University of Luxembourg

Experimental Mathematics 2

Arithmetic Billiards

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Chapter 1

Introduction

In this document, we are going to study arithmetic billiards based on the paper *"three-dimensional arithmetic billiards"* by Antonella Perucca.

1.1 What is arithmetic billiards?

Arithmetic billiards provide a geometrical way to find the greatest common divisor and the least common multiple of n natural numbers using a n-dimensional billiard table. In this paper we focus on the 2-dimensional and 3-dimensional billiards.

1.1.1 2-Dimensional

Consider a billiard table with integer side lengths a and b. A ball is shot into the rectangle at 45° with a constant speed and we neglect friction. We fix an origin with the coordinates (0,0) whose opposite point has the coordinates (a,b). Each so called "step" of the ball will modify every coordinate by increasing or decreasing it by 1.

The ball bounces on the billiard sides and stops only if it lands in a corner. There are two types of trajectories, the corner path and closed path. The closed path is a periodic trajectory where the ball never ends up in a corner. The corner path is a trajectory where the ball enters the rectangle through a corner and ends when it hits another corner. If the trajectroy wouldn't end after the ball hits a corner, the corner path would correspond to a closed path.

The arithmetic billiard path of the ball provides information about the greatest common divisor and least common multiple. The least common multiple $lcm(a, b)$ is the number of unit squares crossed by the arithmetic billiard path. The greatest common divisor $gcd(a, b)$ is the number of unit squares crossed by the first segment of the path up to that point of self-intersection. We can also predict what the ending corner of a corner path will be, this will be discussed in the following chapter.

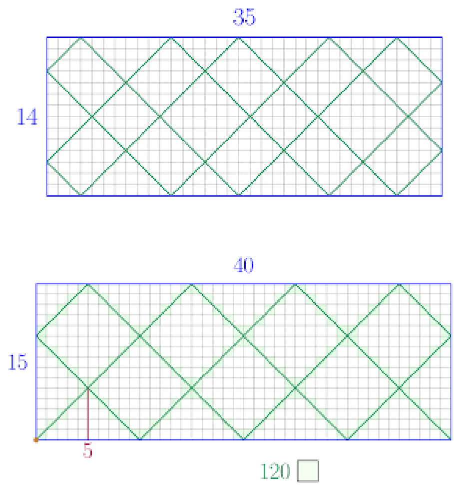


Figure 1.1: closed path and corner path

1.1.2 3-Dimensional

Consider a 3-dimensional billiard table which is parallelepiped and has the integers a,b,c as side lengths.
The ball is moving with a 45° to every face at a constant speed and we neglect friction. We call the face where x=0 and x=a the x-face, where y=0 and y=b the y-face and where z=0 and z=c the z-face.
We fix an origin with the coordinates (0,0,0) whose opposite point has the coordinates (a,b,c). Each so called "step" of the ball will modify every coordinate by increasing or decreasing it by 1.

We call edge points the points that are on the edges and excluding the corners, we call side points the points that are on the sides and not on the edges, and we call interior points the remaining points of the path with integer coordinates. Finally we call boundary points the points of the path which are on the faces (they can be either corners, edge points or side points).

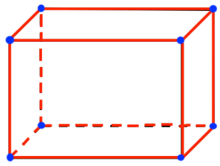


Figure 1.2: edge points (red) and corner points (blue

The trajectory of the ball can be either a corner path or a closed path. A closed path is a trajectory that never reaches a corner and is periodic. A corner path is a trajectory that starts in a corner and ends in another corner. We focus on corner paths in the following. The length of a corner path is $\sqrt{3}lcm(a,b,c)$ or $lcm(a,b,c)$ steps.

In order to get a better understanding of the trajectory of the ball, we introduce projection paths. A projection path the orthogonal projection of the path on of the faces. The projection paths on two parallel faces are clearly the same, so we can focus on the three projections onto the coordinate planes.
Using this we can give another definition of for example edge points. Which will be useful in the third chapter.

In the second part we will concentrate on the study of the ending corner of a corner path in the 2- and 3-dimensional case. And furthermore in the third chapter we will characterise the paths without edge points in the 3-dimensional case.

Chapter 2

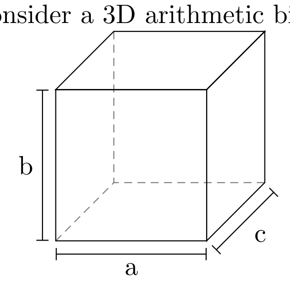
In what corner do we end up?

