group $\mathrm{Aut}(\mathbb{C})$, because every object is mapped to itself while preserving all of its structure. The group $\mathrm{Aut}(\mathbb{C})$, also called 'Möbius group', has subgroups with a lot of applications in mathematics and physics, for example in image processing and cryptography.
An important example of such a subgroup is the modular group, which is central to the theory of many fractals. Actually, the modular group uses the exact same function (only that $a, b, c, d \in \mathbb{Z}$) and has the same properties as defined above for the Möbius transformation, except that the determinant of these functions equals to 1. Thus, the functions of the modular group belong to $SL_2(\mathbb{R})$, which tells us that the images of the function cannot form a spiral twist.

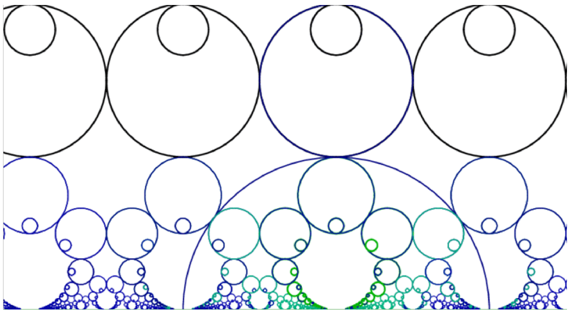

Figure 4: Example of the action of the modular group on the upper half-plane [5]
Note: Here you can also observe that a line is either mapped to another line or a circle, or that a circle is either mapped to a line or another circle.

In our work, we consider the Möbius transformations as described at the beginning and we take a look at subgroups of the Möbius group. The only difference is that we consider $ad - bc = 1$. Thus, if $a, b, c, d \in \mathbb{R}$, then the images of the function cannot form a spiral twist.

1.2 Schottky groups

We then take a look at the Schottky groups: A Schottky group is a subgroup of $SL_2(\mathbb{C})$ that acts on the Riemann sphere by Möbius transformations. Schottky groups basically consist of Möbius transformations and are constructed by pairing shapes like circles (but not necessary only circles). Note that for example the circles can touch, but must be disjoint for the other cases, so as long as the paired shapes are disjoint, any group like this is called a Schottky group. When we have such a group, there also is an associated subset of the plane, namely its 'limit set'. These sets have a lot of symmetries related to these Schottky groups and are often beautiful fractal sets.
Remark: A full understanding of limit sets is beyond the goal of this project.

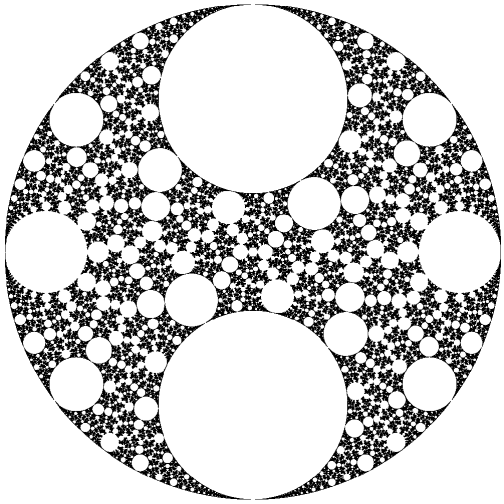

Figure 5: Example of a fractal set [6]

1.3 Goal of this project

The goal of this project is to acquire an experimental understanding of the concept of Möbius transformations, Fractal limit sets and Schottky groups and also includes plotting these subsets.

2 Experimentation

As described in the previous section, we will now proceed to take a look at the different types of Möbius transformations.

2.1 Observations

2.1.1 Orbits of one homography

At the very beginning we tried to plot a Möbius transformation with random parameters satisfying the conditions of a Möbius transformation (determinant,...). This means that we repeatedly applied a Möbius transformation to a specific point, thus creating the orbit of a homography. This means that we try to plot the points with n iterations of the Möbius transformation f and it's inverse function f^{-1}.

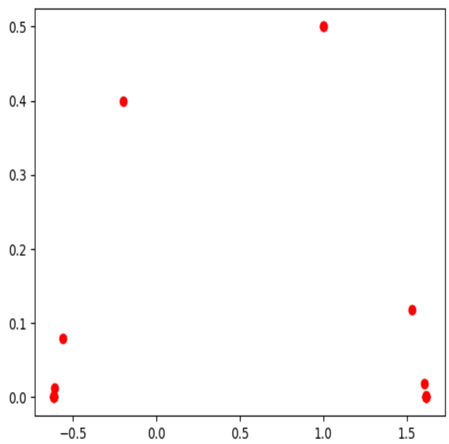

Figure 6: Example of a homography with single orbit

Let's take a first look at an example with the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $z = 1 + 0,5i$ a complex starting point and 10 iterations. The first thing we can see on the *Figure 6* is that after a certain amount of iterations the limit of $\lim_{n\to\infty} f^n(z)$ tends to a specific point. In this case we have two such points, one for $f^n(z)$ and one for $f^{-n}(z)$. These two points are 'fixed points' which we can infer from the Proposition 12.

Remark: The 'pointc()'-function (Code-Reference: Single orbit for one homography) was used to create the *Figure 6*.

Later on we proceeded by superposing multiple orbits using slightly different starting points. This resulted in interesting figures depending on the matrix used. As we will see in subsection 3.3 these can be split up in three categories depending on their trace.

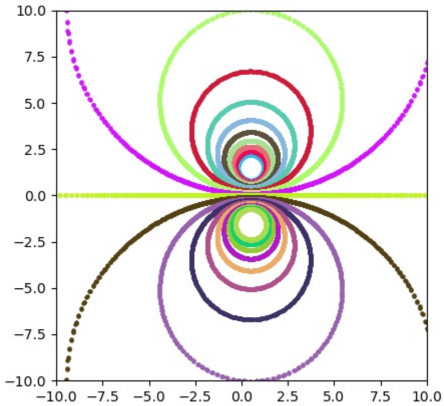

Figure 7: Example of an elliptic map

Figure 7 is an example using the matrix $\begin{pmatrix} 1 & -1 \\ 0.5 & 0.5 \end{pmatrix}$ with the complex starting point $z = 1 + i - k \cdot 0,1i$ with $k \in \{0, 1, 2, ..., 20\}$ (we consider here 20 different starting points) and with 900 iterations of f and f^{-1} for every starting point. This matrix represents an elliptic map. The different starting points are all on a vertical line. We start with $1 + i$ and we slowly decrease the starting points by $-k \cdot 0,1i$ with $k \in \mathbb{N}$. We observe that for each of these points the Möbius transformation creates circles which at first increase in size until we reach the starting point $z = 1$ for which we get a straight line. For the lower half of the figure we get a mirror image of the upper half.

