Faculty of science,Technology and Madecine

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Magic Polyhedra

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1 Intoduction

In this report we are going to present an interesting topic of Magic Polyhedra. The word polyhedra refers to word Polyhedron which is composed of 2 Greek words "poly" (means multiple) and "hedron" (means surface). There's one well known magic figure: magic square.

2 Magic Square

Definition 2.1. A magic square is a square of different size with arrayed numbers which are usually positive integers such that the sum of arrayed numbers in every column, every row and every diagonal is equal to the same number.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 1: Magic square 4×4

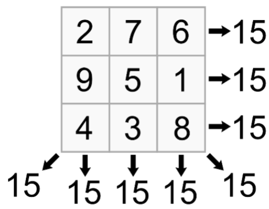

Figure 2: Magic square 3×3

The sum in left square is 34 and 15 in a second one.

2.1 How does it work?

And let's now think how it works. We were thinking about 1 particular solution. That solution is based on the idea of putting the same number in every position. That solution works with most of the magic polyhedrons with some exceptions. And that solution is not so interesting to work with. So, we thought to start working with the square of size 3. As our first step, we replaced all arrayed integers by variables a,b,c,. . . ,i. And wrote equations relating these variables.

a	b	c
d	e	f
g	h	i

$$a + b + c = d + e + f = g + h + i = a + d + g$$
$$= b + e + h = c + f + i = a + e + i = c + e + g \qquad (1)$$

As a next step, we can rewrite equation (1) into multiple equations. And make from them a system of equations. Of course, we can write all possible equations, but it would be useless as some other equations would be a linear combination from first 7 equations.

$$
\begin{cases}
a+b+c = d+e+f \\
a+b+c = g+h+i \\
a+b+c = a+d+g \\
a+b+c = b+e+h \\
a+b+c = c+f+i \\
a+b+c = a+e+i \\
a+b+c = c+e+g
\end{cases}
\Leftrightarrow
\begin{cases}
a+b+c-d-e-f = 0 \\
a+b+c-g-h-i = 0 \\
a+b+c-a-d-g = 0 \\
a+b+c-b-e-h = 0 \\
a+b+c-c-f-i = 0 \\
a+b+c-a-e-i = 0 \\
a+b+c-c-e-g = 0
\end{cases}
$$

$$
\Leftrightarrow
\begin{cases}
a+b+c-d-e-f = 0 \\
a+b+c-g-h-i = 0 \\
b+c-d-g = 0 \\
a+c-e-h = 0 \\
a+b-f-i = 0 \\
b+c-e-i = 0 \\
a+b-e-g = 0
\end{cases}
$$

From Linear Algebra Course, we know that system of equations can be rewritten as a product of matrix and vector. In our case, we can interpret our system as:

$$
\begin{pmatrix}
1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\
0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\
1 & 1 & 0 & 0 & -1 & -0 & -1 & 0 & 0
\end{pmatrix}
*
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

We have spotted that our vector (a, b, c, d, e, f, g, h, i) must belong to the Kernel of the matrix. With a use of computer, we can easily compute the basis of kernel of our matrix.
We find a basis:

$$
\left\{
\begin{matrix}
\begin{pmatrix} 1 & 0 & 0 & \frac{-2}{3} & \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \end{pmatrix} \\
\begin{pmatrix} 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 1 & \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}
\end{matrix}
\right\}
$$

So, we have encountered a problem by getting uncomfortable fractions. We asked computer to compute the kernel of the matrix which was defined over set of rational numbers Q. Consequently, basis was also defined on rational numbers. We don't want to work with fractions as it makes magic square less readable. We defined our matrix then over the set of integers and we have got a more convient basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 1 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \\ 1 \\ -2 \\ -1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

2.2 How do we construct our magic square?

We had to interpret our results in a right way. We can take any vector and rewrite it as square in order to have right variable in the right place. We rewrite all 3 vectors in such way.

1	0	2
2	1	0
0	2	1

0	1	2
3	1	-1
0	1	2

0	0	3
4	1	-2
-1	2	2

We noticed 1 interesting thing. For each square we almost have a basis. That means, that we can choose any numbers and they will not be changed by linear combinations of other vectors.

1	0	2
2	1	0
0	2	1

0	1	2
3	1	-1
0	1	2

0	0	3
4	1	-2
-1	2	2

Avoid negative signs in our basis, we can multiply the last square by -1. So we get:

1	0	2
2	1	0
0	2	1

0	1	2
3	1	-1
0	1	2

0	0	-3
-4	-1	2
1	-2	-2

Now we can choose any 3 numbers and construct our magic square. Let's take 5,7 and 3. We multiply all numbers in 1st, 2nd and 3rd square by 5,7 and 3 respectively.
And we get:

$$5* \begin{array}{|c|c|c|} \hline 1 & 0 & 2 \\ \hline 2 & 1 & 0 \\ \hline 0 & 2 & 1 \\ \hline \end{array} +7* \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 3 & 1 & -1 \\ \hline 0 & 1 & 2 \\ \hline \end{array} +3* \begin{array}{|c|c|c|} \hline 0 & 0 & -3 \\ \hline -4 & -1 & 2 \\ \hline 1 & -2 & -2 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 5 & 7 & 15 \\ \hline 19 & 9 & -1 \\ \hline 3 & 11 & 13 \\ \hline \end{array}$$

Let's have a look at our final magic square, we can see that the sum in every row, column and diagonal equals to 27. For the conclusion of construction, we can take a linear combination of any vectors from basis. If you look carefully at previous squares at the beginning of chapter 2.2, rewriting each vector from our basis gives us magic squares. Construction of magic square 3x3 is finished