2.1 2-Dimensional

Consider a 2D arithmetic billiard with sides of lenght a and b.
We consider a classical setting, where the ball is shot from one corner into the billiard at 45°.
It is possible to predict the ending corner of the ball. Like stated in the article *Arithmetic-billiards-revised*: If a/g and b/g are odd, then the starting corner and the ending corner are opposite; if a/g is even and b/g is odd, then the starting and the ending corner are adjacent to one a -side; if a/g is odd and b/g is even, then the starting and the ending corner are adjacent to one b-side.

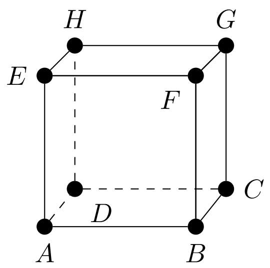
2.2 3-Dimensional

We'll use the information about ending corners of the two-dimensional case in order to predict the ending corner of the three-dimensional case of a corner path.
Consider a 3D arithmetic billiard with sides of lenght a, b and c.



**Note the drawings of cubes represent a general case, a b c can change and do not have to be equal*

We refer to each corner using a letter:



It is possible to determine the ending corner of a ball shot into a 3D-grid, by looking at the parities of

$$\frac{a}{gcd(a,b)}, \frac{b}{gcd(a,b)},$$

$$\frac{c}{gcd(b,c)}, \frac{b}{gcd(b,c)},$$

and

$$\frac{c}{gcd(a,c)} \frac{a}{gcd(a,c)}.$$

For simplicity we write g_1=$gcd(a,b)$, g_2=$gcd(b,c)$, g_3=$gcd(a,c)$.
These couples can either both odd, even and odd or odd and even. Means that there are 3 possible choices for each couple. Since there are 3 of these couples we have 3·3·3 cases. Hence there should be theoretically 27 cases, but this isn't the case, not all are possible.

We suppose wlog $gcd(a,b,c) = 1$. We can say this because it's always possible to find one permutation of a,b,c with the parities we are considering, and if it doesn't work for one permutation it won't work for the other ones either.

Suppose $\frac{a}{g_1}, \frac{b}{g_1}, \frac{c}{g_2}, \frac{b}{g_2}, \frac{c}{g_3}$ all odd and only $\frac{a}{g_3}$ even. This means a is even. But $\frac{b}{g_1}$ is odd, so g_1 is even, and so also b is even. Again, $\frac{b}{g_2}$ is odd, so g_2 is even, so also c is even. So $gcd(a,b,c)$ is at least 2 (as they are all even), this contradicts our assumption.
By symmetry, any case where only one among $\frac{a}{g_1}$, $\frac{b}{g_1}$, $\frac{c}{g_2}$, $\frac{b}{g_2}$, $\frac{c}{g_3}$, $\frac{a}{g_3}$ is even cannot happen.

Then we suppose $\frac{a}{g_1}$, $\frac{b}{g_1}$ are odd, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd.
This implies that a and c are both even. Since $\frac{a}{g_1}$ and $\frac{c}{g_3}$ are odd that implies that g_1 is even. We know that $\frac{b}{g_1}$ is odd which implies that b is as well even. So $gcd(a,b,c)$ is at least 2 (as they are all even), this contradicts our assumption that gcd(a,b,c)=1.
By symmetry, any permutation of a, b, c with parities as above is not possible.

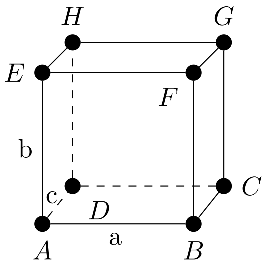
Then we suppose $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd
This implies that a, b, c are all even. So $gcd(a,b,c)$ is at least 2 (as they are all even), this contradicts our assumption that $gcd(a,b,c) = 1$. By symmetry, any permutation of a, b, c with parities as above is not possible.

Now we list the cases that can happen:

1. If $\frac{a}{g_1}$, $\frac{b}{g_1}$ are both odd, $\frac{c}{g_2}$, $\frac{b}{g_2}$ are both odd, $\frac{a}{g_3}$ $\frac{c}{g_3}$ are both odd.
   Then the ending corner will be G. Example: (a,b,c)=(1,1,1).
2. If $\frac{a}{g_1}$, $\frac{b}{g_1}$ are both odd, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is even.
   Then the ending corner will be D. Example: (a,b,c)=(1,5,2).
3. If $\frac{a}{g_1}$, $\frac{b}{g_1}$ are both odd, $\frac{c}{g_2}$ is odd, $\frac{b}{g_2}$ is even, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd.
   Then the ending corner will be F. Example: (a,b,c)=(2,2,1).
4. If $\frac{a}{g_1}$ is even, $\frac{b}{g_1}$ is odd, $\frac{c}{g_2}$, $\frac{b}{g_2}$ are both odd, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd.
   Then the ending corner will be B. Example: (a,b,c)=(2,1,1).
5. If $\frac{a}{g_1}$ is even, $\frac{b}{g_1}$ is odd, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is odd.
   Then the ending corner will be C. Example: (a,b,c)=(2,1,2).
6. If $\frac{a}{g_1}$ is even, $\frac{b}{g_1}$ is odd, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd.
   Then the ending corner will be B. Example: (a,b,c)=(4,1,2).
7. If $\frac{a}{g_1}$ is even, $\frac{b}{g_1}$ is odd, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is even.
   Then the ending corner will be D. Example: (a,b,c)=(1,2,3).
8. If $\frac{a}{g_1}$ is even, $\frac{b}{g_1}$ is odd, $\frac{c}{g_2}$ is odd, $\frac{b}{g_2}$ is even, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd.
   Then the ending corner will be B. Example: (a,b,c)=(4,2,3).
9. If $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$, $\frac{b}{g_2}$ are both odd, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is even.
   Then the ending corner will be H. Example: (a,b,c)=(1,6,2).
10. If $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is even.
    Then the ending corner will be D. Example: (a,b,c)=(1,2,4).
11. If $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$ is odd, $\frac{b}{g_2}$ is even, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is odd.
    Then the ending corner will be E. Example: (a,b,c)=(1,2,5).
12. If $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$ is odd, $\frac{b}{g_2}$ is even, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd.
    Then the ending corner will be E. Example: (a,b,c)=(2,4,1).
13. If $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$ is odd, $\frac{b}{g_2}$ is even, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is even.
    Then the ending corner will be E. Example: (a,b,c)=(1,4,2).

2.3 Comparison of our results with another theorem

In the article *3D − Arithmetic − Billiards* written by Antonella Perruca, another way to predict the ending corner of a ball projected into a 3D-grid is stated.



With each letter we'll refer to a corner (A, B, C, D, E, F, G, H) or a side (a, b, c)

Proposition 2.3.1.

1. *Ending corner is G if $lcm(a, b, c)/a$ and $lcm(a, b, c)/b$ and $lcm(a, b, c)/c$ are all odd ratios.*
2. *Ending corner is D if $lcm(a, b, c)/c$ is the only odd ratio.*
3. *Ending corner is B if $lcm(a, b, c)/a$ is the only odd ratio.*
4. *Ending corner is E if $lcm(a, b, c)/b$ is the only odd ratio.*
5. *Ending corner is H if $lcm(a, b, c)/b$ and $lcm(a, b, c)/c$ are the only odd ratios.*
6. *Ending corner is F if $lcm(a, b, c)/a$ and $lcm(a, b, c)/b$ are the only odd ratios.*
7. *Ending corner is C if $lcm(a, b, c)/c$ and $lcm(a, b, c)/a$ are the only odd ratios.*

Now we want to show that the results we found in 1.2 are equivalent to the results by the article of Antonella Perruca.
This means we need to prove that:

1. $lcm(a, b, c)/a$, $lcm(a, b, c)/b$, $lcm(a, b, c)/c$ are odd if and only if $\frac{a}{g_1}$, $\frac{b}{g_1}$, $\frac{c}{g_2}$, $\frac{b}{g_2}$,$\frac{a}{g_3}$, $\frac{c}{g_3}$ are all odd.

   Proof. In order to prove this, we'll rewrite this proposition using 2-valuation.
   We define $v_2(a) = n_a,\ v_2(b) = n_b,\ v_2(c) = n_c$
   We have:
   $$v_2\left(\frac{lcm(a,b,c)}{a}\right) = v_2(lcm(a,b,c)) - v_2(a) = max(n_a, n_b, n_c) - n_a$$
   From this information we can conclude

   (a)
   $$\frac{lcm(a,b,c)}{a}\ odd \iff max(n_a, n_b, n_c) - n_a = 0 \iff \begin{cases} n_a \le n_b \\ n_a \le n_c \end{cases}$$

   (b)
   $$\frac{lcm(a,b,c)}{b}\ odd \iff max(n_a, n_b, n_c) - n_b = 0 \iff \begin{cases} n_b \le n_a \\ n_b \le n_c \end{cases}$$