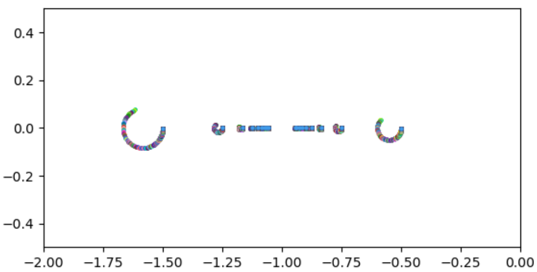

Figure 8: Example of a parabolic map

Figure 8 is an example using the matrix $\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$ with the complex starting point $z = 1 + i - k \cdot 0,1i$ with $k \in \{0, 1, 2, ..., 4000\}$ (we consider here 4000 different starting points) and with 10 iterations of f and f^{-1} for every starting point. This matrix represents a parabolic map.

We observe two lines, which are actually composed of very small arcs, one that goes to the right and one that goes to the left and we can see that these arcs seem to get larger as they distance themselves from these two lines.

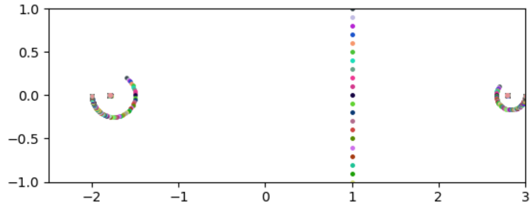

Figure 9: Example of a loxodromic map

Figure 9 is an example using the matrix $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ with the complex starting point $z = 1 + i - k \cdot 0,1i$ with $k \in \{0, 1, 2, ..., 5000\}$ (we consider here 5000 different starting points) and with 10 iterations of f and f^{-1} for every starting point. This matrix represents a loxodromic map.

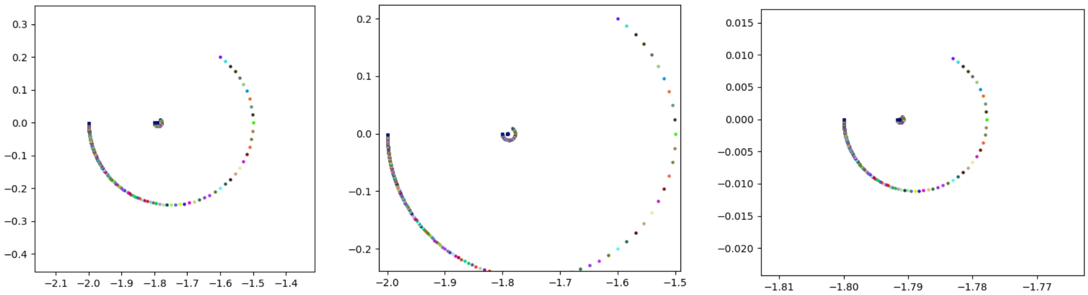

Figure 10: 3 stages of zoom-in of the Figure 9

As we zoom in further into the *Figure 9* we observe that the orbits form circles getting smaller and smaller as we can see in *Figure 10.*
In Figure 9 we observe that the starting points form a vertical line. These points are mapped to two arcs, one on each side of the line. These arcs seem to tend to a fixed point which can be noticed in figure 10. Note that to obtain full circles we would need to have a line of starting points ranging from $-\infty$ to $+\infty$.

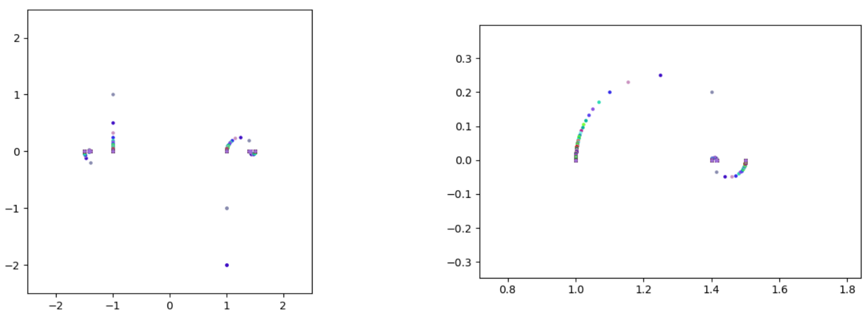

Figure 11: Example of a complex/loxodromic map and a zoom of a certain part of the image

Figure 11 is an example using the matrix $\begin{pmatrix} i & 2i \\ i & i \end{pmatrix}$ with the complex starting point $z = 1 + i - k \cdot 0,1i$ with $k \in \{0, 1, 2, ..., 5000\}$ (we consider here 5000 different starting points) and with 100 iterations of f and f^{-1} for every starting point.
While both figures 10 and 11 are composed of loxodromic maps we can see that *Figure 11* has a spiral twist and *Figure 10* doesn't. This difference comes from the fact that the matrix used to get the Figure 10 belongs to $SL_2(\mathbb{R})$ and thus can't have a spiral twist as opposed to Figure 11 whose matrix belongs to $SL_2(\mathbb{C})$.

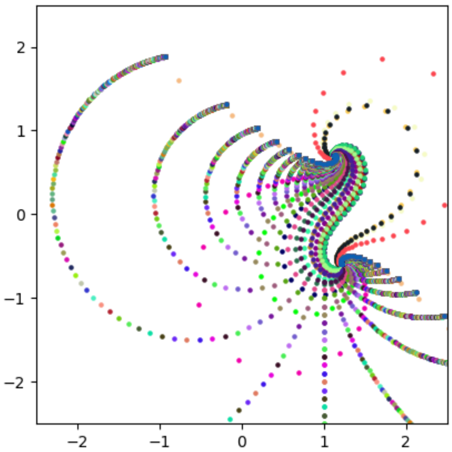

Figure 12: Example of a complex/loxodromic map and determinant close to 1

Figure 12 is an example using the matrix
$$\begin{pmatrix} 1.1545423922881868 + 0.1412093367736713i & -0.22299086383778546 - 0.23952396546400687i \\ 0.13740708608332622 + 0.12010887863498236i & 0.8455503609816553 - 0.15512213290925111i \end{pmatrix}$$
with the complex starting point $z = 1 + i - k \cdot 0,1i$ with $k \in \{0, 1, 2, ..., 1000\}$ (we consider here 1000 different starting points) and with 900 iterations of f and f^{-1} for every starting point. This matrix represents a loxodromic map. Note that here we can clearly perceive two spiral twists.

Remark:

- The examples shown above are only a few of all the possible figures one can obtain. We can obtain other figures when changing the matrix parameters, verifying that the determinant equals to 1 and changing the number of iterations of f and f^{-1} or the number of variations of i.
- Figures 7-12 were made using the 'lc()'-function (Code-Reference: Superposition of multiple orbits).

2.1.2 Orbits of subgroups generated by two elements

We will now proceed to showcase figures of different orbits of subgroups generated by two Möbius transformations.
In the following part we will refer to the iterations that the Python-code has to process. This means that for each iteration we calculate the Möbius transformation for the two given elements for each entry in a specific list. The starting list is composed of a single element and after each iteration we add the resulting points to the list and then we use the new list for the next iteration and we repeat this process until we reach the desired number of iterations.