However, there's an alternative method to construct a magic square of size 3x3. That method was developed by Edouard Lucas. Just choose 3 any positive integers a, b and c. And write the result in a table below:

c-b	c+a+b	c-a
c-a+b	c	c+a-b
c+a	c-a-b	c+b

And let's compare his solution with ours. We can represent his solution as a linear combination of following vectors which we rewrite in a form of cubes:

$$c*\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 1&1&1\\\hline 1&1&1\\\hline\end{array} +b*\begin{array}{|c|c|c|}\hline -1&1&0\\\hline 1&0&-1\\\hline 0&-1&1\\\hline\end{array} +a\begin{array}{|c|c|c|}\hline 0&1&-1\\\hline -1&0&1\\\hline 1&-1&0\\\hline\end{array} =$$

c-b	c+a+b	c-a
c-a+b	c	c+a-b
c+a	c-a-b	c+b

And we need to find out the linear combination of vectors from vectors above in order to have vectors from our kernel. And we find:

$$1*\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 1&1&1\\\hline 1&1&1\\\hline\end{array} +0*\begin{array}{|c|c|c|}\hline -1&1&0\\\hline 1&0&-1\\\hline 0&-1&1\\\hline\end{array} -1*\begin{array}{|c|c|c|}\hline 0&1&-1\\\hline -1&0&1\\\hline 1&-1&0\\\hline\end{array} = \begin{array}{|c|c|c|}\hline 1&0&2\\\hline 2&1&0\\\hline 0&2&1\\\hline\end{array}$$

$$1*\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 1&1&1\\\hline 1&1&1\\\hline\end{array} +1*\begin{array}{|c|c|c|}\hline -1&1&0\\\hline 1&0&-1\\\hline 0&-1&1\\\hline\end{array} -1*\begin{array}{|c|c|c|}\hline 0&1&-1\\\hline -1&0&1\\\hline 1&-1&0\\\hline\end{array} = \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 3&1&-1\\\hline 0&1&2\\\hline\end{array}$$

$$-1*\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 1&1&1\\\hline 1&1&1\\\hline\end{array} -1*\begin{array}{|c|c|c|}\hline -1&1&0\\\hline 1&0&-1\\\hline 0&-1&1\\\hline\end{array} +2*\begin{array}{|c|c|c|}\hline 0&1&-1\\\hline -1&0&1\\\hline 1&-1&0\\\hline\end{array} = \begin{array}{|c|c|c|}\hline 0&0&-3\\\hline -4&-1&2\\\hline 1&-2&-2\\\hline\end{array}$$

And we have proved that every magic square can be rewritten in the way of Edouard Lucas's magic square. Also, his way of construction is the easiest to remember how to make a magic square 3x3. For the conclusion, we can say that every magic constant is equal to 3c. It is an important conclusion for next magic polyhedrons.

2.3 Magic Square of size 4×4,...,n×n

To construct a magic square of bigger size, we can use the matrix-method. Let's construct a 4x4 square:

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

Remark 2.1. It is important to pay attention to variables and their position, 1 little mistake may end with a wrong result.

$$\begin{aligned} &a+b+c+d=e+f+g+h=i+j+k+l=m+n+o+p=a+e+i+m \\ =&b+f+j+n=c+g+k+o=d+h+l+p=a+f+k+p=d+g+j+m \end{aligned} \tag{2}$$

we split (2) into system of equations which we rewrite as a matrix:

```
[ 1  1  1  1 -1 -1 -1 -1  0  0  0  0  0  0  0  0]
[ 1  1  1  1  0  0  0  0 -1 -1 -1 -1  0  0  0  0]
[ 1  1  1  1  0  0  0  0  0  0  0  0 -1 -1 -1 -1]
[ 0  1  1  1 -1  0  0  0 -1  0  0  0 -1  0  0  0]
[ 1  0  1  1  0 -1  0  0  0 -1  0  0  0 -1  0  0]
[ 1  1  0  1  0  0 -1  0  0  0 -1  0  0  0 -1  0]
[ 1  1  1  0  0  0  0 -1  0  0  0 -1  0  0  0 -1]
[ 0  1  1  1  0 -1  0  0  0  0 -1  0  0  0  0 -1]
[ 1  1  1  0  0  0 -1  0  0 -1  0  0 -1  0  0  0]
```

Figure 3: Matrix describing system of equations 4×4

The kernel of that matrix is:

```
[ 1  0  0  0  0  0  0  1  0  1  0  0  0  0  1  0]
[ 0  1  0  0  0  0  0  1  0  0  1  0  1  0  0  0]
[ 0  0  1  0  0  0  0  1  0  0  1  0  1  1 -1  0]
[ 0  0  0  1  0  0  0  1  0 -1  2  0  1  2 -1 -1]
[ 0  0  0  0  1  0  0 -1  0  1 -1  0 -1 -1  1  1]
[ 0  0  0  0  0  1  0 -1  0  0 -1  1  0 -1  1  0]
[ 0  0  0  0  0  0  1 -1  0 -1  0  1  0  1 -1  0]
[ 0  0  0  0  0  0  0  0  1  1 -1 -1 -1 -1  1  1]
```

Figure 4: A kernel of magic square 4×4

2.3.1 How do we construct our magic square 4×4?

1	0	0	0
0	0	0	1
0	1	0	0
0	0	1	0

0	1	0	0
0	0	0	1
0	0	1	0
1	0	0	0

0	0	1	0
0	0	0	1
0	0	1	0
1	1	-1	0

0	0	0	1
0	0	0	1
0	-1	2	0
1	2	-1	-1

0	0	0	0
1	0	0	-1
0	1	-1	0
-1	-1	1	1

0	0	0	0
0	1	0	-1
0	0	-1	1
0	-1	1	0

0	0	0	0
0	0	1	-1
0	-1	0	1
0	1	-1	0

0	0	0	0
0	0	0	0
1	1	-1	-1
-1	-1	1	1

1∗

1	0	0	0
0	0	0	1
0	1	0	0
0	0	1	0

+2∗

0	1	0	0
0	0	0	1
0	0	1	0
1	0	0	0

+3∗

0	0	1	0
0	0	0	1
0	0	1	0
1	1	-1	0

$$+4*\begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 0&0&0&1\\\hline 0&-1&2&0\\\hline 1&2&-1&-1\\\hline\end{array}+5*\begin{array}{|c|c|c|c|}\hline 0&0&0&0\\\hline 1&0&0&-1\\\hline 0&1&-1&0\\\hline -1&-1&1&1\\\hline\end{array}+6*\begin{array}{|c|c|c|c|}\hline 0&0&0&0\\\hline 0&1&0&-1\\\hline 0&0&-1&1\\\hline 0&-1&1&0\\\hline\end{array}$$

$$+7*\begin{array}{|c|c|c|c|}\hline 0&0&0&0\\\hline 0&0&1&-1\\\hline 0&-1&0&1\\\hline 0&1&-1&0\\\hline\end{array}+16*\begin{array}{|c|c|c|c|}\hline 0&0&0&0\\\hline 0&0&0&0\\\hline 1&1&-1&-1\\\hline -1&-1&1&1\\\hline\end{array}=\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6&7&-8\\\hline 16&11&-14&-3\\\hline -12&-9&14&17\\\hline\end{array}$$

$$\begin{aligned}&1+2+3+4=5+6+7-8=16+11-14-3=-12-9+14+17\\=&1+5+16-12=2+6+11-9=3+7-14+14=4-8-3+17\\=&1+6-14+17=4+7+11-12=10\end{aligned}$$

