Printable Planetarium

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Contents

1 Motivation

What really appealed to us in this project was first and foremost the fact that we were learning by ourselves notions that we had never seen before in class. Some of them such as the Euclidean Algorithm we had already seen, discussed some of its applications to real life problems, but never had we yet encountered best approximations or continued fractions, which were key elements to the computations needed in order to model the pieces and print them to assemble them into a planetarium. Of course what we also really liked was simply that this project was the kind of project that we would have appreciated doing on our own, even if it hadn't been offered by the course, because it's just nice to say that you printed a whole planetarium.

2 Continued fraction

2.1 Introduction

We currently do not know precisely when continued fractions were first discovered, but mathematical history shows they have been used even during the first millennium. During the VIIth century, in a work called *Aryabhatiya*, the technique used by Aryabhata, the author, to produce a general solution to a linear problem, is related to continued fractions. The XVIth and XVIIth century though, seem to be at the origin of major breakthroughs in this domain thanks to the understanding of the euclidean algorithm, used to determine the greatest common divisor (gcd) of two integers x and y, this method of computation being quite similar to the one of the continued fraction of $\frac{x}{y}$. The first person though to have used continued fractions as an application to real life problems, is Christiaan Huygens in the attempt of approximating ratios for his gears in order to build his planetarium, which we will discus later in this document.
If the notion of continued fraction is not something we have discussed in our studies until now, it is something we have sometimes used.

Example. The number $1,\overline{3}$, it can also be describe as $1+\frac{1}{3}$. This is a continued fraction.

And we can develop other number in the same way.

Example. The number $0,\overline{729}$ is a rational number that can be written in the form $\frac{27}{37}$, but can also be written as:

$$\frac{27}{37} = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}}}$$

The aim of this section will now be to add further information on continued fraction as well as give some properties that apply to this notion and an algorithm to compute the continued fraction for a given rational number.

2.2 The Gauss map

The Gauss Map, or also known as the continued fraction operator is defined by

$$\begin{array}{rccc} T: & [0,1) & \longrightarrow & [0,1) \\ & x & \longmapsto & \begin{cases} \frac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \end{array}$$

We have two different cases for this function:

- 1st case : x is an irrational number so $T(x)$ is an irrational number and by induction, so is $T^n(x)$ $\forall n \in \mathbb{N}$.
- 2nd case : x is a rational number so by the Divisor Algorithm of Euclid, there exists an n such that $T^n(x) = 0$.

2.3 Definitions

Definition. (Generalized continued fraction). A generalized continued fraction is a fraction of the form:

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \quad \ddots \quad + \cfrac{b_{n-1}}{a_{n-1} + \cfrac{b_n}{a_n}}}}}$$

where the a_i and b_i are complex numbers.

Example. A continued fraction of π is (formula due to Euler) :

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \quad \ddots}}}}$$

.

Definition. (Simple continued fraction). A regular or simple continued fraction a is a fraction of the form:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \quad \ddots \quad + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

where a_0 is an integer and a_i are positive integers.The continued fraction representation of x is written $x = [a_0; a_1, a_2, \cdots, a_n]$.

Remark. This type of continued fraction is the mainly relevant type of continued fraction for Huygens planetarium. The fraction can also be denoted as $[a_0, a_1, \cdots, a_n]$ where $a_i \in \mathbb{Z}, \forall i \in \mathbb{Z}$, and $\{x\} = x - a_0 \in [0, 1)$.

Example. e can be written as a generalized continued fraction or a regular one:

$$e = 2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cfrac{5}{6 + \ddots}}}} = 1 + \cfrac{1}{0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 = \cfrac{1}{1 + \cfrac{1}{4 + \ddots}}}}}}}.$$

Definition. When the expression of the continued fraction contains a finite number of a_i, it is called finite. On the contrary when the expression does not contains a finite number of a_i it is called an infinite continued fraction. When the terms repeat in the expression of the continued faction, it is called periodic.