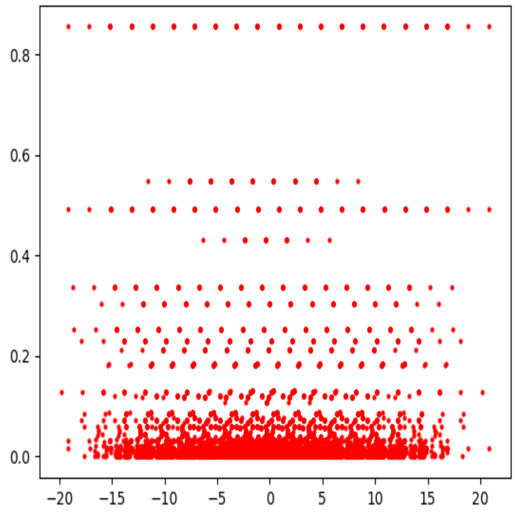

Figure 13: Example of a composition of two parabolic maps

Figure 13 is an example using the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ with two random starting points with 10 iterations. We observe that the points tend to the real axis. The matrices used above were provided to us by our tutor.

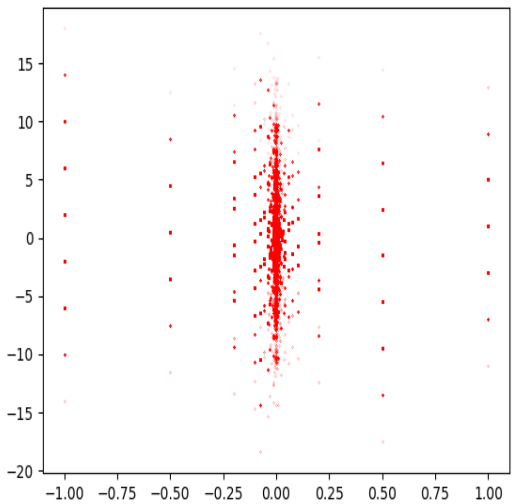

Figure 14: Example of a composition of two loxodromic maps

In *Figure 14* we use the matrices $\begin{pmatrix} -3i & 1 \\ 5 & 2i \end{pmatrix}$ and $\begin{pmatrix} i & 1 \\ -3 & 2i \end{pmatrix}$ with starting point $z = 1 + i$ with 11 iterations. We observe that the points tend to the imaginary axis. We considered matrices that we found by chance.

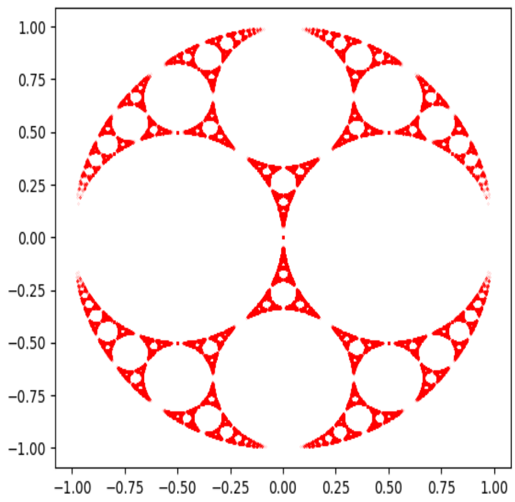

Figure 15: Example of a composition of two parabolic maps

In *Figure 15* we use the matrices $\begin{pmatrix} 1 & 0 \\ -2i & 1 \end{pmatrix}$ and $\begin{pmatrix} 1-i & 1 \\ 1 & 1+i \end{pmatrix}$ with starting point $z = 0 + 0 \cdot i$ with 12 iterations. Upon closer inspection we notice that the patterns repeat themselves. In fact we observe fractal behavior. In other words the patterns repeat themselves on a smaller scale for each repetition. Hence we can recognize the fractal limit set of the group.

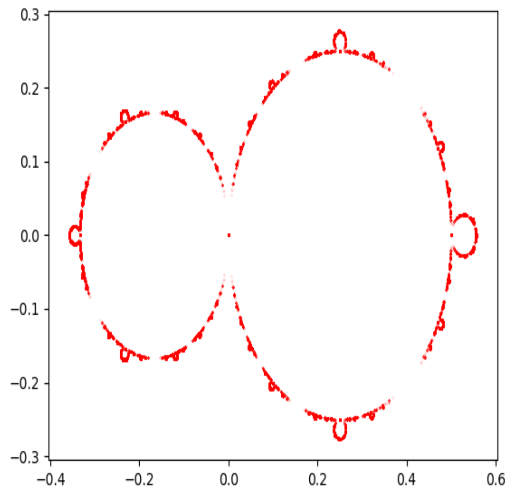

Figure 16: Example of a composition of a loxodromic and a parabolic map

In *Figure 16* we use the matrices $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix}$ with starting point $z = 0 + 0 \cdot i$ with 10 iterations. The matrices used to create the figure 16 were found by chance. In other words we just tried random matrices with determinant equal to 1. You can notice that the subset generated by two elements forms two circles tangent at the starting point $z = 0 + 0 \cdot i$. You can also observe that there are multiple curves along the path of the ellipses.

If we take a closer look at the largest curve on the right side you can clearly notice that it has the same outline as the right ellipses. This is a perfect example of the fractal behavior of the fractal limit set. In fact the aforementioned curve also possesses smaller ellipses which are a part of it.

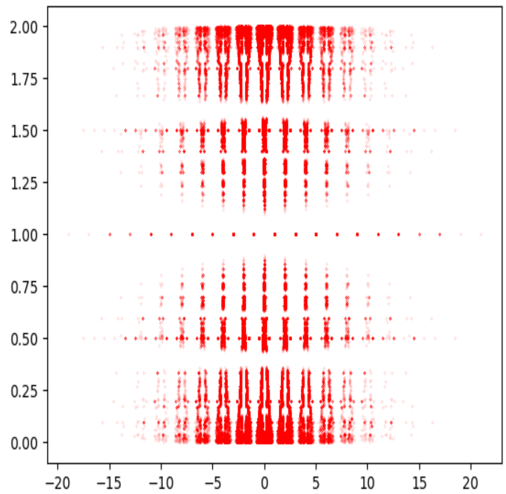

Figure 17: Another example of a composition of two parabolic maps

In *Figure 17* we use the matrices $\begin{pmatrix} 2 & -i \\ -i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ with starting point $z = 1 + i$ with 10 iterations. This figure was created by using a random matrix and a matrix used before in figure 13 and we notice that the points are going outwards seen from the starting point $z = 1 + i$ and they seem to form a part of an ellipse.