2.4 Properties of magic squares

All magic squares of size nxn can be modified. Some modifications will still give us magic squares. There are 3 such modifications:
We can add any constant to each number in the square. This helps us to have only positive integers if we want to. That modification changes also the sum of the numbers. Let's add 5 to each number in one of our previous square:

$$\begin{array}{|c|c|c|}\hline 5&7&15\\\hline 19&9&-1\\\hline 3&11&13\\\hline\end{array}\quad\xrightarrow{+5}\quad\begin{array}{|c|c|c|}\hline 10&12&20\\\hline 24&14&4\\\hline 8&16&18\\\hline\end{array}$$

Therefore, that modification works for square of any size. We show :

$$\begin{aligned}(a_1+x)+(b_1+x)+\cdots+(n_1+x)&=(a_2+x)+(b_2+x)+\cdots+(n_2+x)\\&=\cdots\\&=(a_n+x)+(b_n+x)+\cdots+(n_n+x)\\\Leftrightarrow(a_1)+(b_1)+\cdots+(n_1)+n*x&=(a_2)+(b_2)+\cdots+(n_2)+n*x\\&=\cdots\\&=(a_n)+(b_n)+\cdots+(n_n)+n*x\\\Leftrightarrow(a_1)+(b_1)+\cdots+(n_1)&=(a_2)+(b_2)+\cdots+(n_2)\\&=\cdots\\&=(a_n)+(b_n)+\cdots+(n_n)\end{aligned}$$

Moreover, we can multiply all number by any constant. This also works with squares of any size.

$$\begin{aligned} x(a_1) + x(b_1) + \cdots + x(n_1) &= x(a_2) + x(b_2) + \cdots + x(n_2) \\ &= \cdots \\ &= x(a_n) + x(b_n) + \cdots + x(n_n) \end{aligned}$$

Here we suppose that $x \neq 0$,so we can divide all equations by x and it brings us back to default equation.
But if x = 0, then all numbers in square are 0, and it is also magic square. We conclude that our solutions correspond to a vector sub-space.
Finally, we can rotate and reflect our magic square:

10	12	20
24	14	4
8	16	18

$\xrightarrow{vertical\ +\ horizontal\ reflection}$

18	16	8
4	14	24
20	12	10

10	12	20
24	14	4
8	16	18

$\xrightarrow{Clockwise\ rotation\ by\ 90\ degrees}$

8	24	10
16	14	12
18	4	20

And we can use complex numbers inside our magic square. So, we can interpret our magic square as a double magic square where sum of real part of all numbers is different from sum of imaginary part of all numbers.

3 Magic Tetrahedron

Definition 3.1. A tetrahedron is a polyhedron composed of four triangular faces and six straight edges.

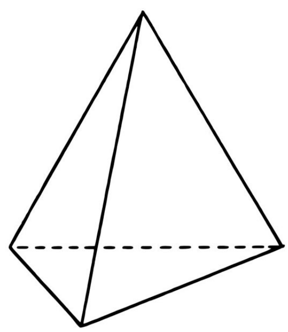

Figure 5: Tetrahedron 3D

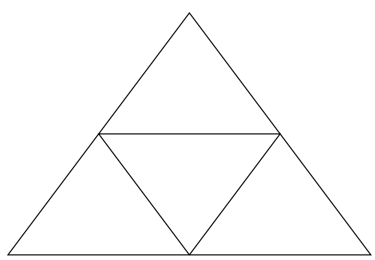

Figure 6: Tetrahedron net

3.1 How does it work?

Firstly, we associate each variable to each corner of all triangles. Secondly, we write all necessary equations, system of equations and matrix associated to a system of equations

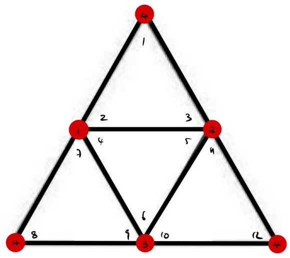

Figure 7: Tetrahedron 2D with numbers

$$"1"+"2"+"3"="4"+"5"+"6"="7"+"8"+"9"="10"+"11"+"12"$$
$$="1"+"8"+"12"="2"+"4"+"5"="3"+"5"+"11"="6"+"9"+"10"$$

we split equation into system of equations which we rewrite as a matrix:

```
[ 1  1  1 -1 -1 -1  0  0  0  0  0  0]
[ 1  1  1  0  0  0 -1 -1 -1  0  0  0]
[ 1  1  1  0  0  0  0  0  0 -1 -1 -1]
[ 1  0  1 -1  0  0 -1  0  0  0  0  0]
[ 1  1  0  0 -1  0  0  0  0  0 -1  0]
[ 1  1  1  0  0 -1  0  0 -1 -1  0  0]
[ 0  1  1  0  0  0  0 -1  0  0  0 -1]
```

Figure 8: A system of equations of tetrahedron

the kernel of that matrix is:

```
[ 1  0  0  0  0  1  1  0  0  0  1  0]
[ 0  1  0  0  0  1  0  0  1 -1  1  1]
[ 0  0  1  0  0  1  1  0  0  0  0  1]
[ 0  0  0  1  0 -1 -1  0  1  0  0  0]
[ 0  0  0  0  1 -1  0  0  0  1 -1  0]
[ 0  0  0  0  0  0  0  1 -1  1  0 -1]
```

Figure 9: A kernel of tetrahedron

3.2 How do we construct magic tetrahedron?

We do the same steps as for magic squares. We just choose randomly 4 numbers to multiply them with each vector in basis to obtain linear combination for final construction.

