Bachelor in Mathematics

Experimental Mathematics Lab

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Tuning your piano, and why your are bound to fail doing so

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1 Introduction and first definitions

The goal of this project is to study and compare different musical tuning systems in order to prove that it is physically impossible to find a way to perfectly tune a piano, or any other instrument. Throughout this report, we will therefore define and analyze a few music theory concepts using mathematics.

The content of this first section will cover basic definitions that will allow us to introduce the subject and study tuning systems in the next section.

First, let us define a unit of measurement that will help us quantify the change of size in intervals and thus compare the different ways to tune an instrument.

Definition 1.1. *Cents* are a logarithmic unit of interval measurement. A *cent* is defined as the logarithmic division of the 12-tone equal temperament semitone into 100 equal parts, that is that a semitone of the 12-tone equal tempered scale is equal to 100 cents. We denote cents as ¢ and we can compute the value in cents of the interval between any two frequencies or notes as follows:

$$¢ = 1200\frac{\ln(\frac{f_2}{f_1})}{\ln 2} = 1200 \cdot \log_2 \frac{f_2}{f_1}$$

Let us now define a tuning system that will then be used as a reference for other systems.

Definition 1.2. *Just intonation* defines the attempt to tune all intervals in a scale as fractions, or whole number ratios, of frequencies. An interval tuned in such a way is called *pure*, or a *just interval.*

Example 1.3. Let us consider a reference pitch which vibrates at 200 Hz, that is that it vibrates at 200 cycles per second. One way to visualize this is to picture a string vibrating at this frequency. If we wanted to play another note vibrating at 400 Hz, that is an octave higher than our reference pitch, we would then need to make the chord twice as short, which corresponds to a ratio of 2/1.

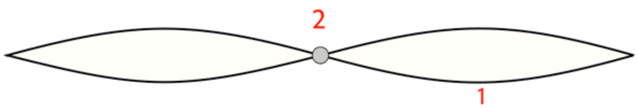

Figure 1: Creation of an octave on a string [2]

If we now consider a pitch vibrating at 300 Hz, we then obtain a ratio of 3/2 from our reference pitch. This corresponds to a perfect fifth, which sounds particularly good.

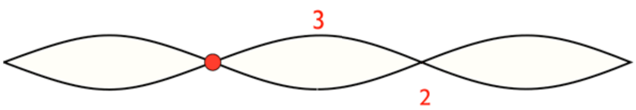

Figure 2: Creation of a pure fifth on a string [2]

Let us now give a concrete example using notes. Let us consider C as our reference pitch (that is, we are in the key of C), then C is denoted as 1/1. In terms of intervals, that corresponds to an unison. If we now consider a C that is an octave above, this would be denoted as 2/1. If we then consider the perfect fifth above, which corresponds to G, we obtain a ratio of 3/2, as explained earlier, while the perfect fourth, F, would be denoted as 4/3.

Extending this idea to other intervals, we get the following table:

Interval	**Frequency ratio from starting point**
Unison	1/1
Semitone	16/15
Tone	9/8
Major third	5/4
Minor third	6/5
Perfect fourth	4/3
Tritone	7/5
Perfect fifth	3/2
Major sixth	5/3
Minor sixth	8/5
Major seventh	15/8
Minor seventh	9/5
Octave	2/1

Table 1: Intervals and their corresponding ratios in just intonation

Definition 1.4. *Wolf interval* A wolf interval exists in certain temperament scheme circles of fifths.The dissonance is in sharp contrast to the harmony.It is an imperfect fifth that makes the other fifths perfect.A traditional belief says that its discord is like the howl of a wolf.

Definition 1.5. A *comma* is an interval representing the difference in pitch between two enharmonic notes tuned according to the same tuning system. There exist multiple types of commas, but the two most important ones are the *syntonic comma* and the *Pythagorean comma*.

Definition 1.6. The *syntonic comma* is defined as the difference between the Pythagorean major third (a frequency ratio equal to 81/64, as we are gonna see in the next section) and the just major third (which we have defined as a ratio of 5/4). This comma is equal to the frequency ratio $81/80 = 1.0125$. This is equivalent to approximately 21.5 cents, a unit of measurement that we are going to define later on. This comma Such an interval between two notes would be noticeable to anyone, and would sound like out-of-tune variations of the same note since the difference is rather close.