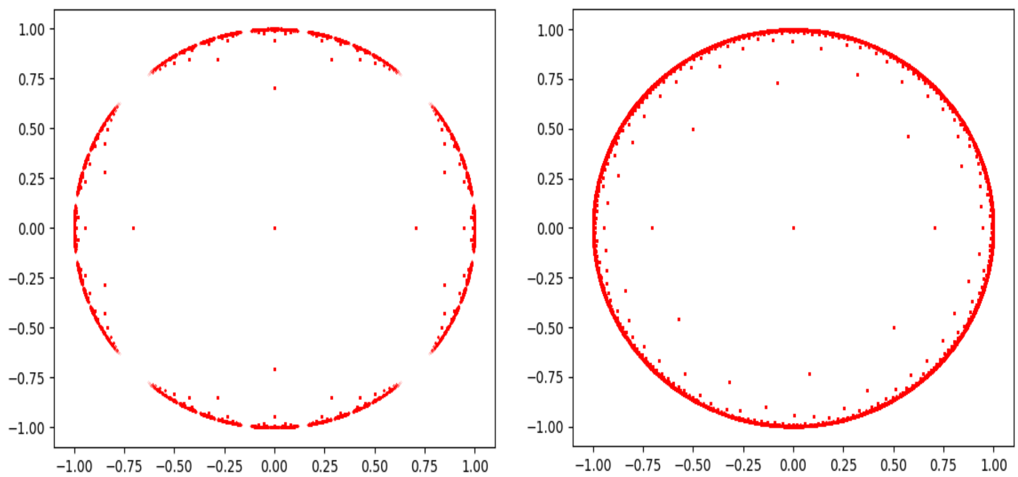

Figure 18: Left: Example of a composition of two loxodromic maps without conjugation
Right: Example of a composition of two loxodromic maps with conjugation

In *Figure 18* we use the matrices $\begin{pmatrix} \sqrt{2} & i \cdot e^{i \cdot \frac{\pi}{4}} \\ -i \cdot e^{-i \cdot \frac{\pi}{4}} & \sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$ with starting point $z = 0 + 0 \cdot i$ with 12 iterations for the right figure and the matrices $\begin{pmatrix} \sqrt{2} & i \\ -i & \sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$ with 12 iterations for the left figure. The matrices used are matrices used to get 'Indra's necklace' as one can read in chapter 6 of the book [15].
Actually we use the matrix $\begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$ for both figures, the only difference is that we use the conjugated matrix of $\begin{pmatrix} \sqrt{2} & i \\ -i & \sqrt{2} \end{pmatrix}$ for the right figure.
The conjugation is obtained by the following calculation:
$$\begin{pmatrix} e^{i \cdot \pi/8} & 0 \\ 0 & e^{-i \cdot \pi/8} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} e^{-i \cdot \pi/8} & 0 \\ 0 & e^{i \cdot \pi/8} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & i \cdot e^{i \cdot \pi/4} \\ -i \cdot e^{-i \cdot \pi/4} & \sqrt{2} \end{pmatrix}$$
It is actually an anticlockwise rotation by $\frac{\pi}{4}$. We observe that by taking the conjugated matrix, the circle becomes a closed path and thus has no more holes like on the left figure.

2.2 Other examples

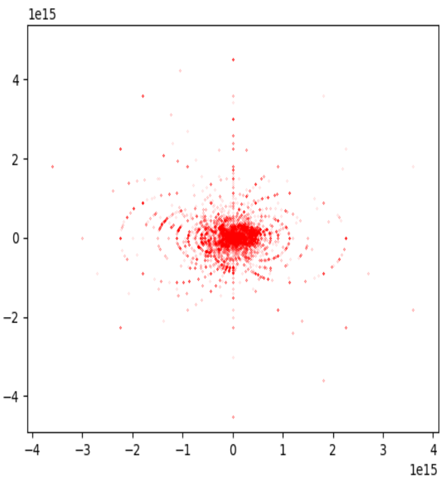

Figure 19

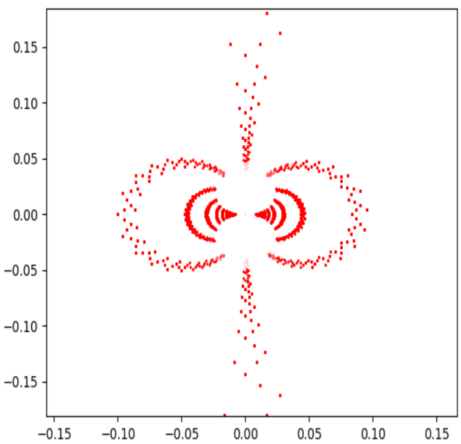

Figure 20

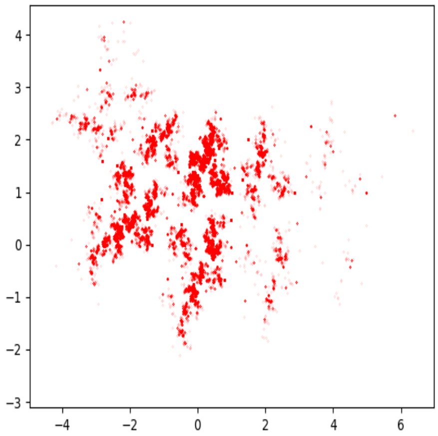

Figure 21

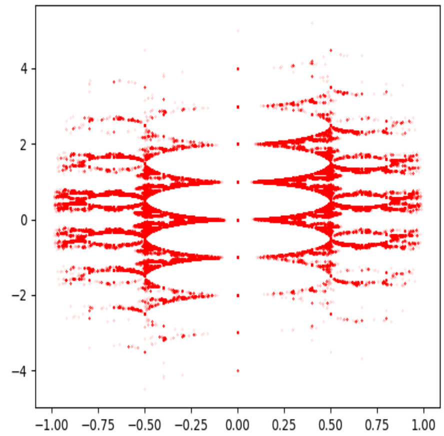

Figure 22

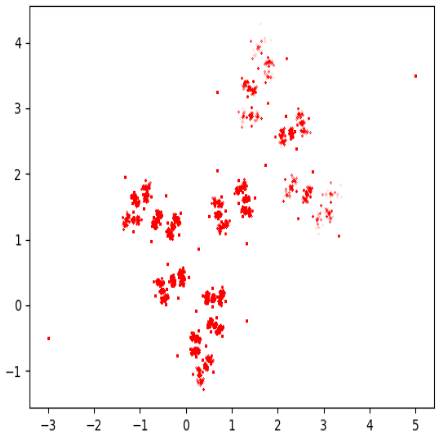

Figure 23

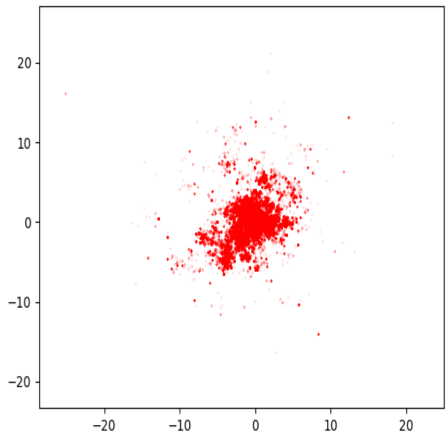

Figure 24

Figure List				
Figure number	Matrix A	Matrix B	Number of iterations	Starting points
19	0, -1, 1, 1	2i+1, i, 2, 1	13	1+i
20	1, 0, 11, 1	1, 0, 2i, 1	12	0+0i, 1+i, 1+0i, 0+i, -1+i, 1-i
21	2, 4-i, -i, -2i	$\frac{1}{4}$+i, -1, 1, 0	10	1+i
22	1, 0, 1, 1	2i+1, i, 2, 1	14	0+0i
23	i, 0, 1, -i	-i+1, i, -1, 1	14	-3-0.5i
24	$\frac{1}{2}$, i, $\frac{3}{4}$·i, $\frac{1}{2}$	-i+1, i, -1, 1	12	0+0i