1	0	0	0
0	1	1	0
0	0	1	0

0	1	0	0
0	1	0	0
1	-1	1	1

0	0	1	0
0	1	1	0
0	0	0	1

0	0	0	1
0	-1	-1	0
1	0	0	0

0	0	0	0
1	-1	0	0
0	1	-1	0

0	0	0	0
0	0	0	1
-1	1	0	-1

As it was mentioned before, with red color, we want to show that there are some numbers which will be 0 if we don't take each vector from our kernel basis. We take random numbers from -10 to 10 multiplied with each vector from kernel.

$$3* \begin{array}{|c|c|c|c|} \hline 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline \end{array} +7* \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 1 & -1 & 1 & 1 \\ \hline \end{array} +9* \begin{array}{|c|c|c|c|} \hline 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array}$$

$$+1* \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 1 \\ \hline 0 & -1 & -1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array} +8* \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & 0 \\ \hline 0 & 1 & -1 & 0 \\ \hline \end{array} +11* \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline -1 & 1 & 0 & -1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 3 & 7 & 9 & 1 \\ \hline 8 & 10 & 11 & -4 \\ \hline 12 & -3 & 2 & 20 \\ \hline \end{array}$$

Now we have magic tetrahedron where sum inside each triangle is 19 and sum of all numbers in each corner is also 19 (see the computations below).

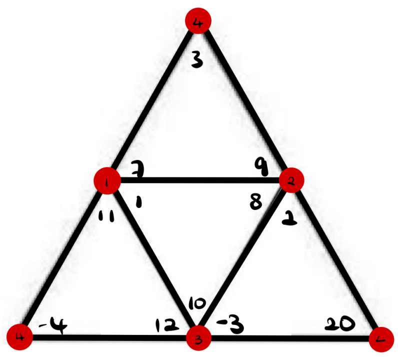

Figure 10: Final version of magic tetrahedron.

$$3 + 7 + 9 = 1 + 8 + 10 = 11 - 4 + 12 = -3 + 2 + 20$$
$$= 7 + 1 + 11 = 12 + 10 - 3 = 9 + 8 + 2 = 20 + 3 - 4 = 19$$

3.3 Properties of magic tetrahedron.

Magic tetrahedron has the same properties as a magic square of any size. We can rotate, it will stay "magic". We may add any constant and multiply with any constant. And we can put same number in every corner and tetrahedron will be magic. Complex numbers are also welcome to be written in.

4 Magic Cube

We cannot really define what is exactly magic cube because it has a lot of possible interpretations depending on the size and nature. We have found 2 interpretations which we find interesting.

4.1 Magic cube interpretations

For the left cube, we put a magic square on each side of the cube with the additional condition that the sum of numbers having a common corner of the cube is the same as the sum in any row and any column of the magic square. For the right cube, the sum of the 9 numbers belonging to each side and the sum of the 9 numbers belonging to diagonal of the whole cube are the same. It's easier to understand when you have a look at the equations and picture of right cube.

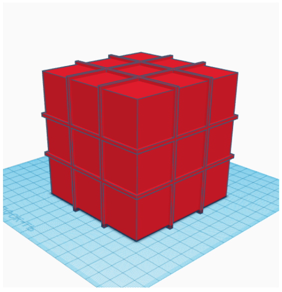

Figure 11: Magic side-cube 3D

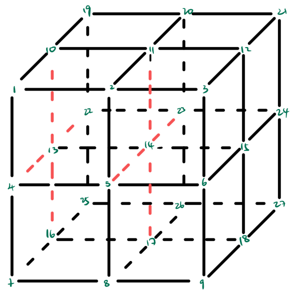

Figure 12: Magic skeleton-cube 3D

4.2 How does skeleton-cube work?

We start working with skeleton-cube. As always, we associate variables and we write necessary equations.

$$\begin{aligned}
"1"+"2"+"3"+"4"+"5"+"6"+"7"+"8"+"9" &= "10"+"11"+"12"+"13"+"14"+"15"+"16"+"17"+"18" \\
&= "19"+"20"+"21"+"22"+"23"+"24"+"25"+"26"+"27" \\
&= "1"+"10"+"19"+"4"+"13"+"22"+"7"+"16"+"25" \\
&= "2"+"11"+"20"+"5"+"14"+"23"+"8"+"17"+"26" \\
&= "3"+"12"+"21"+"6"+"15"+"24"+"9"+"18"+"27" \\
&= "1"+"10"+"19"+"5"+"14"+"23"+"9"+"18"+"27" \\
&= "3"+"12"+"21"+"5"+"14"+"23"+"7"+"16"+"25" \\
&= "1"+"10"+"19"+"2"+"11"+"20"+"3"+"12"+"21" \\
&= "4"+"13"+"22"+"5"+"14"+"23"+"6"+"15"+"24" \\
&= "7"+"16"+"25"+"8"+"17"+"26"+"9"+"18"+"27" \\
&= "1"+"2"+"3"+"13"+"14"+"15"+"25"+"26"+"27" \\
&= "19"+"20"+"21"+"13"+"14"+"15"+"7"+"8"+"9" \\
&= "1"+"11"+"21"+"4"+"14"+"24"+"7"+"17"+"27" \\
&= "3"+"6"+"9"+"11"+"14"+"17"+"19"+"22"+"25"
\end{aligned}$$

