Rubik's Snakes on a plane

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Abstract

This article is about the planar configurations of the mathematical toy called Rubik's Snake (or Rubik's Twist), where all Snake pieces have a triangular face lying in one same plane. Any such configuration with n corners can be obtained from the straight Snake with exactly n twists of the physical toy (in such a way that the Snake does not self-intersect during the construction). We also discuss a number of other interesting properties, in particular we classify all convex subsets of the plane that can be precisely covered by such a planar configuration. The proofs only require school mathematics.

Fortunately for the reader, they are not going to share a flight with poisonous Snakes but they are about to explore the mathematical toy called *Rubik's Snake* (or *Rubik's Twist*), and more precisely the configurations of this toy that may be considered as *two-dimensional*. Investigating three-dimensional configurations is beyond the scope of this paper, and we leave this as a challenge for the interested reader.

1 The two-dimensional Rubik's Snake

The usual version of the Rubik's Snake consists in 24 pieces of the same size in the form of thick half squares. In our exploration all these triangles will be parallel, thus it makes sense to consider the Rubik's Snake as lying inside the plane. The configurations of the toy that we consider are those for which, placing the toy on a table, all pieces touch the table with a triangular face: looking at the projection of the toy on the table, the two-dimensional figure consists of 24 triangles.

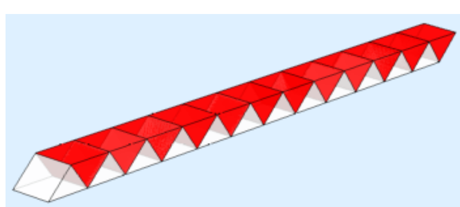

Figure 1: The straight Snake.

The 24 triangular pieces of the two-dimensional Rubik's Snake come in a row, and usually have two alternating colors. Each piece is connected to the previous one and to the next one, with the exception of the two end pieces. The connection is made by joining legs of triangles. Any two neighboring pieces have a common leg and they build either a parallelogram (*untwisted connection*) or half a square (*twisted connection*), and in the latter case we say that the Snake makes a *corner*.

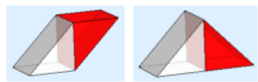

Figure 2: Untwisted and twisted connection respectively.

By *twisting*/*untwisting* a connection we can change the connection type, and this counts as a *move*. What happens is that one of the two triangles at the connection stays fixed, while the other one gets reflected at a line. The Snake pieces from the fixed triangle onwards stay fixed, while the pieces from the flipped triangle onwards all get reflected at the same line. Overall, one part of the Snake stays fixed while the other part gets reflected: it is not important which part is which because it is always possible to flip the whole Snake.

However, we cannot twist/untwist any connection because the Snake, being a toy, cannot intersect itself. The issue is precisely the following: the part of the Snake that we reflect could intersect, after the reflection, the part of the Snake that we leave fixed.

2 Describing the planar configurations

There are 23 connections in a row hence we can describe a planar configuration of the Rubik's Snake as a 23-tuple of zeroes and ones (0 and 1 stand for an untwisted and twisted connection respectively). Alternatively, we can indicate the twisted connections as a subset of the numbers from 1 to 23.

The *straight Snake* (without twisted connections, see Figure 1) thus corresponds to a 23-tuple consisting only of zeroes and to the empty subset of $\{1, 2, 3, \ldots, 23\}$. And, for example, the configuration in Figure 3 can be described in the following two ways:

$$(1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1)$$

$$(1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23)\,.$$

It is helpful to superpose a square grid to a planar Rubik's Snake, letting each triangular piece be half of a grid square, see Figure 4. *The color scheme of the Rubik's Snake matches the chessboard pattern of the square grid* because the next triangle piece lies in one neighboring square. In particular, two triangle pieces of different colors cannot share the hypothenuse.

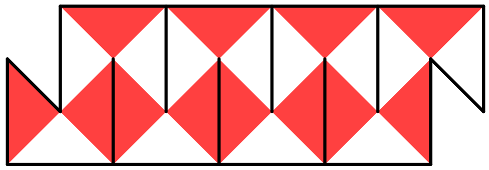

Figure 3: The planar configuration with 16 twists.