Remark:

- The matrix of *Figure 15* can be found in the book 'Indra's Pearls: The Vision of Felix Klein' [15] on page 201.
- The matrix of *Figure 18* can be found in the book 'Indra's Pearls: The Vision of Felix Klein' [15] on page 175.
- The 'fcalc()'-function (Code-Reference: Subgroups generated by two elements) was used to create the figures 13-24.

3 Mathematical background

3.1 Definitions

Definition 1. Group action:
Let G be a group and A a set with $g \in G$ and $a \in A$:

- **Left group action:** G operates on the left on A if we have the following application

$$\begin{aligned} G \times A &\to A \\ (G, a) &\to g \cdot a \end{aligned}$$

  verifying:

  - $\forall a \in A \ : \ a \cdot e = a$
  - $\forall g, g' \in G, \ \forall a \in A \ : \ g \cdot (g' \cdot a) = (gg') \cdot a$

So have group action if the function behaves well to the rule of G.
In our case, we only consider the left group action and we have that $G = SL_2(\mathbb{C})$ and $E = \hat{\mathbb{C}}$ (with G the Möbius group).
So, if G acts on E and $a \in E$, then the orbit of a is the set of points of E of the form $g \cdot a$ where $g \in G$. The action is by homographies as proved in *Proposition 10* and *Proposition 11*.

Definition 2. Generated subgroup:
Let $(G, *)$ be a group and A a non-empty subset of G and let G' be the set of all the subgroups of G with $A \in G'$. We have $G' \neq \varnothing$, because $G \in G'$.
Hence we call generated subgroup by A the subgroup: $\langle A \rangle := \bigcap\limits_{H \in G'} H$
Now we take a look at the elements of a generated subgroup:
We consider the same group $(G, *)$ with A a non-empty subset of G. Let $g \in G$, so there exists $n \in \mathbb{N}^*$ and the elements $x_i \in A$ or $x_i^{-1} \in A$ with $i = \{1, 2, ..., n\}$ such that $x = x_1 \cdot x_2 \cdot ... \cdot x_n$, then $x \in \langle A \rangle$.
Therefore: $\langle A \rangle = \left\{ x_1 \cdot x_2 \cdot ... \cdot x_n \ : \ n \in \mathbb{N}^* \ , \ x_i, x_i^{-1} \in A \ , \ \forall i \in \{1, 2, ..., n\} \right\}$
Note that the generated subgroup by the $g_i \in G$ is the smallest subgroup of G containing all the g_i. In this project we take a look at subgroups of $SL(n, \mathbb{C})$ generated by one or two elements.

Definition 3. A function $f : X \to Y$ is surjective if:

- $\forall y \in Y \ \exists x \in X \ f(x) = y$.

Definition 4. A function $f : X \to Y$ is injective if one of these two equivalent conditions is verified:

- $\forall x \in X$ and $\forall x' \in X$ we have: $f(x) = f(x') \Rightarrow x = x'$
- $\forall x \in X$ and $\forall x' \in X$ we have: $x \neq x' \Rightarrow f(x) \neq f(x')$

Definition 5. If a function is injective and surjective, then the function is called bijective.

Proposition 6. *Another method to prove bijection*
Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. If we can construct $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $f \circ g = id$ and $g \circ f = id$, then f is bijective.

Definition 7. Fixed point
Let $f : X \to X$ be a function.
A fixed point of f is a point $z \in X$ that satisfies $f(z) = z$.

3.2 Homographies

Matrix of a homography

A homography is represented by the function $f(z) = \frac{az+b}{cz+d}$. Now using simple matrix algebra, we can define a (2×2)-matrix of $f(z)$ composed by the variables a, b, c and d, namely $M_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Proposition 8. *Every homography is a holomorphic function:*

Proof.

$$\begin{aligned} \textit{Let } f : \mathbb{C} &\to \mathbb{C} \qquad\qquad \text{with } f \leftrightarrow M_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \\ z &\mapsto \frac{az+b}{cz+d} \end{aligned}$$

We need to prove that f is differentiable on the complex plane:

$$\begin{aligned} \text{We calculate: } f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{\frac{a(z+h)+b}{c(z+h)+d} - \frac{az+b}{cz+d}}{h} \\ &= \lim_{h\to 0} \frac{(az+ah+b)(cz+d) - (az+b)(cz+ch+d)}{(cz+ch+d)(cz+d)h} \\ &= \lim_{h\to 0} \frac{acz^2 + achz + bcz + adz + adh + bd - (acz^2 + bcz + achz + bch + adz + bd)}{(cz+ch+d)(cz+d)h} \\ &= \lim_{h\to 0} \frac{(ad-bc)h}{(cz+ch+d)(cz+d)h} \\ &= \frac{ad-bc}{(cz+d)^2} \in \mathbb{C} \qquad C.E. : cz+d \neq 0 \Leftrightarrow z \neq \frac{-d}{c} \end{aligned}$$

Conclusion: f is complex derivable $\forall z \in \mathbb{C} \setminus \{\frac{-d}{c}\}$ and so every homography is a holomorphic function.

$\square$

Proposition 9. *Homographies are bijective*

Proof. A homography is injective.
Method 1 (Definition 1,2 and 3):

$$\begin{aligned}
&f(z) = f(z') \\
\Leftrightarrow\ &\frac{az+b}{cz+d} = \frac{az'+b}{cz'+d} \ with\ z \neq \frac{-d}{c} \ and\ z' \neq \frac{-d}{c} \\
\Leftrightarrow\ &(az+b)(cz'+d) = (az'+b)(cz+d) \\
\Leftrightarrow\ &aczz' + adz + bcz' + bd = aczz' + adz' + bcz + bd \\
\Leftrightarrow\ &adz - adz' + bcz' - bcz = 0 \\
\Leftrightarrow\ &ad(z-z') - bc(z-z') = 0 \\
\Leftrightarrow\ &(ad-bc)(z-z') = 0 \ \ since\ ad - bc = 1\ (SL_2(\mathbb{C})) \\
\Leftrightarrow\ &z = z'
\end{aligned}$$

Thus: A homography is injective

□

Proof. A homography is surjective.

$$\begin{aligned}
&Let\ x = g(y)\ with\ g(y) = \frac{dy-b}{-cy+a} \\
&We\ need\ to\ prove\ that\ f(x) = y \\
&Thus\ f(x) = f(g(y)) = \frac{a\cdot g(y)+b}{c\cdot g(y)+d} = \frac{a\cdot\frac{dy-b}{-cy+a}+b}{c\cdot\frac{dy-b}{-cy+a}+d} = \frac{\frac{ady-ab-bcy+ab}{-cy+a}}{\frac{cdy-cb-cdy+ad}{-cy+a}} = \frac{(ad-bc)y}{ad-bc} = y
\end{aligned}$$

Thus: A homography is surjective

□

Hence: A homography is injective and surjective, so it is a bijection of $\hat{\mathbb{C}}$ on $\hat{\mathbb{C}}$.