Then we split equations into system of equations which we rewrite as a matrix:

```
[ 1  1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1 -1  0  0  0  0  0  0  0  0  0]
[ 1  1  1  1  1  1  1  1  1  0  0  0  0  0  0  0  0  0 -1 -1 -1 -1 -1 -1 -1 -1 -1]
[ 0  1  1  0  1  1  0  1  1 -1  0  0 -1  0  0 -1  0  0 -1  0  0 -1  0  0 -1  0  0]
[ 1 -1  0  1 -1  0  1 -1  0  1 -1  0  1 -1  0  1 -1  0  1 -1  0  1 -1  0  1 -1  0]
[ 1  0 -1  1  0 -1  1  0 -1  1  0 -1  1  0 -1  1  0 -1  1  0 -1  1  0 -1  1  0 -1]
[ 0  0  0  1  1  1  1  1  1 -1 -1 -1  0  0  0  0  0  0 -1 -1 -1  0  0  0  0  0  0]
[ 1  1  1 -1 -1 -1  0  0  0  1  1  1 -1 -1 -1  0  0  0  1  1  1 -1 -1 -1  0  0  0]
[ 1  1  1  0  0  0 -1 -1 -1  1  1  1  0  0  0 -1 -1 -1  1  1  1  0  0  0 -1 -1 -1]
[ 0  1  1  1  0  1  1  1  0 -1  0  0  0 -1  0  0  0 -1 -1  0  0  0 -1  0  0  0 -1]
[ 1  1  0  1  0  1  0  1  1  0  0 -1  0 -1  0 -1  0  0  0  0 -1  0 -1  0 -1  0  0]
[ 0  0  0  1  1  1  1  1  1  0  0  0 -1 -1 -1  0  0  0  0  0  0  0  0  0 -1 -1 -1]
[ 1  1  1  1  1  1  0  0  0  0  0  0 -1 -1 -1  0  0  0 -1 -1 -1  0  0  0  0  0  0]
[ 0  1  1  0  1  1  0  0  0  0 -1  0  0 -1  0  0 -1  0  0  0 -1  0  0 -1  0  0 -1]
[ 1  1  0  1  1  0  1  1  0  0 -1  0  0 -1  0  0 -1  0 -1  0  0 -1  0  0 -1  0  0]
```

Figure 13: Matrix describing system of equations of skeleton-cube

The kernel of that matrix is:

```
[ 1  0  0  0  0  0  2  0  0  0  0  2  0  0  1 -2  1  1  0  0  0  2  1 -1  0  1  0]
[ 0  1  0  0  0  0  2  0  0  0  0  2  0  0  1 -1  1  0  0  0  0  3  1 -2 -1  0  2]
[ 0  0  1  0  0  0  2  0  0  0  0  2  0  0  1  0  1 -1  0  0  0  4  1 -3 -3  1  3]
[ 0  0  0  1  0  0  2  0  0  0  0  3  0  0  1 -2  1  0  0  0  0  3  1 -3 -1  1  2]
[ 0  0  0  0  1  0  2  0  0  0  0  3  0  0  1 -1  1 -1  0  0  0  4  0 -3 -2  1  3]
[ 0  0  0  0  0  1  2  0  0  0  0  3  0  0  1  0  1 -2  0  0  0  4  1 -4 -3  1  4]
[ 0  0  0  0  0  0  3  0  0  0  0  4  0  0  1 -2  1 -1  0  0 -1  4  1 -3 -2  1  3]
[ 0  0  0  0  0  0  0  1  2  0  0  4  0  0  1  2  0 -4  0  0 -1  4  1 -3 -3  1  4]
[ 0  0  0  0  0  0  0  0  3  0  0  4  0  0  1  3  0 -5  0  0 -1  4  1 -3 -4  2  4]
[ 0  0  0  0  0  0  0  0  0  1  0 -1  0  0  0 -1  0  1  0  0  0 -2  0  2  2  0 -2]
[ 0  0  0  0  0  0  0  0  0  0  1 -1  0  0  0  0 -1  1  0  0  0 -1  0  1  1  0 -1]
[ 0  0  0  0  0  0  0  0  0  0  0  0  1  0 -1 -1  0  1  0  0  0 -1  0  1  1  0 -1]
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  1 -1  0 -1  1  0  0  0  0 -1  1  0  1 -1]
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0 -1 -2  0  2  1  0 -1]
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1 -1 -1  0  1  1 -1  0]
```

Figure 14: The kernel for magic skeleton-cube

4.3 How do we construct magic skeleton-cube?

As you can see, we have 15 basis-vectors in the kernel. For the next figures, the number of vectors will increase. And it is very difficult to compute linear combinations manually. So we used a for-loop and while-loop which computed linear combinations for us. For such linear combination, numbers were chosen randomly in range [-20,20] and multiplied with each vector from kernel. While-loop was used to obtain different numbers in every position. Finally, we write all numbers in skeleton cube and now it is a magic skeleton-cube (see the equations below).

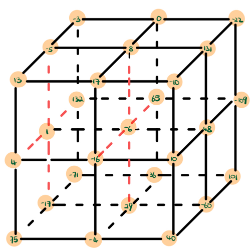

Figure 15: The magic skeleton-cube

$$
\begin{aligned}
13+17-10+4-16+10+75-4+40 &= 8-5+131+1-6+48-17+29-60\\
&= 0-3-22+132+65-109-71+36+101\\
&= 13+4+75-5+1-17-3+132-71\\
&= 17-16-4+8-6+29+0+65+36\\
&= 10-10+40+131+48-60-22-109+101\\
&= 13-5-3+65-6-16+101-60+40\\
&= 131-22-10+65-6-16+75-17-71\\
&= 0-3-22+48-6+1+75-4+40\\
&= 13+17-10+1-6+48+101+36-71\\
&= 13+4+75+8-6+29-22-109+101\\
&= 8-3-10+132-6+10+29+40-71\\
&= 13+17-10-5+8+131-22+0-3\\
&= 4-16+10+1-6+48+132+65-109\\
&= 75-4+40-17+29-60-71+36+101 = 129
\end{aligned}
$$

4.4 Properties of the magic skeleton-cube

As for all previous magic polyhedrons, we can add any constant, multiply by any constant, rotate and/or reflect and use complex numbers and the skeleton-cube will remain magic. Now we pass to side-cube.

4.5 How does side-cube work?

As always, we associate 54 variables. We write equations (see them below). Make a system of equations and compute its kernel. But we had 1 additional restriction for side-cube. Variable "5" must be the same as "14", "23", "32","41","50". See properties of side-cube for the explanation.