Now that we have seen how pure intervals are defined, we can ask ourselves if there exist other tuning systems altering the size of pure intervals.

2 Temperaments

Definition 2.1. In music theory, a *temperament* is a tuning system which compromises to a certain extent the size of pure intervals to meet the new requirements of the system. *Tempering* is the process of altering the size of intervals by making them slightly narrower or wider than pure.

Temperament is important for keyboard instruments; it allows the player to play only the pitches assigned to the various keys and lack any way to alter pitch of a note in performance. Tempering involves transferring a proportion of the imperfect sounds to the more perfect ones in order to compensate in some way for the false intervals of instruments whose sounds are fixed.
We are now able to define a few important temperaments.

2.1 Pythagorean tuning

Definition 2.2. *Pythagorean tuning* defines a tuning system in which all intervals are based on the frequency ratio of a pure perfect fifth, that is 3/2.

Modern authors of music theory have mostly credited the method to Pythagoras, whereas Ptolemy and later Boethius credited Eratosthenes with the division of the tetrachord by merely two intervals, known as *semitonium*, *tonus* in Latin.

Up until the beginning of the 16^{th} century, musicians used the *Pythagorean tuning.*

While the Pythagorean system appears to be ideal because of the purity of the fifths, some people believe that other intervals, particularly the major third, are so horribly out of tune that major chords are a dissonance.

Definition 2.3. The *Pythagorean comma* are minimal intervals in musical tuning. They are the intervals between enharmonically equivalent notes such as C and $B\#$, or D and C.
The frequency ratio is

$$\frac{(1.5)^{12}}{2^7} = \frac{521441}{524288} \approx 1.01364$$

or approximately 23.46 cents, or roughly a quarter of a semitone. Among musical temperaments, the Pythagorean comma is often used to describe tempering.

Fraction of a comma	**Ratio**
1/315 (nearly Pythagorean tuning)	9 : 4
1/11 (1/12 Pythagorean comma)	2 : 1
1/6	9 : 5
1/5	7:4
1/4	5:3
2/7	8:5
1/3	3:2
1/2	4:3

Table 2: Intervals and their corresponding ratios in Pythagorean tuning

Any scale made up entirely of pure perfect fifths (3:2) and octaves is known as the Pythagorean scale (2:1). It was used to tune tetrachords, which were formed into octave-spanning scales in Greek music.

However, in 12-tone Pythagorean temperament, one is limited by 12-tones per octave, making it impossible to perform most music using the Pythagorean method, which corresponds to enharmonic notation; instead, the decreased sixth becomes a *wolf fifth*.

- Method

  12-tone Pythagorean temperament is built on a series of perfect fifth intervals, each tuned in the 3:2 ratio, the next simplest after 2:1. Starting with D (D-based tuning), six additional notes are produced by moving the same ratio up six times, and the other notes are produced by moving the same ratio down:

  $$Eb–Bb–F–C–G–D–A–E–B–F\#–C\#–G\#$$

  This sequence of eleven 3:2 intervals covers a wide frequency range, on a piano keyboard, it encompasses 77 keys. Notes with frequencies that differ by a factor of two are given the same name, it is common practice to divide or multiply the frequencies of some of these notes by 2 or a power of 2.

  The goal of this adjustment is to move the 12 notes into a smaller frequency range, specifically between the base note D and the D above it. A note with twice its frequency. The basic octave is the name given to this interval, on a piano keyboard, an octave has only 12 keys.

  The A is set to a frequency that is 3/2 times that of D. If D is tuned to $288Hz$, then A is tuned to $432Hz$. Similarly, the E above A is tuned to have a frequency that is 3/2 times that of A or 9/4 times that of$D$.With A at $432Hz$, this puts E at $648Hz$.

  E is outside the basic octave, its frequency is more than twice that of the base note D. So it is common to halve its frequency to bring it into the octave.

  As a result, E is tuned to $324Hz$, which is a 9/8 higher than D. The ratio 27:16 is adjusted by the B at 3/2 above the E, and so on.