Proof. A homography is a bijection of $\hat{\mathbb{C}}$ on $\hat{\mathbb{C}}$.
Method 2 (Definition 4):
We construct $g(z) = \frac{dz-b}{-cz+a}$
Then we see that: $f \circ g(z) = f(g(z)) = \frac{a\cdot\frac{dz-b}{-cz+a}+b}{c\cdot\frac{dz-b}{-cz+a}+d} = \frac{\frac{adz-ab-bcz+ab}{-cz+a}}{\frac{cdz-cb-cdz+ad}{-cz+a}} = \frac{(ad-bc)z}{ad-bc} = z$
and: $g \circ f(z) = g(f(z)) = \frac{d\cdot\frac{az+b}{cz+d}-b}{-c\cdot\frac{az+b}{cz+d}+a} = \frac{\frac{adz+bd-bcz-bd}{cz+d}}{\frac{-acz-bc+acz+ad}{cz+d}} = \frac{(ad-bc)z}{ad-bc} = z$
Thus $f \circ g(z) = z$ and $g \circ f(z) = z$ and so f is bijective. Note: $g(z)$ is the inverse of f. □

Proposition 10. $M_{f_1 \circ f_2} = M_{f_1} \cdot M_{f_2}$

Proof.
Note: M_{f_i} is defined as the matrix of a function f_i with $i \in \mathbb{N}^*$ (i could be any number it is only used to differentiate different functions.)
Let $f_1(z) = \frac{az+b}{cz+d}$ and $f_2(z) = \frac{ez+f}{gz+h}$, so $M_{f_1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$ and $M_{f_2} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in SL_2(\mathbb{C})$

Let's calculate $f_1 \circ f_2(z)$:
$f_1 \circ f_2(z) = f_1(f_2(z)) = \frac{a \cdot f_2(z)+b}{c \cdot f_2(z)+d} = \frac{a \cdot \frac{ez+f}{gz+h}+b}{c \cdot \frac{ez+f}{gz+h}+d} = \frac{aez+af+gbz+bh}{cez+cf+gdz+dh} = \frac{(ae+gb)z+(af+bh)}{(ce+gd)z+(cf+dh)}$
$\Rightarrow M_{f_1 \circ f_2} = \begin{pmatrix} ae+gb & af+bh \\ ce+gd & cf+dh \end{pmatrix} \in SL_2(\mathbb{C})$

Let's calculate $M_{f_1} \cdot M_{f_2}$: $M_{f_1} \cdot M_{f_2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+gb & af+bh \\ ce+gd & cf+dh \end{pmatrix} \in SL_2(\mathbb{C})$
Thus: $M_{f_1 \circ f_2} = M_{f_1} \cdot M_{f_2}$

□

Proposition 11. $M_{f^{-1}} = (M_f)^{-1}$

Proof.
Let's calculate $f^{-1}(z)$: $f(z) = \frac{az+b}{cz+d}$ with $f(z) = y$
Let's calculate the inverse of f:

$$\begin{aligned} \text{So:} \quad z &= \frac{ay+b}{cy+d} \\ &\Leftrightarrow z(cy+d) = ay+b \\ &\Leftrightarrow czy+dz = ay+b \\ &\Leftrightarrow (cz-a)y = b-dz \ |\cdot(-1) \ \ |:(-cz+a) \quad \text{with } z \neq \frac{a}{c} \\ &\Leftrightarrow y = \frac{dz-b}{-cz+a} \end{aligned}$$

So: $f^{-1}(z) = \frac{dz-b}{-cz+a} \quad \Rightarrow M_{f^{-1}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbb{C})$
Let's calculate $(M_f)^{-1}$:
$M_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \quad So (M_f)^{-1} = \frac{1}{\det(M_f)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Note: $\det M_f = 1$,because $f \in SL_2(\mathbb{C})$
Thus: $\quad M_{f^{-1}} = (M_f)^{-1}$

□

3.3 Dynamics of homographies

Recall that a fixed point is a point z that satisfies $f(z) = z$.
In our case here, f is a homography and we consider that $ad - bc = 1$ ($SL_2(\mathbb{C})$), thus:
If $z \in \mathbb{C}$:

$$\begin{aligned} f(z) = z &\Leftrightarrow \frac{az+b}{cz+d} = z \\ &\Leftrightarrow az + b = z(cz+d) \\ &\Leftrightarrow az + b = cz^2 + dz \\ &\Leftrightarrow cz^2 + dz - az - b = 0 \\ &\Leftrightarrow cz^2 + (d-a)z - b = 0 \end{aligned}$$

If $c \neq 0$, then we recover an order 2 equation with discriminant:

$$\begin{aligned} \triangle &= (d-a)^2 - 4c(-b) \\ &= d^2 - 2ad + a^2 + 4bc \\ &= a^2 + 2ad + d^2 - 4ad + 4bc \\ &= (a+d)^2 - 4(ad - bc) \\ &= Tr(M_f)^2 - 4 \end{aligned}$$

Hence, we find either one fixed point in $\mathbb{C}$ for $\triangle = 0$ or two fixed points in $\mathbb{C}$ for $\triangle \neq 0$. If $c = 0$, but $a - d \neq 0$, then we have an order 1 equation, leading to a unique fixed point in $\mathbb{C}$ and a fixed point at infinity. Finally, if $c = 0$ and $a - d = 0$, there are no fixed points in $\mathbb{C}$ and ∞ is the only fixed point.
Fixed points of a homography:
We give more details in two particular examples.

First case. $\triangle = 0$
By the definition of $\triangle$, we know that our solution is given by $z = \frac{-d+a}{2c}$
We can rewrite z:
$z = \frac{-d+a}{2c} = \frac{-2d+d+a}{2c} = \frac{-2d+Tr(M_f)}{2c}$
Hence, we consider two cases:

- $Tr(M_f) = 2 \Longrightarrow z = \frac{-2d+Tr(M_f)}{2c} = \frac{-2d+2}{2c} = \frac{-d+1}{c}$
  In this case our solution is of the form $\frac{-d+1}{c}$ and $ad - bc = 1$ and $Tr(M_f) = a + d = 2$ need to be fulfilled.
- $Tr(M_f) = -2 \Longrightarrow z = \frac{-2d+Tr(M_f)}{2c} = \frac{-2d-2}{2c} = \frac{-d-1}{c}$
  In this case our solution is of the form $\frac{-d-1}{c}$ and $ad - bc = 1$ and $Tr(M_f) = a + d = -2$ need to be fulfilled.