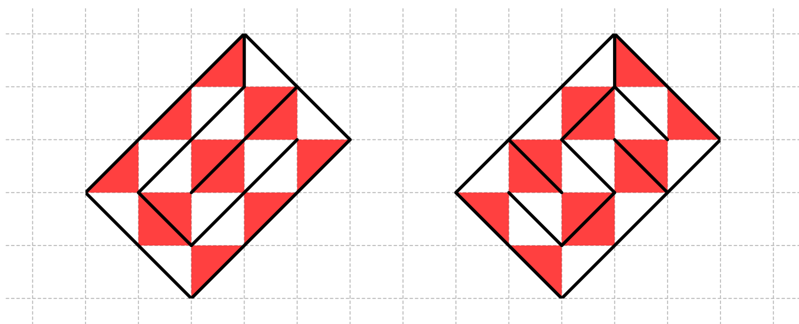

Figure 4: Colors of planar configurations follow a chessboard pattern. The two configurations fill in different ways the same rectangle.

Let us thus draw the Snake in a different way on our squared sheet, shifting the picture so that the middle of the hypothenuse of a triangle piece is at a vertex of a grid square. This new picture shows that a planar Snake configuration can be seen as a path following edges of squared paper in which *each edge is only traversed once*. Mathematicians call these *Eulerian paths* (we refer to [1] for a gentle introduction). Moreover, each new step of the path must turn right or left, and it can never proceed forward, as neighboring pieces of the Snake do not share the hypothenuse. The precise statement, that the interested reader can prove by induction, would be: *Any planar Snake configuration corresponds to an Eulerian path of finite length on the square grid such that consecutive steps are never parallel. On the other hand, each Eulerian path on the square grid for which no two consecutive steps are parallel can be associated to at least one Snake.* It is easy to observe that for some Eulerian paths there can be more than one associated Snake, because the ends of the Snake may have space to be twisted without changing the path leading to them: an example is the dashed piece in Figure 5.

A *closed Rubik's Snake* is a planar configuration in which the first and last piece have a common leg (else, we say that the Snake is *open*). One first remark: *in a closed Snake, with some mathematical cutting and glueing, each piece can be made one of the end pieces of the Snake.* Indeed, for a closed Snake we can

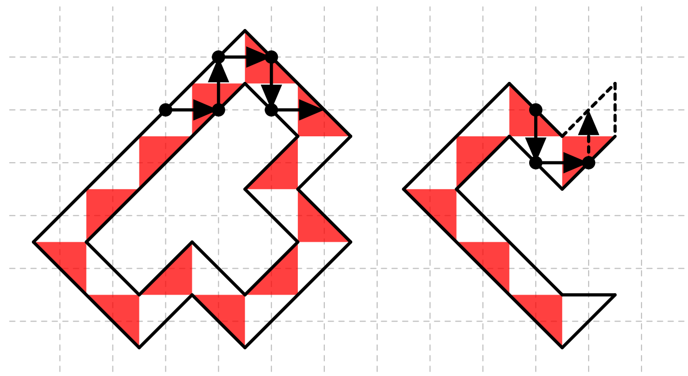

Figure 5: Connecting the middle of the hypothenuse of neighbouring Snake pieces we obtain an Eulerian path on the square grid.

glue the two end pieces, and we can cut any connection, resulting again in a closed Snake (to understand this, imagine that a closed Snake is a periodic path for which we can freely choose the start).

By means of the Eulerian path from Figure 5, we can prove that *if a Rubik's Snake of length n has a closed planar configuration, then n is a multiple of* 4. Indeed, n is the number of steps of the corresponding Eulerian path. As the path is closed, it consists of an even number of steps. Moreover, we have the same number of horizontal and vertical steps, as we alternate between them. Moreover, as the path is closed, the number of steps going rightwards (respectively, upwards) is the same as the number of steps going leftwards (respectively, downwards). Thus n is divisible by 2 twice.

3 Constructing a planar configuration

Suppose to have a 23-tuple of zeroes and ones (or a subset of $\{1, 2, 3, \ldots, 23\}$) that corresponds to a planar configuration of the triangle pieces where the 24 triangles do not intersect (more precisely, any two of them can only share one vertex or one side), see for example Figure 3.

Are we sure that we can produce the given configuration with the toy? In other words, starting from the straight Snake, can we build this configuration by twisting/untwisting connections so that, during the construction, the Snake never intersects itself? And, in the affirmative, what is the minimal number of moves for such a construction?