Method. Computing a regular fraction of a finite rational number. In the case of an integer x, we just take $a_0 = x$ and all the $a_i = 0$. If x is not an integer, every rational number can be written as a finite fraction. We denote the denominator by d and the numerator by n. To compute each a_i we compute the euclidean division of n by d, a_i is the quotient and n becomes d and d becomes the rest of the division.

Remark. A more detailed algorithm is given in the section "Calculation for a Planetarium" (5.1).

[Wik22]

2.4 Convergents and properties of a continued fraction

Definition. (Convergents of a continued fraction). The fractions $\frac{p_n}{q_n} = [a_0; a_1, \cdots, a_n]$ are the convergents of the continued fraction of x where $n \in \mathbb{N}$, p_n, $q_n \in \mathbb{Z}$, $q_n \geq 1$ and $\gcd(p_n, q_n) = 1$.

[Hen06]

Proposition. *1. In order to compute p_n and q_n, we use the following algorithm:*

$$\begin{aligned}
&p_{-1} = 1 \\
&p_0 = a_0 \\
&q_{-1} = 0 \\
&q_0 = 1 \\
&n \geq 1 \\
&p_n = p_{n-2} + a_n p_{n-1} \\
&q_n = q_{n-2} + a_n q_{n-1}
\end{aligned}$$

2.

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n$$

3.

$$\begin{aligned}&\{x\} = \frac{p_n + p_{n-1}T^n(t)}{q_n + q_{n-1}T^n(t)}\\ &n \geq 1\\ &\{x\} = x - a_0\end{aligned}$$

Remark. Those 3 properties are basic properties of continued fractions. T^n describes the n-th iteration of of the Gauss map.

Proof. 1. Let x be a rational number.
We define the matrices A_i such that $\forall i \in \{1, \cdots, n\}$:

$$A_i = \begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix}.$$

We have :

$$M_n = A_1 \cdot A_2 \cdot A_3 \cdots A_n.$$

Proposition. *(Möbius transformation). If a matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

with a, b, c and d $\in \mathbb{Z}$ *and* $\det(A) = ad - bc = \pm 1$. *The we can define the Möbius transformation:*

$$A : \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R} \cup \{\infty\} : x \longmapsto \frac{ax+b}{cx+d}.$$

Remark. Notice that for each A_i, we have that $\det(A_i) = 0 \times a_i - 1 \times 1 = -1$. So we can define the Möbius transformation:

$$A_i(x) = \frac{1}{x + a_i}.$$

In particular, we have:

$$A_i(0) = \frac{1}{a_i}.$$

We compute now $A_{i-1} \cdot A_i(0)$:

$$A_{i-1} \cdot A_i(0) = \begin{pmatrix} 0 & 1 \\ 1 & a_{i-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix}(0) = \begin{pmatrix} 0 & 1 \\ 1 & a_{i-1} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ a_i \end{pmatrix} = \frac{1}{\frac{1}{a_i} + a_{i-1}} = \frac{1}{a_{i-1} + \frac{1}{a_i}}.$$

By iterating this process we obtain:

$$M_n(0) = A_1 \cdot A_2 \cdot A_3 \cdots A_n(0) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \quad \ddots \quad + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

And this is the regular continued fraction of x.
Since for all i we have that A_i are matrices with integers, their product M_n is also a matrix with only integers. We can write M_n as:

$$M_n = \begin{pmatrix} u_n & p_n \\ v_n & q_n \end{pmatrix}$$

and thus:

$$M_n(0) = \begin{pmatrix} u_n & p_n \\ v_n & q_n \end{pmatrix}(0) = \frac{p_n}{q_n}.$$

Moreover:

$$M_n = A_1 \cdot A_2 \cdot A_3 \cdots A_n = M_{n-1} \cdot A_n.$$

So we have that:

$$M_n = \begin{pmatrix} u_{n-1} & p_{n-1} \\ v_{n-1} & q_{n-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & a_n p_{n-1} + u_{n-1} \\ q_{n-1} & a_n q_{n-1} + v_{n-1} \end{pmatrix} = \begin{pmatrix} u_n & p_n \\ v_n & q_n \end{pmatrix}.$$

with $u_n = p_{n-1}$ and $v_n = q_{n-1}$.
And we have that for $n \geq 1$:

- $p_{-1} = 1$, $p_0 = 0$ and $p_n = a_n p_{n-1} + p_{n-2}$
- $q_{-1} = 0$, $q_0 = 1$ and $q_n = a_n q_{n-1} + q_{n-2}$.