  Beginning at the same point and proceeding in the opposite direction, G is tuned 3/2 below D, which means it has a frequency that is 2/3 that of D. With D at $288Hz$, this puts G at $192Hz$. To get it into the basic octave, this frequency is doubled to $384Hz$.

  However, there is an issue with extending this tuning: no stack of 3:2 intervals (perfect fifths) will fit exactly inside any stack of 2:1 intervals. Consider this stack, which was created by adding one additional note to the previous one.

  $$Ab–Eb–Bb–F–C–G–D–A–E–B–F\#–C\#–G\#$$

  This will be similar in size to a stack of seven octaves, but not identical. It will be about a quarter of a semitone larger, which is known as the Pythagorean comma. As a result, when Ab and Gb are brought into the fundamental octave, they will not match up as predicted.
  The table below shows the conventional name of the interval from D,the base note.

The formula to compute its frequency ratio, its size in cents, and the difference in cents between its size and the size of the corresponding one in the equally tempered scale for each note in the basic octave.

Note	Interval	Ratio
Ab	diminished fifth	1024/729
Eb	minor second	256/243
Bb	minor sixth	128/81
F	minor third	32/27
C	minor seventh	16/9
G	perfect fourth	4/3
D	unison	1/1
A	perfect fifth	3/2
E	major second	9/8
B	major sixth	27/16
F#	major third	81/64
C#	major seventh	243/128
G#	augmented fourth	729/512

Figure 3: The diatonic scale according to the Pythagorean tuning

2.2 Meantone temperaments

Meantone temperament is an attempt to avoid Pythagorean tuning or just intonation difficulties. In truth, there are various meantone temperament systems, but they all have the same propensity to narrow the fifths in order to keep the purity of the *thirds*.

Definition 2.4. A *meantone temperament* is a temperament obtained by flattening the fifths and making them slightly narrower than a perfect fifth (that is that their ratio is slightly less than 3/2), which allows to push the major thirds a little closer to a 5/4 ratio. As there are several ways to compromise the fifths here, there are multiple meantone temperaments.

The method is known as meantone temperament because each whole tone is the same size, with $192C$ being the average of($182C$ or 10/9) and ($203C$ or 9/8) whole tones. Pure major thirds emerge as a result of this.

Two pure fifths make up a major full tone:

$$702 + 702 \equiv 204 \pmod{1200}$$

If half of the syntonic comma is narrowed to get the mean, each fifth must be narrowed by a fourth of the comma. Because of this, the system is also known as one-quarter comma meantone tuning. Pietro Aron is credited with the first meantone tuning (1523).

Meantone tuning produces nine relatively excellent keys (Bb, F, C, G, D, A, g, d, and a), however the tempering also reduces the fifths to 696.6 *cents*. The incomplete fifth's impurity, on the other hand, is only noticeable when it occurs as an open interval. Regardless, the difference between the final fifth to be tempered and an equally tempered fifth is 38 C (113.4 C). Between the E flat and the G sharp, a large fifth (738 C) is commonly placed.

The major second is formed by two fifths minus one octave, while the major third is formed by four fifths minus two octaves.

The percentage of the syntonic comma by which the fifths are tempered is often used to describe meantone temperaments. The most common type, *quarter-comma meantone*, tempers the fifths by $\frac{1}{4}$ of a syntonic comma, resulting in four fifths producing a just major third, a syntonic comma lower than a Pythagorean major third.
Third-comma meantone tempers by $\frac{1}{3}$ of a syntonic comma, resulting in three fifths producing a just major sixth and thus a just minor 3^{rd}.

A linear temperament is distinguished by the width of its generator in a meantone temperament, which is often measured in cents.

Meantone temperaments can be specified in a number of ways, including the fraction (logarithmically) of a syntonic comma by which the fifth is flattened , the equal temperament in which the meantone fifth is found, the width of the tempered perfect fifth in cents, or the ratio of the whole tone to the diatonic semitone.

2.3 Equal Temperament

From the end of the 19^{th} century, Equal Temperament gradually replaced the various well temperaments as the standard tuning system. The semi-tones are 100 cents wide, allowing modulation to any key in a tuning system that is the same in all keys. Nevertheless, equal temperament can be traced back much further than this.