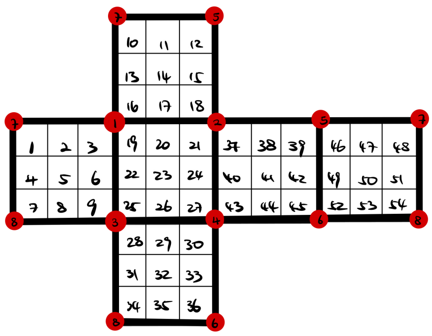

Figure 16: Magic side-cube net

”1” + ”2” + ”3” = ”4” + ”5” + ”6” = ”7” + ”8” + ”9”
= ”10” + ”11” + ”12” = ”13” + ”14” + ”15” = ”16” + ”17” + ”18”
= ”19” + ”20” + ”21” = ”22” + ”23” + ”24” = ”25” + ”26” + ”27”
= ”28” + ”29” + ”30” = ”31” + ”32” + ”33” = ”34” + ”35” + ”36”
= ”37” + ”38” + ”39” = ”40” + ”41” + ”42” = ”43” + ”44” + ”45”
= ”46” + ”47” + ”48” = ”49” + ”50” + ”51” = ”52” + ”53” + ”54”
= ”1” + ”4” + ”7” = ”2” + ”5” + ”8” = ”3” + ”6” + ”9”
= ”10” + ”13” + ”16” = ”11” + ”14” + ”17” = ”12” + ”15” + ”18”
= ”19” + ”22” + ”25” = ”20” + ”23” + ”26” = ”21” + ”24” + ”27”
= ”28” + ”31” + ”34” = ”29” + ”32” + ”35” = ”30” + ”33” + ”34”
= ”37” + ”40” + ”43” = ”38” + ”41” + ”44” = ”39” + ”42” + ”45”
= ”46” + ”49” + ”52” = ”47” + ”50” + ”53” = ”48” + ”51” + ”54”
= ”1” + ”5” + ”9” = ”3” + ”5” + ”7” = ”10” + ”14” + ”18”
= ”12” + ”14” + ”16” = ”19” + ”23” + ”27” = ”21” + ”23” + ”25”
= ”28” + ”32” + ”36” = ”30” + ”32” + ”34” = ”37” + ”41” + ”45”
= ”39” + ”41” + ”43” = ”46” + ”50” + ”54” = ”48” + ”50” + ”52”
= ”1” + ”10” + ”48” = ”3” + ”16” + ”19” = ”7” + ”34” + ”54”
= ”9” + ”25” + ”28” = ”18” + ”21” + ”37” = ”27” + ”30” + ”43”
= ”12” + ”39” + ”46” = ”36” + ”45” + ”52”

we split equation into system of equations which we rewrite as a matrix:

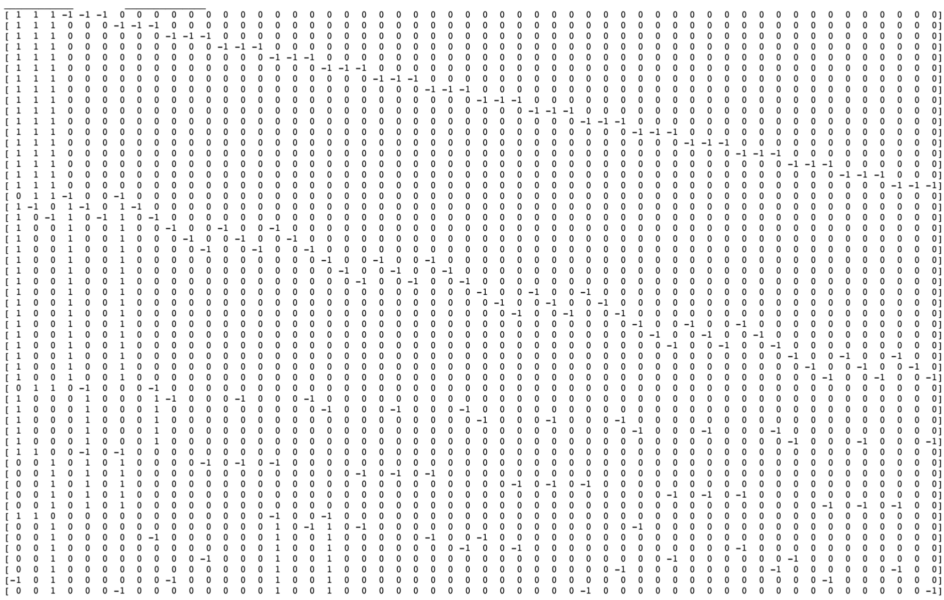

Figure 17: Matrix describing equations of side-cube.

The kernel of that matrix is:

```
[ 1  0  2  2  1  0  0  2  1  0  0  3  4  1 -2 -1  2  2  2  0  1  0  1  2  1  2  0  1  0  2  2  1  0  0  2  1  0  2  1  2  1  0  1  0  2 -1  2  2  4  1 -2  0  0  3]
[ 0  1  2  3  1 -1  0  1  2  0  0  3  4  1 -2 -1  2  2  2  0  1  0  1  2  1  2  0  0  0  3  4  1 -2 -1  2  2  0  1  2  3  1 -1  0  1  2 -2  2  3  6  1 -4 -1  0  4]
[ 0  0  3  4  1 -2 -1  2  2  0  0  3  4  1 -2 -1  2  2  1  0  2  2  1  0  0  2  1  1  0  2  2  1  0  0  2  1 -1  2  2  4  1 -2  0  0  3 -2  2  3  6  1 -4 -1  0  4]
[ 0  0  0  0  0  0  0  0  0  1  0 -1 -2  0  2  1  0 -1 -1  0  1  2  0 -2 -1  0  1  1  0 -1 -2  0  2  1  0 -1  0  0  0  0  0  0  0  0  0  1  0 -1 -2  0  2  1  0 -1]
[ 0  0  0  0  0  0  0  0  0  0  1 -1 -1  0  1  1 -1  0 -1  0  1  2  0 -2 -1  0  1  1  0 -1 -2  0  2  1  0 -1 -1  1  0  1  0 -1  0 -1  1  1 -1  0 -1  0  1  0  1 -1]
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1 -1 -1  0  1  1 -1  0 -1  0  1  2  0 -2 -1  0  1  1 -2  1  0  0  0 -1  2 -1 -1  1  0  1  0 -1  0 -1  1]
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1 -1 -1  0  1  1 -1  0  0  1 -1 -1  0  1  1 -1  0  1 -1  0 -1  0  1  0  1 -1]
```

Figure 18: A kernel of magic cube net

4.6 How do we construct magic side-cube?

With the help of computer and for-loop, we compute the linear combination ("position of vector" * vector) using all vectors in basis. The sum of integers on each column, row, diagonal and corner is 18.