   (c)
   $$\frac{lcm(a,b,c)}{c}\ odd \iff max(n_a, n_b, n_c) - n_c = 0 \iff \begin{cases} n_c \le n_b \\ n_c \le n_a \end{cases}$$

We do the same for the other part of the proposition.
We have:

$$v_2\left(\frac{b}{gcd(a,b)}\right) = n_b - min(n_a, n_b)$$

And the same holds for $\frac{a}{g_1}$, $\frac{b}{g_1}$, $\frac{c}{g_2}$, $\frac{b}{g_2}$,$\frac{a}{g_3}$, $\frac{c}{g_3}$.
This allows us to write the proposition we want to proove in another form:

$$\begin{cases} n_a \leq n_b \\ n_a \leq n_c \\ n_b \leq n_a \\ n_b \leq n_c \\ n_c \leq n_b \\ n_c \leq n_a \end{cases} \iff \begin{cases} n_a \leq n_b \\ n_b \leq n_a \\ n_b \leq n_c \\ n_c \leq n_b \\ n_c \leq n_a \\ n_a \leq n_c \end{cases}$$

Proving this is simple since we notice that both sides are actually equal and simplify into

$$n_a = n_b = n_c$$

□

2. lcm(a, b, c)/a is the only odd ratio if and only if we either have
   - if $\frac{a}{g_1}$ is even, $\frac{b}{g_1}$ is odd, $\frac{c}{g_2}$, $\frac{b}{g_2}$ are both odd, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd
   - if $\frac{a}{g_1}$ is even, $\frac{b}{g_1}$ is odd, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd
   - if $\frac{a}{g_1}$ is even, $\frac{b}{g_1}$ is odd, $\frac{c}{g_2}$ is odd, $\frac{b}{g_2}$ is even, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd

   Proof. As in the proof above, we have:

   (a)
   $$\frac{lcm(a,b,c)}{a} \; odd \iff \begin{cases} n_a \leq n_b \\ n_a \leq n_c \end{cases}$$

   (b)
   $$\frac{lcm(a,b,c)}{b} \; even \iff n_a > n_b \; or \; n_c > n_b$$

   (c)
   $$\frac{lcm(a,b,c)}{c} \; even \iff n_a > n_c \; or \; n_b > n_c$$

   And

   (a)
   $$\frac{a}{gcd(a,b)} \; even \iff n_b < n_a$$

   (b)
   $$\frac{b}{gcd(a,b)} \; odd \iff n_b \leq n_a$$

   This allows us to rewrite our proposition as:

   $$\begin{cases} n_a \geq n_b \\ n_a \geq n_c \\ n_a > n_b \; or \; n_c > n_b \\ n_a > n_c \; or \; n_b > n_c \end{cases} \iff \begin{cases} n_b < n_a \\ n_c < n_a \\ n_b = n_c \; or \; n_c < n_b \; or \; n_b < n_c \; (this\ line\ can\ be\ removed) \end{cases}$$

   The implication (⇐) is clear, as all the conditions are satisfied.

   Conversely, we consider all the different cases:

   (a) $n_a > n_b$ and $n_a > n_c$ implies what we want to proove.
   (b) $n_a > n_b$ and $n_b > n_c \Rightarrow n_a > n_c$
   (c) $n_c > n_b$ and $n_a > n_c \Rightarrow n_a > n_b$

(d) $n_c > n_b$ and $n_b > n_c \Rightarrow$ This is impossible.

□

3. $lcm(a,b,c)/c$ is the only odd ratio if and only if we either have
   - $\frac{a}{g_1}$, $\frac{b}{g_1}$ are both odd, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is even
   - if $\frac{a}{g_1}$ is even, $\frac{b}{g_1}$ is odd, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is even.
   - if $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is even

   This proof is the same as *proof 2*.

4. $lcm(a,b,c)/b$ is the only odd ratio if and only if we either have
   - if $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$ is odd, $\frac{b}{g_2}$ is even, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is odd
   - if $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$ is odd, $\frac{b}{g_2}$ is even, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd.
   - if $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$ is odd, $\frac{b}{g_2}$ is even, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is even.

   This proof is the same as *proof 2*.

5. $lcm(a,b,c)/b$ and $lcm(a,b,c)/c$ are the only odd ratios if and only if we have $\frac{a}{g_1}$ is odd, $\frac{b}{g_1}$ is even, $\frac{c}{g_2}$, $\frac{b}{g_2}$ are both odd, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is even.