Notice that to build a configuration where there are t twisted connections (equivalently, t corners), then we clearly need at least t moves for constructing it from the straight Snake. Luckily, t moves always suffice, so in other words *we only have to twist connections to produce any planar configuration from the* *straight Snake*, provided of course that we carefully choose the order of the connections to be twisted (else the Snake may intersect itself during the construction). Thus we have: *the minimal number of moves to construct any planar configuration is precisely the number of Snake corners in the given configuration.*

We prove all the above assertions by reversing time. Namely, being able to construct the given planar configuration from the straight Snake in t moves is the same as saying that we are able to construct the straight Snake from the given configuration in t moves. So we are left to prove the following assertion: *Given any planar configuration of the Snake (that has at least two pieces and it is not the straight Snake) it is possible to untwist some twisted connection (in such a way that the Snake does not intersect itself).* Then the new configuration has one twisted connection less, and when there are no more twisted connections left, then we have obtained the straight Snake. The proof of the above assertion is postponed to the last section.

Consequently, the instructions for constructing (from the straight Snake) any planar configuration of the Rubik's Snake could be given as an ordered subset of $\{1, 2, 3, \ldots, 23\}$, so that one knows which connections have to be twisted first.

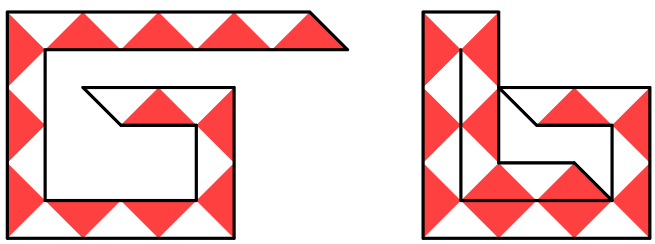

Figure 6: We cannot obtain the latter configuration from the former with only two moves.

It is important to notice that *the above result holds for Rubik's Snakes with an arbitrary number of pieces.* Also notice that the result does not say that if two planar configurations only differ by c connections, then it is possible to obtain one from the other with c moves. A counterexample is given in Figure 6, and this also shows that, if we badly choose the order of the twists to construct the second configuration from the straight Snake, then the Snake will intersect itself during the construction.

4 Convex planar configurations of Rubik's Snakes

A subset of the plane is said to be *convex* if any two points of it can be joined with a straight segment that lies entirely in the subset. For instance, this is the

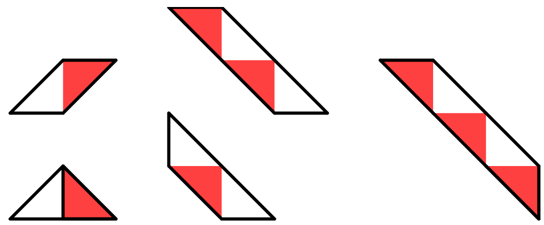

Figure 7: Open Snakes where both end pieces have two free sides: both configurations of the Snake with 2 pieces, and the straight Snake with 3, 4, or 5 pieces.

case for (the area bounded by) a triangle, a rectangle or a circle. What are the possible convex shapes that a Rubik's Snake can precisely cover with a planar configuration? We classify such convex shapes, for Snakes with an arbitrary number of pieces, according to properties of the end pieces. An end piece has either one or two *free sides*, meaning that either one or two of its sides do not touch the other Snake pieces.

- *CASE 0:* The Snake with only one piece is a triangle.

Now suppose that there are at least two pieces.

- *CASE 1: Both end pieces have two free sides.* If there are only two pieces and the connection is twisted, then we have a triangle. Else, we must have the straight Snake: the convex shape is a parallelogram or an isosceles trapezoid, according to whether there is an even or odd number of pieces. See Figure 7.

- *CASE 2: Precisely one end piece has two free sides.* The convex configurations are the right trapezoids (the height being the hypothenuse of a Snake piece) covered by Snakes folded in the shape of a "U". See Figure 8.

- *CASE 3: Open Snakes such that both end pieces only have one free side (which must be a leg).* The convex configurations are rectangles with "broken corners". Figure 9 portrays the three possibilities: the rectangle is missing half a Snake piece on two opposite corners, and we have an hexagon; the rectangle is missing half a Snake piece on two adjacent corners, and we have a pentagon or an hexagon; the rectangle is missing two Snake pieces on one corner (large "broken corner"), and we have a pentagon. The angles in these polygons are 90° or 135°.