By the middle of the 1500s, this tuning was accepted as the standard for lutes, a compromise solution that made sense because it was the only way to tune a fretted string instrument. Among the oldest written references is one by Abbott Girolamo Roselli in 1588 and so it is ironic that keyboard instruments weren't adopted until much later.

If we compare Equal Temperament with a well-tempered temperament, we see a more contentious difference.

If we examine the tunings one after the other, Equal Temperament is identical in each key, while Werckmeister's intervals vary within different keys, resulting in 'colouring'.

Having travelled through a myriad of different tunings and temperaments, it is a strange coincidence that we have arrived in the 21^{st} century at a tuning similar to Pythagorean tuning. Based on their close proximity to one another, the 5ths are extremely close and the 3rds are both sharper than their natural ratios and are only 8 cents apart, so modern Equal Temperament is a better tuning and temperament to play Medieval music than all those in between.

Definition 2.5. *Equal temperament* defines any tuning system that divides the octave into equal intervals. Pitch is perceived as the logarithm of frequency, which means that there is an equal step size when two adjacent notes are of the same frequency.

In Western music, the equal tempered scale has been the most commonly used scale since the 18^{th} century.

This scale divides the octave into 12 equal semitones with a frequency ration equal to the 12^{th} root of 2 ($\sqrt[12]{2} \approx 1.05946$). This is equivalent to saying that each semitone is equal to 100 cents.

Figure 4: Octave divided into 12 equal semitones [9]

When used without qualification, equal temperament refers to 12-TET. That resulting smallest interval, $\frac{1}{12}$ the width of an octave, is called a semitone or half step.

440 Hz has not always been the standard pitch. In the past few hundred years, it has changed and has generally risen.

The Unfretted String Orchestra can adjust the tuning of all notes except the open strings, as well as the vocal group without mechanical tuning restrictions, and sometimes use a tuning closer to the pitch for acoustic reasons. Other musical instruments, such as some wind instruments, keyboard instruments, and fret instruments, usually only have approximately equal temperament, and technical limitations prevent precise tuning. Some wind instruments that can easily and spontaneously bend their timbre, especially the trombone, use tuning similar to string ensemble and vocal group.

3 Comparing the different tuning systems

Let us now compare the different tuning systems previously defined. In order to do so, we first write a program. We will then analyze the numerical results thus obtained.

3.1 Python comparison program

```python
import math
from fractions import Fraction

#we define a function to compute the value in cents of any frequency ratio
def cents(n, d):
    r = Fraction(n, d)
    r = 1200*math.log(r, 2)
    r = round(r)
    return r

def Comparison():
    print('Tuning Systems :')
    print('1 : Just Intonation')
    print('2 : Equal Temperament')
    print('3 : Pythagorean Tuning')
    print('4 : 1/4 Comma Meantone')
    a = str(input('Enter the tuning system of reference:'))
    b = str(input('Enter the tuning system you want to compare the first
   one to:'))
    x = []
    y = []
    Comp = []
    if a == '1':
        x = cJustInto
    elif a == '2':
        x = cEqualTemp
```

```
26     elif a == '3':
27         x = cPytha
28     elif a == '4':
29         x = cCommaMean
30     if b == '1':
31         y = cJustInto
32     elif b == '2':
33         y = cEqualTemp
34     elif b == '3':
35         y = cPytha
36     elif b == '4':
37         y = cCommaMean
38 
39     for i in range(13):
40         Comp.append(x[i] - y[i])
41     print(Comp)
42 
43 
44 
45 #we create a list with the frequency ratios of the different intervals of a
     chromatic scale in just intonation
46 JustInto = [[1,1], [16,15], [9,8], [6,5], [5,4], [4,3], [64,45], [3,2],
    [8,5], [5,3], [9,5], [15,8], [2,1]]
47 
48 #we convert the ratios of the previous lists into cents
49 cJustInto = [cents(item[0], item[1]) for item in JustInto]
50 
51 cEqualTemp = [0, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, 1100,
    1200]
52 
53 Pytha = [[1,1], [256,243], [9,8], [32,27], [81,64], [4,3], [729,512],
    [3,2], [128,81], [27,16], [16,9], [243,128], [2,1]]
54 
55 cPytha = [cents(item[0], item[1]) for item in Pytha]
56 
57 cCommaMean = [0, 117, 193, 310, 386, 503, 579 ,697, 814, 890, 1007, 1083,
    1200]
```

The goal of this program is to compare the values (in cents) of the intervals of a scale tuned according to two different tuning systems.