-16	5	140
199	43	-113
-54	81	102

16	7	106
133	43	-47
-20	79	70

77	6	46
12	43	74
40	80	9

-19	72	76
138	43	-52
10	14	105

-87	87	129
259	43	-173
-43	-1	173

19	0	110
134	43	-48
-24	86	67

Figure 19: Side-cube net with all written integers.

Verification of magicness of side-cube.

$$
\begin{aligned}
&16+7+106=133+43-47=-20+79+70=5+140-16=199+43-113=102+81-54\\
&=77+6+46=12+43+74=40+80+9=19+110+0=134+43-48=-24+86+67\\
&=-19+72+76=138+43-52=10+14+105=-87+87+129=259+43-173=-43-1+173\\
&=16+133-20=7+43+79=106-47+70=-16+199-54=5+43+81=140-113+102\\
&=77+12+40=6+43+80=46+74+9=19+134-24=0+43+86=110-48+67\\
&=-19+138+10=72+43+14=76-52+105=-87+256-43=87+43-1=129-173+173\\
&=16+43+70=106+43-20=-16+43+102=77+43+9=46+43+40=19+43+67\\
&=110+43-24=-19+43+105=76+43+10=-87+43+173=129+43-43=106+77-54\\
&=70+40+19=9+10+110=102+46-19=140+76-87=16-16+129=105-43+67\\
&=173-20-24=140+43-54=129
\end{aligned}
$$

4.7 Properties of the magic magic side-cube

One restriction was mentioned previously about the same number at the center of each side. This is necessary because, in previous magic polyhedrons, we established that for magic square 3x3, the sum of all numbers must 3 times the number at the middle. For side-cube, we have in total 6 magic squares. Different numbers at the middle imply that the sum in every row, column on each side will be different and it won't be a magic square anymore. The side-cube has the same properties as magic square 3x3 due to its construction.

5 Magic Dodecahedron

Definition 5.1. A dodecahedron is a polyhedron which counts twelve sides. Each side is a form of penthagon. This figure counts 20 corners and 30 edges. In order to make a magic dodecahderon, we came up with an idea of finding a solution where the magic constant will be the same. Let's find out if it is possible.

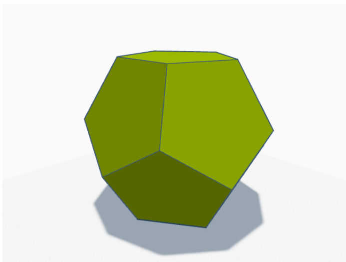

Figure 20: Magic dodecahedron 3D

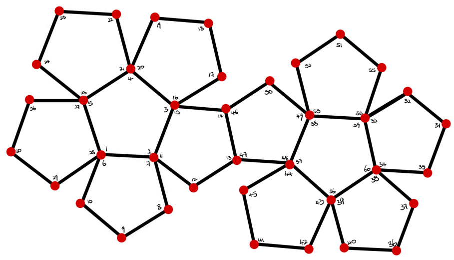

Figure 21: Magic dodecahedron 2D

5.1 How does dodecahedron work?

We write all necessary equations relating 60 variables. We can write all important equations, or we can split them in two parts. One part includes equations with five variables and other part considers all equations with three variables. But, it's better to proceed with all equations simultaneously.

$$
\begin{aligned}
&''1'' + ''2'' + ''3'' + ''4'' + ''5'' = ''6'' + ''7'' + ''8'' + ''9'' + ''10''\\
&= ''11'' + ''12'' + ''13'' + ''14'' + ''15'' = ''16'' + ''17'' + ''18'' + ''19'' + ''20''\\
&= ''21'' + ''22'' + ''23'' + ''24'' + ''25'' = ''26'' + ''27'' + ''28'' + ''29'' + ''30''\\
&= ''31'' + ''32'' + ''33'' + ''34'' + ''35'' = ''36'' + ''37'' + ''38'' + ''39'' + ''40''\\
&= ''41'' + ''42'' + ''43'' + ''44'' + ''45'' = ''46'' + ''47'' + ''48'' + ''49'' + ''50''\\
&= ''51'' + ''52'' + ''53'' + ''54'' + ''55'' = ''56'' + ''57'' + ''58'' + ''59'' + ''60''
\end{aligned}
$$

$$
\begin{aligned}
&''1'' + ''6'' + ''28'' = ''2'' + ''7'' + ''11'' = ''3'' + ''15'' + ''16''\\
&= ''4'' + ''20'' + ''21'' = ''5'' + ''25'' + ''27'' = ''8'' + ''12'' + ''41''\\
&= ''9'' + ''42'' + ''40'' = ''10'' + ''29'' + ''36'' = ''13'' + ''47'' + ''45''\\
&= ''14'' + ''46'' + ''17'' = ''18'' + ''50'' + ''52'' = ''19'' + ''22'' + ''51''\\
&= ''23'' + ''55'' + ''32'' = ''24'' + ''26'' + ''31'' = ''30'' + ''37'' + ''35''\\
&= ''56'' + ''43'' + ''39'' = ''44'' + ''48'' + ''57'' = ''58'' + ''49'' + ''53''\\
&= ''59'' + ''33'' + ''54'' = ''60'' + ''34'' + ''38''
\end{aligned}
$$

When we asked computer to find a solution where magic constant is the same for every corner and every penthatgon, computer reported a mistake. That shows that the solution, which we were looking for, does not exist.