- *CASE 4: Closed convex Snakes.* The convex configurations are full rectangles (the side lengths are integer multiples of the hypothenuse of a Snake piece). See Figure 4.

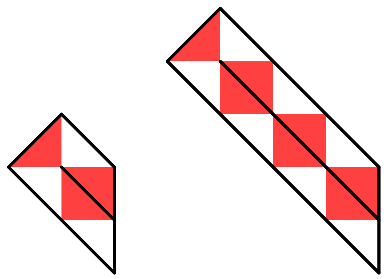

Figure 8: Convex shapes from Snakes such that precisely one end piece has two free sides.

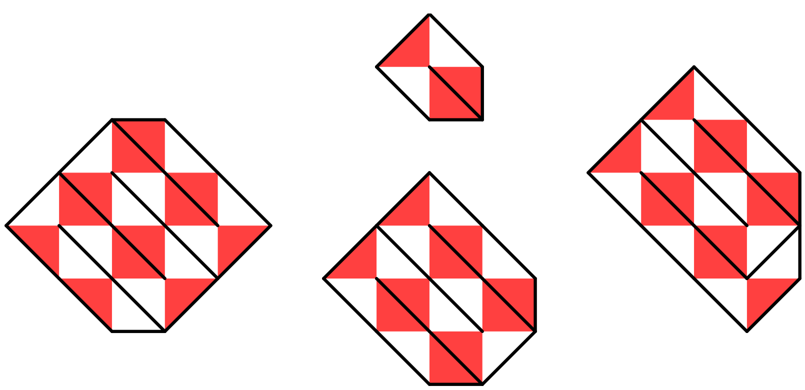

Figure 9: Convex shapes of open Snakes where both end pieces have only one free side.

In Figure 4 we see that different Snake configurations can give rise to the same convex shape. Also notice that not all convex shape types can be realized with a Snake of a given number of pieces (for example, with the standard toy consisting of 24 pieces). To see this, consider that the number of pieces must be (taking the hypothenuse of the triangular pieces as unit length) four times the area of the convex shape.

5 The two remaining proofs

5.1 The proof of the result on untwisting a connection

Suppose that the Snake has at least two pieces and it is not the straight Snake. We prove that it is possible to untwist some twisted connection so that, doing this, the Snake does not intersect itself (hence the move is doable on the actual toy).

Draw a grid of squares as in Figure 4, and consider the smallest rectangle made of grid segments that contains the Snake. This rectangle touches the Snake only at end pieces or Snake corners.

The *easy case* is the following: there is a rectangle side touching no end pieces. Then we can untwist the Snake at any corner touching that side. Roughly speaking, the only effect of the untwist is that part of the Snake is moved outside the rectangle and there is no risk of self-intersection. To be precise, a piece touching the chosen rectangle side and moved by the untwisting will not exit the rectangle but half of it will move to a part not occupied by the Snake and half of it will occupy the same region as before, see Figure 10.

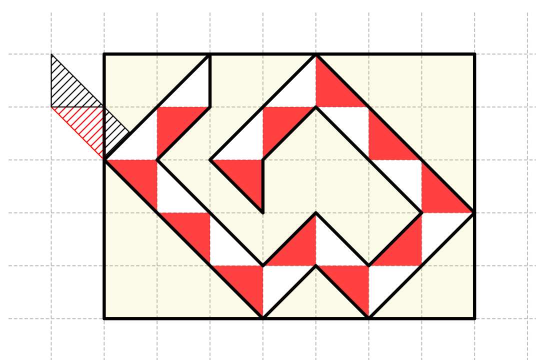

Figure 10: The easy case of the proof of the untwisting result.

In the remaining case, which we call the *tricky case*, the end pieces must be at two opposite rectangle corners as, by minimality of the rectangle, each rectangle side touches the Snake. We then consider a new rectangle, namely the smallest one containing the Snake without its *initial segment* (namely, the pieces between the starting point and the first corner of the Snake).