First, we import two packages that will later be of use to us (lines 1-2). We then define a function called 'cents', which takes two integers as arguments, transform them into a fraction and computes the corresponding value of this ratio in cents (lines 6-7). This function finally rounds to the closest integer the value obtained (line 8) before returning it (line 9).

At line 11, we now define the function that we call 'Comparison'. It will constitute the core of our program. From lines 12 to 16 the function only prints out some information for the user. It displays the tuning systems that the user can choose from and how to properly enter them later on. The two variables a and b introduced at lines 17 and 18 allow the user to choose the tuning systems he or she wants to compare. As indicated before, 1 corresponds to just intonation, 2 to equal temperament, 3 to pythagorean tuning and 4 corresponds to 1/4-comma meantone.

We then introduce three empty lists x, y and Comp. The list x will be replaced with the list of the values (in cents) of the size of the intervals according to the tuning system first chosen, while the list y will be replaced with the list of the values (in cents) of the size of the intervals according to the tuning system the user wants to compare the first one to. This is exactly what happens at lines 22-37.

At line 39, we finally define a small for-loop which will append to the previously empty list called 'Comp' the difference in size of each interval tuned according to the two tuning systems we are comparing. The range of the for-loop goes from 0 to 13 since we are considering 13 intervals (including unison) in our chromatic scale. The 'Comparison' function ends right after this, by printing the complete 'Comp' list.

Finally, we define the different lists that are being used in the 'Comparison' function. We start with the list of the frequency ratios of the different intervals of the chromatic scale in the key of C in just intonation (line 46), and we convert those ratios in cents by applying the 'cents' function previously defined to each element of the 'JustInto' list. At line 51, we define a list of the size, in cents, of the intervals according to equal temperament. At lines 53-55, we apply the same procedure as in lines 46-49, but, this time, for the pythagorean tuning, and lastly we write a list of the size of the intervals, in cents, according to the 1/4-comma meantone (line 57). In this last line, we chose to work with the size in cents directly since the formulas of the ratios were difficult to implement and handle in Python. We did the exact computations the 'cents' function would have done for us, therefore, this will not affect the final results in any way.

3.2 Comparison of just intonation and equal temperament

Every comparison is going to be defined in the key of C in the tables. Pursuing, we are going to define the tuning systems in every key.

The colors in the tables define if the differences in pitch are noticeable or not. The orange color stands for all the positive values where the differences in pitch can be distinguished and the blue color stands for all the negative values where the differences in pitch can be distinguished. Every value which is not colored is because these differences are so small that they are not able to be differentiated.

Let us compare Just intonation to Equal temperament and see what happens.
We get the following results (all results are rounded):

Interval	**Difference between just intonation and equal temperament (in cents)**
Unison	0
Semitone	12
Tone	4
Minor third	16
Major third	-14
Perfect fourth	-2
Tritone	10
Perfect fifth	2
Minor sixth	14
Major sixth	-16
Minor seventh	18
Major seventh	-12
Octave	0

Table 3: Comparison of just intonation and equal temperament

Let's have a little synopsis of our results.

What we already know is that every difference of cents under the value of 5 cents is not able to be distinguished. If we look at our table we can say that Just intonation and Equal temperament cannot be distinguished in our intervals unison and octave where the difference of these two temperaments lies at 0 cents. This is because they are both defined from the same frequency ratio, which then makes it normal (for every tuning system) that the differences in pitch in these two intervals are identical and 0. The same goes for tone, because the difference of these two temperaments is 4 cents, which lies under the value of 5 cents.

Then, if we look at minor third and major third we see that the values are quite larger than 5 cents, which means that the equivalences in pitch of these two temperaments can be distinguished well.

But, the highest difference of our two tuning systems to recognize is in our interval minor seventh as our value of difference lies at 18 cents.

The difference we hear between the different tones in just intonation and in equal temperament is therefore substantial.