Note: If $\triangle = 0$ we get only one fixed point. This is the case when $Tr(M_f) = \pm 2$.

Example 1. Let us first consider the case when $Tr(M_f) = 2$:
In this case, the fixed point is $\frac{a-1}{c}$.

- $a - 1 \neq 0$ and $c = 0$.
  $ad - bc = 1$ and $a + d = 2$ need to be fulfilled, thus we find that $ad = 1$ (because $c = 0$) and because $a \neq 1$, then $d \neq 1$ too.
  Then $\frac{a-1}{c} = \frac{a-1}{0} = \infty$ and $z = \frac{-d+1}{c} = \infty$ So: ∞ is a fixed point.

- $a - 1 = 0$ and $c \neq 0$:
  $ad - bc = 1$ and $a + d = 2$ need to be fulfilled, thus we find that $bc = 0$ (because if $a = 1$, than by $a + d = 2$ we get that $d = 1$) and so $b = 0$ (because $c \neq 0$).
  Then $\frac{a-1}{c} = 0$ and $z = \frac{-d+1}{c} = 0$ So: 0 is a fixed point.

Example 2. Let us then consider the case when $Tr(M_f) = -2$:
In this case, the fixed point is $\frac{a+1}{c}$.

- $a + 1 \neq 0$ and $c = 0$:
  $ad - bc = 1$ and $a + d = -2$ need to be fulfilled, thus we find that $ad = 1$ (because $c = 0$) and because $a \neq -1$, than $d \neq -1$ too.
  Then $\frac{a+1}{c} = \frac{a+1}{0} = \infty$ and $z = \frac{-d-1}{c} = \infty$ So: ∞ is a fixed point.

- $a + 1 = 0$ and $c \neq 0$:
  $ad - bc = 1$ and $a + d = -2$ need to be fulfilled, thus we find that $bc = 0$ (because if $a = -1$ then by $a + d = -2$ we get that $d = -1$) and so $b = 0$ (because $c \neq 0$).
  Then $\frac{a+1}{c} = 0$ and $z = \frac{-d-1}{c} = 0$ So: 0 is a fixed point.

Second case. $\triangle \neq 0$
By the definition of $\triangle$, we know that our solutions are given by $z = \frac{-(d-a) \pm y}{2c}$ where y is a square root of the complex number $Tr(M_f)^2 - 4$. So, if $Tr(M_f)^2 - 4 = r \cdot e^{i \cdot \theta}$, then $y = \sqrt{r} \cdot e^{\frac{\theta}{2} i}$.
The conditions that need to be fulfilled are: $a + d \neq \pm 2$ *and* $ad - bc = 1$.
Note: If $\triangle \neq 0$ we get two fixed points ($z_1 = \frac{-(d-a) - \sqrt{r} \cdot e^{\frac{\theta}{2} i}}{2c}$ *and* $z_2 = \frac{-(d-a) + \sqrt{r} \cdot e^{\frac{\theta}{2} i}}{2c}$) This is the case when $Tr(T) \neq \pm 2$.

Classification of homographies

The trace of M_f also plays a major role when we want to distinguish between different Möbius transformations. It is calculated like this: $Tr(M_f) = a + d$
We can distinguish the following Möbius maps:

- Loxodromic maps: $Tr(M_f) \in \mathbb{C} \setminus [-2, 2]$
- Elliptic maps: $Tr(M_f) \in \left]-2, 2\right[$
- Parabolic maps: $Tr(M_f) = \pm 2$

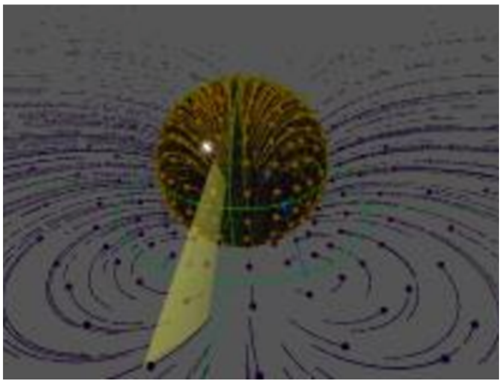

Figure 25: Example of an elliptic Möbius transformation with arbitrary fixed points [7]

We saw in this section (3.3) that the trace of M_f determines how many fixed points we have. Hence, when we have a loxodromic map, then we have exactly 1 fixed point and when we have an elliptic or a parabolic map, then we have exactly 2 fixed points.

Proposition 12. *If* $\lim_{n\to\infty} f^n(z) = l$, *then* $f(l) = l$. $(z, l \in \mathbb{C})$

Proof.
The condition that allows us to go to the limit is that f must be a continuous function.
We know that f is a continuous function, because we proved that f is holomorphic.
We know that:

- $f(z) = \frac{az+b}{cz+d}$ and so $f(l) = \frac{al+b}{cl+d}$
- If $\lim_{n\to\infty} f^n(z) = l$, then it is also true that $\lim_{n\to\infty} f^{n+1}(z) = l$

So: $\lim_{n\to\infty} f^{n+1}(z) = \lim_{n\to\infty} \frac{a\cdot f^n(z)+b}{c\cdot f^n(z)+d} = \frac{al+b}{cl+d} = f(l)$
This finally implies that $f(l) = l$. □

4 Python code

<u>**Single orbit for one homography:**</u>

```
import cmath
import numpy as np
import matplotlib.pyplot as plt


def determinanttest(a, b, c, d):  # Test if the det is 1
    if a * d - b * c == 1:
        return True
    else:
        return False


def tz(a, b, c, d, z):  # main function used for moebius transformations
    if z == cmath.inf:  # check if z is equal to infinity
        if c != 0:      # check if c is different from 0
            return a / c    # the homography is equal to infinity in this case
        else:
            return cmath.inf    # if c = 0 and z = infinity then the homography is equal to infinity
    else:
        num = (a * z) + b   # calculating the numerator of the homography
        den = (c * z) + d   # calculating the denominator of the homography
        if den == 0:        # check if the denominator is 0
            return cmath.inf    # if the previous line is true then the homography is equal to 0
        else:
            return num / den    # returning the homography


def tzinv(a, b, c, d, z):  # inverse function used for moebius transformations
    if z == cmath.inf:
        if c != 0:
            return d / -c
        else:
            return cmath.inf
    else:
        num = (d * z) - b
        den = (-c * z) + a
        if den == 0:
            return cmath.inf
        else:
            z = num / den
            return z


def repeat(f, a, b, c, d, n, z):       # composite function for T^n
    if n == 0:                         # no repetition
        return z
    else:
        if n == 1:                      # function is called upon once
            return f(a, b, c, d, z)
        else:
            return f(a, b, c, d, (repeat(f, a, b, c, d,  n-1, z)))      # recursive part repeating the def with n-1


def vrep(ly):              # plots a graph using a list of points
```

```python
    plt.plot(np.real(ly), np.imag(ly), 'ro', markersize=5)                    # plotting graph
    plt.show()
    return


def pointc(a, b, c, d, z, n, m):              # creating a list of points using the previous functions
    # if not determinanttest(a, b, c, d):
    #     return "Error"
    # else:
    ly = []  # list creation
    for i in range(n):
        ly.append(repeat(tz, a, b, c, d, i, z))  # adding points to the list using the composite function
    for j in range(m):
        ly.append(repeat(tzinv, a, b, c, d, j, z))
    return ly


def final(a, b, c, d, z, n, m):               # function combining the other functions
    vrep(pointc(a, b, c, d, z, n, m))
    return
```