   Proof. Using the same reasoning as in the second proof we can rewrite this proposition as:

$$\begin{cases} n_b \geq n_a \\ n_b \geq n_c \\ n_c \geq n_a \\ n_c \geq n_b \\ n_c > n_a \text{ or } n_b > n_a \end{cases} \iff \begin{cases} n_a < n_c \\ n_a < n_b \\ n_c = n_b \end{cases}$$

   The implication ($\Leftarrow$) is clear, as all the conditions are satisfied.
   Conversely, we consider all the different cases:

   (a) $n_c > n_a$ implies what we want to prove.

   (b) $n_b > n_a$ implies what we want to prove.

□

6. $lcm(a,b,c)/a$ and $lcm(a,b,c)/b$ are the only odd ratios if and only if we have $\frac{a}{g_1}$, $\frac{b}{g_1}$ are both odd, $\frac{c}{g_2}$ is odd, $\frac{b}{g_2}$ is even, $\frac{a}{g_3}$ is even and $\frac{c}{g_3}$ is odd.

   This proof is the same as *proof 5*.

7. lcm(a, b, c)/a and lcm(a, b, c)/b are the only odd ratios if and only if we have $\frac{a}{g_1}$ is even, $\frac{b}{g_1}$ is odd, $\frac{c}{g_2}$ is even, $\frac{b}{g_2}$ is odd, $\frac{a}{g_3}$ is odd and $\frac{c}{g_3}$ is odd.

   This proof is the same as *proof 5*.

Chapter 3

When are there no edge points?

3.1 Theorem

In this section we want to characterise those 3D billiards that have no edge points.
First, recall the definition of an edge point.

Definition 3.1.1. *A point on a trajectory is called an edge point when on one of the projectures on the x, y ,z planes is in the corner and the other two are on the edge.*

Theorem 3.1.1. *Any corner path in a 3D billiard of edges of lenght a, b, c has no edge points if and only if*

$$lcm(a, b) = lcm(a, c) = lcm(b, c)$$

Proof. Suppose first thatwe have no edge point in any corner path.
We have a corner point on the z-face after $lcm(a, b)$ steps. This also has to be a corner on the y-face so: $lcm(a, b) = k_1 \cdot lcm(a, c)$ with k_1 an integer an which implies $lcm(a, c)|lcm(a, b)$. We have a corner point on the y-face after lcm(a,c). This also has to be a corner on the x-face so: $lcm(a, c) = k_2 \cdot lcm(b, c)$ with k_2 an integer which implies$lcm(b, c)|lcm(a, c)$. We have a corner point on the x-face after lcm(b,c). This also has to be a corner on the z-face so: $lcm(b, c) = k_3 \cdot lcm(a, b)$ withk_3 an integer which implies $lcm(a, b)|lcm(b, c)$.
This implies lcm(a,b) = lcm(a,c) = lcm(b,c)

Conversely, suppose $lcm(a, b) = lcm(a, c) = lcm(b, c)$:
We reach a corner on a projection after $lcm(x, y)$ (with x and y being any choicen of a,b and c with $x \neq y$) steps. Since we know that $lcm(a, b) = lcm(a, c) = lcm(b, c)$, each time the ball hits a corner on one projection, the ball will also hit a corner on the two other projections. This implies that there are no edge points by definition.

□

3.2 Example

Consider the example (6,10,15).
We notice

$$lcm(6, 10) = lcm(10, 15) = lcm(6, 15) = 30$$

So according to the theorem there shouldn't be any edge points, which is confirmed graphically by the *Figure 3.1*.

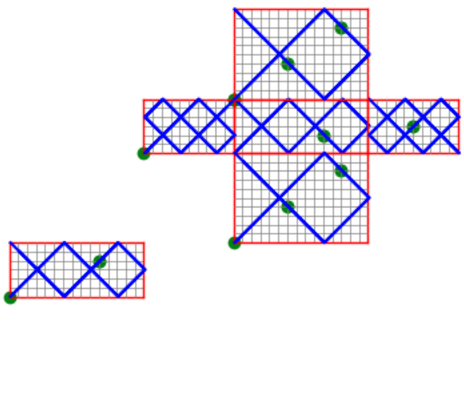


Figure 3.1: (6,15,10)

Chapter 4

References and Acknowledgements

4.1 References

1. $http://www.antonellaperucca.net/Arithmetic-Billiards.pdf$
2. *Three-Dimensional Arithmetic Billiards* (hasn't been published yet)
3. $https://en.wikipedia.org/wiki/Arithmetic_billiards$
4. $Pythonprogram$
5. $https://math.uni.lu/eml/projects/2019_Winter/arithmeticBilliards.html$

4.2 Acknowledgements

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