This is good to know, but what is the most important thing here?
Let us have a closer look at the major third and the perfect fifth. These two intervals are the most important ones, as they are used especially in Classical music, but not only.

One big problem we can see is that our major third is too big, as it goes way over 5 cents with approximately -14 cents. -14 is a large number, as the difference between these two tuning systems is really big to hear.
Our perfect fifth has a value of approximately 2 cents, which is a low value. It is close to being indistinguishable, considering the difference of the two temperaments is close to 0 cents.

3.3 Comparison of 1/11 comma meantone and equal temperament

The comparison of 1/11 comma meantone and equal temperament is very interesting and a bit different than others.

When we compare 1/11 comma meantone and equal temperament, we get the following results:

Interval	Difference between 1/11 comma meantone and equal temperament (in cents)
Unison	0
Semitone	0
Tone	0
Minor third	0
Major third	0
Perfect fourth	0
Tritone	0
Perfect fifth	0
Minor sixth	0
Major sixth	0
Minor seventh	0
Major seventh	0
Octave	0

Table 4: Comparison of 1/11 comma meantone and equal temperament [10]

What we obtain here is that equal temperament and 1/11 comma meantone tuning are almost exactly the same. This means that there is no difference in pitch to hear. In other words, every interval where we compare the difference of both of the tuning systems together has a value of 0 cents, which means that they are equal.

What is really interesting here is that these two tuning systems are not the same, yet they seem to not have a difference between each other if we calculate the difference in pitch between them. They sound the same.

If we look now at our most important intervals, which are major third and perfect fifth we see that they seem to be quite perfect as they have the value 0.

4 Comparing the different keys within a tuning system

In the previous section, for simplicity, all the comparisons were done in the key of C. But let us now focus on the variations we can observe within a tuning system. Indeed, the size of the intervals of a scale will change depending on the key, and that is what the next program will allow us to study.

4.1 Python shift program

```python
import math
from fractions import Fraction

#defining a function to compute the value in cents of any frequency ratio
def cents(n, d):
    r = Fraction(n, d)
    r = 1200*math.log(r, 2)
    r = round(r)
    return r

```

```
11 KeyShift = {'C': 0, 'C#': 1, 'Db': 1, 'D': 2, 'D#': 3, 'Eb': 3, 'E': 4, 'F': 5, 'F#': 6, 'Gb': 6, 'G': 7, 'G#': 8, 'Ab': 8, 'A': 9, 'A#': 10, 'Bb': 10, 'B': 11}
12
13 def Shift():
14     print('Tuning Systems :')
15     print('1 : Just Intonation')
16     print('2 : Equal Temperament')
17     print('3 : Pythagorean Tuning')
18     print('4 : 1/4 Comma Meantone')
19     a = str(input('Enter the tuning system of your choosing:'))
20     b = str(input('Enter the key of your choosing:'))
21     x = []
22     y = []
23     if a == '1':
24         x = cJustInto
25     elif a == '2':
26         x = cEqualTemp
27     elif a == '3':
28         x = cPytha
29     elif a == '4':
30         x = cCommaMean
31
32     if b in KeyShift.keys():
33         y = [item - x[KeyShift[b]] for item in x]
34         for i in range(KeyShift[b]):
35             y[i] = y[i] + 1200
36         y.append(1200)
37         y.sort()
38         print(y)
39
40
41 #we create a list with the frequency ratios of the different intervals of a major scale in 'C' in just intonation
42 JustInto = [[1,1], [16,15], [9,8], [6,5], [5,4], [4,3], [64,45], [3,2], [8,5], [5,3], [9,5], [15,8]]
43
44 #we convert the ratios of the previous lists into cents
45 cJustInto = [cents(item[0], item[1]) for item in JustInto]
46
47 cEqualTemp = [0, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, 1100]
48
49 Pytha = [[1,1], [256,243], [9,8], [32,27], [81,64], [4,3], [729,512], [3,2], [128,81], [27,16], [16,9], [243,128]]
50
51 cPytha = [cents(item[0], item[1]) for item in Pytha]
52
53 cCommaMean = [0, 117, 193, 310, 386, 503, 579 ,697, 814, 890, 1007, 1083]
```

This program essentially works in the same way as the previous program. At lines 1-2, we import the same packages as before, and at line 5 we define the 'cents' function again. At the end of the program (lines 42-53), we create the same lists that we previously used to define the size of the intervals in the different tuning systems we want to study.