We split all equations to have a system of equations, which can be rewritten as a matrix:

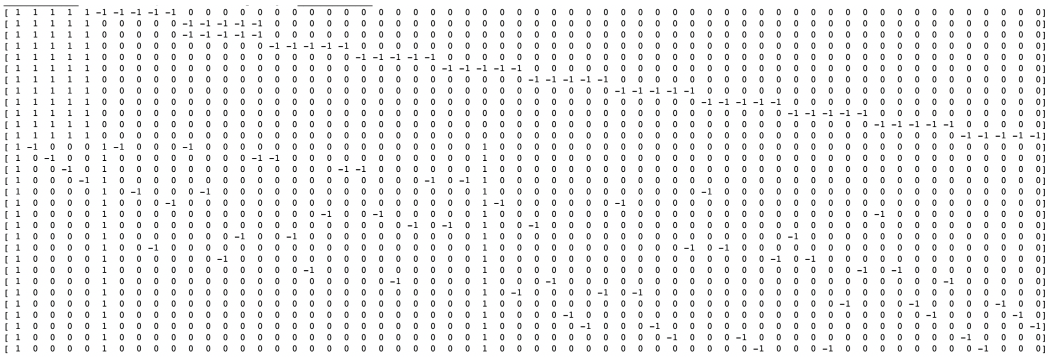

Figure 22: A matrix corresponding to system of equations of magic dodecahedron

The kernel of that matrix is:

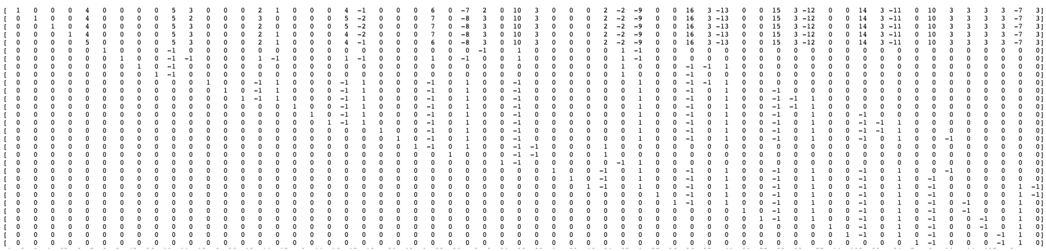

Figure 23: A kernel of the matrix above

5.2 How do we construct magic dodecahedron?

We do the same operations as for all other magic polyhedrons: we compute a linear combination by using old good for-loop and while-loop.

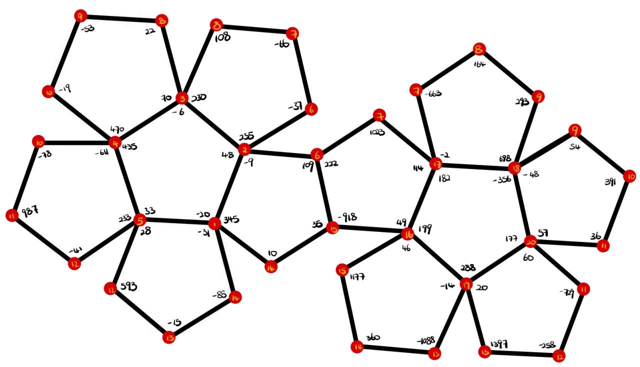

Figure 24: Our final result of magic dodecahedron

$$435 + 33 - 20 + 48 - 6 = 28 + 593 - 31 - 95 - 15 = 109 + 35 + 1 - +345 - 9$$
$$= 230 + 108 + 255 - 66 - 37 = 470 + 70 + 22 - 53 - 19 = 987 + 233 - 78 - 611 - 41$$
$$= 391 + 36 + 57 + 54 - 48 = 1397 + 20 + 60 - 258 - 729 = 1177 + 360 + 46 - 14 - 1088$$
$$= 1023 + 222 + 114 + 49 - 918 = 698 + 164 + 293 - 2 - 663 = 288 + 199 + 182 + 177 - 356$$
$$= 490$$

$$345 - 20 - 31 = 255 + 48 - 9 = 230 + 70 - 6 = 435 + 470 - 611 = 233 + 33 + 28 = 222 + 109 - 37$$
$$= 1023 - 663 - 66 = 108 + 23 + 164 = 293 + 54 - 53 = 391 - 19 - 78 = 987 + 36 - 729 = 593 - 41 - 258$$
$$= 1397 - 1088 - 15 = 1360 + 10 - 85 = 1177 + 35 - 918 = 199 + 49 + 46 = 182 + 114 - 2 = 698 - 356 - 48$$
$$= 177 + 57 + 60 = 288 + 20 - 14 = 294$$

As you may have already noticed, magic constant is not the same for penthagons and corners. The difference will be explained in next session.

5.3 Properties of the magic dodecahedron

The magic dodecahedron is the first example where putting the same non-zero number everywhere will not make it magic,. Because we need to satisfy the general condition 5*x = 3*x, in that case x = 0. According to our previous computations, we have the sum in each penthagon 490 and 294 on each side. We can divide by 98 these numbers and we get then 5 and 3. It can be explained by the fact that we can take all variables and sum them up. We can do it by taking every penthagon or every corner. We have then 12 pentagons and 20 corners. We obtain the ratio 20/12 = 5/3. This means that for any numbers (including complex), the sum inside penthagon will always be 5 times x and for every corner 3 times x. So for other polyhedrons, whose construction includes different amount of variables for magic constant, we will obtain the same problem, unless all variables are equal to 0.

6 3D printed figures

We have also thought about representation of our magic polyhedrons in real life. We have come up with an idea of using 3D-printer to create them. We will print 3D models of both magic cubes, tetrahedron and dodecahedron. Each figure will not exceed the volume of 1L, they can be placed in a cube box of 10x10x10 cm. All numbers on each figure will not be greater than 100 and such that magic constant will be positive. Special thanks to our supervisors for helping us.

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[1] Tetrahedron - Wikipedia. En.wikipedia.org. (2022). Retrieved 21 May 2022, from *https://en.wikipedia.org/wiki/Tetrahedron.*

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[3] Durer's Magic Square — Aiming High Teacher Network. Aiming High Teacher Network —. (2022). Retrieved 21 May 2022, from *https://aiminghigh.aimssec.ac.za/durers − magic − square/.*

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[5] 3D-figures - Tinkercad. Tinkercad.com (2022). Retrieved 21 May 2022, from *https://www.tinkercad.com/dashboard*