The initial segment must intersect the boundary of the smaller rectangle, and it can do it in two ways, which we discuss separately. Figure 11 depicts the two possibilities, and shows untwists that do not lead to a self-intersection of the Snake: in Figure 11 the initial segment has a dashed pattern, the smaller rectangle is drawn in a darker shade of yellow, the symmetry line for the untwists are dashed blue.

- *The initial segment crosses no corner of the smaller rectangle.* There is another side of the smaller rectangle that touches only in its prolongation the initial segment and does not touch the last piece. There is at least one Snake corner on this side, and we may safely untwist that corner.

- *The initial segment crosses a corner of the smaller rectangle.* This time, any corner of the (whole) Snake touching the smaller rectangle can be safely untwisted.

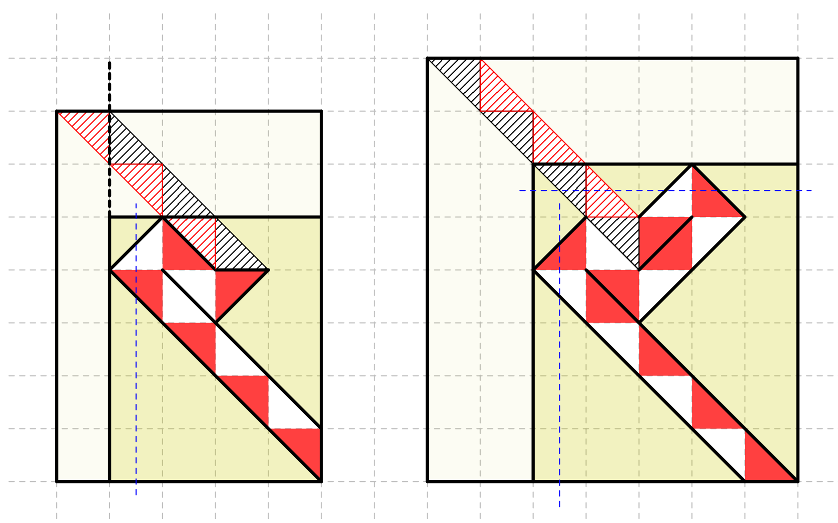

Figure 11: The tricky case of the proof of the untwisting result.

5.2 Proof of the classification of the convex planar configurations

Recall the cases of the classification, excluding the trivial case (CASE 0), and let n be the number of Snake pieces. With $n = 2$, we may get either a triangle or a parallelogram (CASE 1), so now suppose that $n \geq 3$.

CASE 1. Consider the left picture of Figure 12, the Snake starting up left. By convexity (as the initial piece has two free sides) the Snake lies between the upper horizontal and the diagonal ray. The first and second connection are

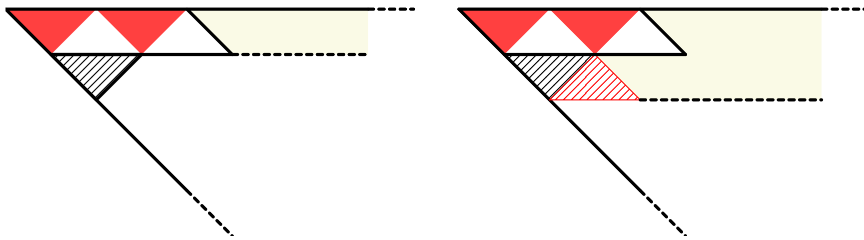

Figure 12: Proof of the classification of the convex planar configurations, CASES 1 and 2.

untwisted, else the third piece would exit the sector. In fact, all connections are untwisted (resulting in the straight Snake). Else, the dashed triangle in the picture should be, by convexity, part of the Snake. It would be the end piece but it would have only one free side, contradicting the assumption of CASE 1.

CASE 2. Consider the right picture of Figure 12, the Snake starting up left. Supposing that we don't have the straight Snake then, similarly to the previous case, the dashed triangle in the picture is the end piece.

If there is no piece of the Snake below the n-th, that is below the (now larger) yellow stripe, the Snake must be (a longer version of) the trapezoid configuration portrayed in Figure 8. We thus conclude the discussion of CASE 2 showing that this is actually the only possibility.