Let us now have a closer look at the parts where this program differs from the first one. At line 11, we define a dictionary in which the keys are the keys that the user will later on be able to enter, and the values associated are the shift that the following function will have to apply.

At line 13, we finally define a function called 'Shift' that will allow us to compute the size of the intervals in a given key for a given tuning system. From lines 14 to 22, the function is the same as the 'Comparison' one from the previous section, except that the variable b allows the user to enter a key and that only the list x will be replaced by one of the lists defined at lines 42-53. This is what happens from lines 23 to 30.

We then make sure that the user chose a valid key (line 32), and if it the case, we create a list made from the shifted elements of the original list which contained the size of the intervals in the key of C for the tuning system in question. After that, we only have to add 1200 (which correspond to the size of an octave) to the first items of the new list since they are now negative (because of the shift).

Lastly, we append 1200 (the octave) to this list, as we chose not to add it to the lists in lines 42-53. If we did, we would have gotten an interval twice because of the shift. At line 37, we proceed to sort the items of the list and at line 38 we print the given list, which gives us a list of the size of the intervals in a given key, for a given tuning system.

4.2 Just intonation

Let us now use the program introduced right above to compare the size of the intervals depending on the keys we consider our scale in. We compile in the following table the results obtained for just intonation.

Let us remind that 1 denotes the unison, m2 the minor second, M2 the major second, m3 the minor third, M3 the major third, 4 the perfect fourth, A4 the tritone, 5 the perfect fifth, m6 the minor sixth, M6 the major sixth, m7 the minor seventh, M7, the major seventh and 8 the octave.

In the following tables, we chose to color in blue the differences in size lesser than -5 while we colored in orange the differences in size greater than 5. This is due to the fact that we know that any difference in pitch greater than 5 (in absolute value) is noticeable to anyone. The reference we chose to compare these sizes with is the size of the intervals in the key of C. This means that any value colored in blue corresponds to an interval that is too small, and one colored in orange is too big.

Key	*1*	*m2*	*M2*	*m3*	*M3*	*4*	*A4*	*5*	*m6*	*M6*	*m7*	*M7*	*8*
C	0	112	204	316	386	498	610	702	814	884	1018	1088	1200
C#/Db	0	92	204	274	386	498	590	702	772	906	976	1088	1200
D	0	112	182	294	406	498	610	680	814	884	996	1108	1200
D#/Eb	0	70	182	294	386	498	568	702	772	884	996	1088	1200
E	0	112	224	316	428	498	632	702	814	926	1018	1130	1200
F	0	112	204	316	386	520	590	702	814	906	1018	1088	1200
F#/Gb	0	92	204	274	408	478	590	702	794	906	976	1088	1200
G	0	112	182	316	386	498	610	702	814	884	996	1108	1200
G#/Ab	0	70	204	274	386	498	590	702	772	884	996	1088	1200
A	0	134	204	316	428	520	632	702	814	926	1018	1130	1200
A#/Bb	0	70	182	294	386	498	568	680	792	884	996	1066	1200
B	0	112	224	316	428	498	610	722	814	926	996	1130	1200

Table 5: Size of the intervals (in cents) of the chromatic scale, depending on keys and according to just intonation

The intervals we want to focus our analysis on are the minor and major thirds and the fifth, since they are the most commonly used in western music (whether classical or modern). Here, we notice that the minor third is often too narrow while the major one is a bit too big. The fifth is also a bit too big or too small depending on the key we choose. Those are situations we want to avoid, meaning this tuning system is really flawed.

4.3 Equal temperament

Key	*1*	*m2*	*M2*	*m3*	*M3*	*4*	*A4*	*5*	*m6*	*M6*	*m7*	*M7*	*8*
C	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
C#/Db	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
D	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
D#/Eb	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
E	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
F	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
F#/Gb	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
G	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
G#/Ab	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
A	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
A#/Bb	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
B	0	100	200	300	400	500	600	700	800	900	1000	1100	1200

Table 6: Size of the intervals (in cents) of the chromatic scale, depending on keys and according to equal temperament

If we now analyze equal temperament, we notice that the size of the intervals never changes, regardless of the key. Thus, equal temperament is the most consistent tuning system of all. The intervals might not be perfect and differ a bit from the ones used in just intonation, but they won't change and will always sound the same to the listener. This consistency is one of the major reasons why we currently use this tuning system to tune our instruments.