Superposition of multiple orbits:

```python
import Math_exp_project as Mp
import matplotlib.pyplot as plt
import numpy as np
import cmath


def sp(ly, q):                 # plotting a graph with multiple orbits using a list of lists
    fig = plt.figure()           # creating a graph
    for i in range(q):
        rgb = np.random.rand(3, )    # choosing a random colour
        ax = fig.add_subplot(111)    # adding the axis to the graph
        ax.scatter(np.real(ly[i]), np.imag(ly[i]), 5, color=rgb)    # adding the elements of the list ly to the graph
        plt.setp(ax.get_xticklabels(), visible=True)    # making the the tick labels visible on the graph
        plt.setp(ax.get_yticklabels(), visible=True)
    plt.xlim(-10, 10)   # setting the range of the x axis
    plt.ylim(-10, 10)   # setting the range of the y axis
    plt.gca().set_aspect('equal', adjustable='box')     # the x and y dimensions are set to be
    plt.show()                                          # of the same length in data coordinates
    return


def lc(a, b, c, d, z, n, m, q):          # using the previous function and two matrices to create a graph
    ly = []                              # creating an empty list
    for i in range(q):
        ly.append(Mp.pointc(a, b, c, d, z-0.1j*i, n, m))    # adding elements to the list using
    return sp(ly, q)                                        # the pointc function of the previous code
```

```

def calc(a, b, c, d, e, f):  # matrices multiplying different matrices; used mainly to create matrices for various tests
    if a*d - b*c == 0:        # checking if the determinant is 0
        return "Error"
    else:
        x = np.array([[a, b], [c, d]])  # defining a matrix
        y = np.linalg.inv(x)            # defining the inverse matrix of y
        z = np.array([[cmath.exp(e), 0], [0, cmath.exp(f)]])  # defining a matrix
        r = np.dot(np.dot(x, z), y)                          # multiplying the three matrices in the order x*y*z
    return np.ndarray.item(r, 0), np.ndarray.item(r, 1), np.ndarray.item(r, 2), np.ndarray.item(r, 3)   # returning
    # the elements of the resulting matrix
```

Subgroups generated by two elements:

```
import matplotlib.pyplot as plt
import numpy as np
import Math_exp_project as Mp
import random


def ttest(x):                 # using the trace to determine the type of the map for testing purposes only
    if np.imag(x) != 0:     # if the trace is an imaginary number then we have a loxodromic map
        return "Loxodromic map"
    else:
        if x > 2 or x < -2:     # if the trace is not between 2 and -2 (excluded) then the map is also loxodromic
            return "Loxodromic map"
        else:
            if x == 2 or x == -2:       # if the trace is equal to either 2 or -2 then the map is parabolic
                return "Parabolic map"
            else:
                if x < 2 or x > -2:     # if the trace is between 2 and -2 (excluded) then the map is elliptic
                    return "Elliptic map"
        return


def trtest(a, d, e, h):          # output for the Trace test; likewise only for testing purposes and not included in the
    print("La Trace de A est:", a+d, ttest(a+d), "La Trace de B est:", e+h, ttest(e+h))  # main function
    return


def mapping(a, b, c, d, e, f, g, h, li):        # function calculating the different points using matrices and a list
    ly0 = []                                    # of starting points
    ly1 = []                                    # creation of 4 empty lists
    ly2 = []
    ly3 = []
    for i in range(len(li)):
        ly0.append(Mp.tz(a, b, c, d, li[i]))    # using the tz and tzinv function of the first code with the
        ly1.append(Mp.tzinv(a, b, c, d, li[i]))     # first  matrix to add new elements to the lists
        ly2.append(Mp.tz(e, f, g, h, li[i]))    # using the tz and tzinv function
```

```
    of the first code with the
        ly3.append(Mp.tzinv(e, f, g, h, li[i]))     # second  matrix to add new
    elements to the lists
    ly = ly0 + ly1 + ly2 + ly3      # combing all four lists into one list
    return ly                       # returning the aforementioned list


def fcalc(a, b, c, d, e, f, g, h, li, n):       # iterative function calling upon
    the 'mapping' function
    lx = [*li]                                  # defining a list with the same
    elements as the input list
    for i in range(n):
        lx = [*lx, *mapping(a, b, c, d, e, f, g, h, lx)]    # adding new elements
    to the list using the previous
        lx = list(set(lx))      # function; the new list only contains the
    elements of the other lists and no lists
    print(len(lx))              # to avoid duplicate elements in the list we
    transform the list into a set (which by
    vrep(lx)                    # definition does not possess duplicate elements
    and then we transform the set back
    return                      # into a list; then we call the vrep function to
    create a graph


def vrep(ly):                                   # plotting a graph using a list
    plt.plot(np.real(ly), np.imag(ly), 'ro', markersize=0.1)
    plt.show()
    return


def randl(m):                   # creating a list with random complex numbers for
    testing purposes
    lx = []
    for i in range(m):
        for n in range(m):
            lx.append(random.random()+random.random()*1j)
    return lx
```

5 Conclusion

At the beginning of this project we created images using a single matrix and proceeded to describe these while splitting them into different groups depending on their trace. In fact as we've seen in the mathematical part 3.3 we can have either loxodromic, parabolic or elliptic maps. For each type of map we got a unique image. Later on we did the same thing only this time we used two matrices to calculate the orbits of these matrices. Thus we created figures 13-24 which are not only quite beautiful but also show us the fractal limit sets. Although we quickly discovered that by using random matrices we didn't get a lot of nice figures, mostly random clusters of points, we still managed to get at least one nice figure using random matrices. We also managed to reproduce some interesting examples of fractals from the Book 'Indra's Pearls: The Vision of Felix Klein' [15]. After delighting in this brief insight about fractal limit sets and Schottky groups, there's only on thing left to say:

God help us – for art is long, and life so short

-Johann Wolfgang von Goethe, Faust, First Part

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Remark:
The title page picture was created by Gil Moes and Mirza Muharemovic using 'Chaotica'.