The last connection, between the n-th and $(n-1)$-th pieces, on the right of the former, must be untwisted. There can not be another piece right below the $(n-1)$-th, since it would have a free short edge, which is not allowed for pieces in the mid of the Snake. Prolonging the hypotenuse of the $(n-1)$-th piece and imposing convexity, we deduce that there cannot be other pieces below the dotted ray, thus the remainder of the Snake must remain above the black dashed line, extending the long side of piece $n-1$.

CASES 3 and 4. Consider the smallest rectangle (oriented such that the hypothenuse of a triangle piece is parallel to a rectangle side) containing the Snake, see Figure 13. By minimality, all rectangle sides touch the Snake. In CASE 4 any rectangle side must be covered by hypothenuses of the Snake pieces (hence we conclude). Indeed, consider a piece touching a rectangle side (away from the rectangle corners) at its right angle: the connections at this piece must be untwisted, else the Snake would exit the rectangle. Now consider a piece with the hypothenuse on a rectangle side, without loss of generality the upper side: if there is no piece parallel to it and next left, then by convexity its connection on the left is untwisted (and similarly by replacing left with right). We can reason analogously in CASE 3 because, as the Snake is open and by convexity, the initial leg and the end leg of the Snake must touch the rectangle sides.

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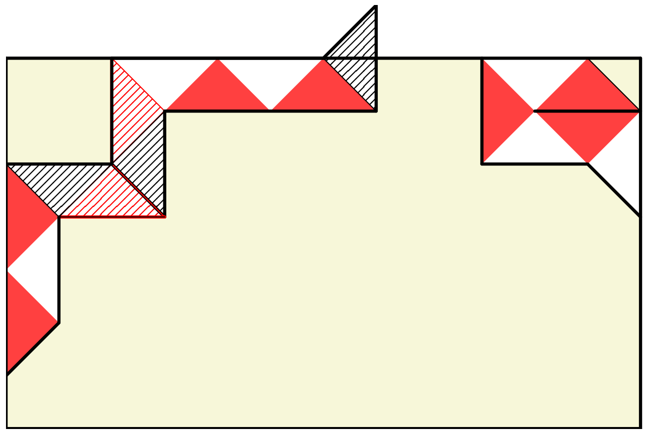

Figure 13: Figure for the proof of CASES 3 and 4.

Schammo for some pictures and very interesting discussions.

Exercises for the reader

1. Given a Rubik's Snake consisting of n pieces, what is the largest number of twisted connections that we may have in a planar configuration?

   Solution: For $n = 1, 2, 3, 4$ the answer is $n-1$. For $n \geq 5$ notice that there cannot be three twisted connections in a row. Writing $n = 3q+r$ for some non-negative integers q and r with $r = 0, 1, 2$, then the requested number is at most $2q$ if $r = 0, 1$, and at most $2q + 1$ if $r = 2$. Moreover, there is a configuration with this number of twisted connection, similar to the one in Figure 3.

2. Given a Rubik's Snake consisting of n pieces, for which n can we have the planar configuration which looks like an empty square frame?

   Solution: Those n which are of the form $4t$ with $t > 1$ an odd integer. Indeed, on each side of a square frame there are s pieces on the exterior and $s - 1 \geq 1$ pieces on the interior, so the total number of pieces must be $4(2s - 1)$ with $s \geq 2$.

3. Consider any closed polygon consisting of s equal squares attached by their sides. Take a Rubik's Snake with $n = 4s$ pieces, such that the hypothenuse of the triangle pieces is the side length of the above squares. Prove that there is a closed planar configuration of the Snake covering precisely the given polygon. Hint: Look at Figure 14.

   Solution: For $s = 1$ the assertion is clear, as we can fold the Snake with 4 pieces to a square. By iteration, suppose that the assertion holds for a

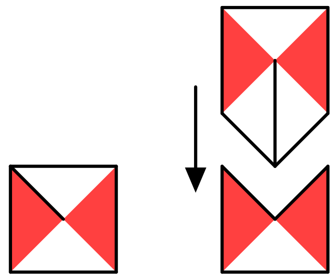

Figure 14: The construction for Exercise 3.

polygon made by s squares and consider a polygon made by $s+1$ squares, obtained by adding one new square. Starting from the closed Snake realizing the smaller polygon, consider its triangle piece touching the new square (without loss of generality it is an end piece, as remarked in the paper). We may replace this piece by 5 triangles pieces as in Figure 14 as to fill the new square.

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