4.4 Pythagorean tuning

Key	*1*	*m2*	*M2*	*m3*	*M3*	*4*	*A4*	*5*	*m6*	*M6*	*m7*	*M7*	*8*
C	0	90	204	294	408	498	612	702	792	906	996	1110	1200
C#/Db	0	114	204	318	408	522	612	702	816	906	1020	1110	1200
D	0	90	204	294	408	498	588	702	792	906	996	1086	1200
D#/Eb	0	114	204	318	408	498	612	702	816	906	996	1110	1200
E	0	90	204	294	384	498	588	702	792	882	996	1086	1200
F	0	114	204	294	408	498	612	702	792	906	996	1110	1200
F#/Gb	0	90	180	294	384	498	588	678	792	882	996	1086	1200
G	0	90	204	294	408	498	588	702	792	906	996	1110	1200
G#/Ab	0	114	204	318	408	498	612	702	816	906	1020	1110	1200
A	0	90	204	294	384	498	588	702	792	906	996	1086	1200
A#/Bb	0	114	204	294	408	498	612	702	816	906	996	1110	1200
B	0	90	180	294	384	498	588	702	792	882	996	1086	1200

Table 7: Size of the intervals (in cents) of the chromatic scale, depending on keys and according to pythagorean tuning

Concerning pythagorean tuning, we can observe that the size of the fifth is uniform throughout the keys, and is equal to the one used in just intonation. Nevertheless, the minor and major thirds are either too big or too small, which is flaw we cannot neglect despite the perfection of the fifth.

4.5 Quarter-comma meantone

Key	*1*	*m2*	*M2*	*m3*	*M3*	*4*	*A4*	*5*	*m6*	*M6*	*m7*	*M7*	*8*
C	0	117	193	310	386	503	579	697	814	890	1007	1083	1200
C#/Db	0	76	193	269	386	462	580	697	773	890	966	1083	1200
D	0	117	193	310	386	504	621	697	814	890	1007	1124	1200
D#/Eb	0	76	193	269	387	504	580	697	773	890	1007	1083	1200
E	0	117	193	311	428	504	621	697	814	931	1007	1124	1200
F	0	76	194	311	387	504	580	697	814	890	1007	1083	1200
F#/Gb	0	118	235	311	428	504	621	738	814	931	1007	1124	1200
G	0	117	193	310	386	503	620	696	813	889	1006	1082	1200
G#/Ab	0	76	193	269	386	503	579	696	772	889	965	1083	1200
A	0	117	193	310	427	503	620	696	813	889	1007	1124	1200
A#/Bb	0	76	193	310	386	503	579	696	772	890	1007	1083	1200
B	0	117	234	310	427	503	620	696	814	931	1007	1124	1200

Table 8: Size of the intervals (in cents) of the chromatic scale, depending on keys and according to quarter-comma meantone

Lastly, the results we obtain for quarter-comma meantone are really similar to the ones we had for pythagorean tuning, except for a small discrepancy in the key of F# for the fifth.

As for pythagorean tuning, we cannot really ignore these problems, which add up to the ones we encounter when we compare this temperament to others.

5 Conclusion

Now that we have numerically and musically compared a few temperaments, we can say that there is no perfect way to tune an instrument, and we are thus bound to fail doing so. While just intonation offers pure intervals that sound really nice, it is inconsistent over different keys. Pythagorean tuning and meantone temperaments were nice alternatives and offered almost pure fifths over keys, but the other intervals were too dissonant. This justifies the choice for equal temperament nowadays, since the intervals are consistent throughout keys which makes it the most adjustable tuning system. But it is also wrong in many ways as the intervals are often far from being pure when using equal temperament.

We were therefore able to find and justify the best approach to tune a piano, but we will never be able to find a perfect tuning system, as we proved in this report.

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