Thus we have proven the first part of the proposition.

2.

Remark. For rational numbers, the continued fraction is finite and the last convergent (the n-th convergent) is the rational number and thus the error is 0.

Proposition. *If A and B are two square matrix of order n. Then*

$$\det(AB) = \det(A)\det(B).$$

So we have that:

$$\det(M_n) = \det(A_1) \cdot \det(A_2) \cdots \det(A_n) = (-1)^n.$$

From this we can deduce that:

$$u_n q_n - v_n p_n = p_{n-1} q_n - p_n q_{n-1} = (-1)^n.$$

Which confirmed the second part of the proposition. And we can also deduce that $\gcd(p_n, q_n) = 1$.

3. We replace the matrix A_n by:

$$\tilde{A}_n = \begin{pmatrix} 0 & 1 \\ 1 & a_n = T^n(x) \end{pmatrix},$$

and we compute $\tilde{M}_n$ given by:

$$\begin{aligned}
\tilde{M}_n &= A-1 \cdot A_2 \cdots A_{n-1} \cdot \tilde{A}_n \\
&= M_{n-1} \cdot \tilde{A}_n \\
&= \begin{pmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & a_n = T^n(x) \end{pmatrix} \\
&= \begin{pmatrix} p_n & p_{n-2} + p_{n-1}\left(a_n + T^n(x)\right) \\ q_n & q_{n-2} + q_{n-1}\left(a_n + T^n(x)\right) \end{pmatrix} \\
&= \begin{pmatrix} p_n & p_{n-2} + p_{n-1}a_n + p_{n-1}T^n(x) \\ q_n & q_{n-2} + q_{n-1}a_n + q_{n-1}T^n(x) \end{pmatrix} \\
&= \begin{pmatrix} p_n & p_n + p_{n-1}T^n(x) \\ q_n & q_n + q_{n-1}T^n(x) \end{pmatrix}.
\end{aligned}$$

And for $x = 0$:

$$\tilde{M}_n(0) = \frac{p_n + p_{n-1}T^n(x)}{q_n + q_{n-1}T^n(x)}.$$

But we have :

$$\begin{aligned}
\tilde{M}_n(0) &= A-1 \cdot A_2 \cdots A_{n-1} \cdot \tilde{A}_n(0) \\
&= A-1 \cdot A_2 \cdots A_{n-1} \cdot \frac{1}{a_n + T^n(x)} \\
&= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \quad \ddots \quad + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + T^n(x)}}}}} \\
&= x,
\end{aligned}$$

and thus we have that:

$$x = \frac{p_n + p_{n-1}T^n(x)}{q_n + q_{n-1}T^n(x)}.$$

And we have proven the third part of the proposition

□

[Ami10]

2.5 Margin of error

The margin of error at the i-th convergent is:

$$\left|x - \frac{p_i}{q_i}\right|.$$

The greater the i, the smaller the error is. Now to compute the maximal error, we replace the matrix A_n by:

$$\tilde{A}_n = \begin{pmatrix} 0 & 1 \\ 1 & a_n = T^n(x) \end{pmatrix},$$

and from the point 3) of the proposition in the previous section we have that:

$$x = \frac{p_n + p_{n-1}T^n(x)}{q_n + q_{n-1}T^n(x)}.$$

From that we deduce that the margin of error is given by:

$$\begin{aligned}
\left|x - \frac{p_n}{q_n}\right| &= \left|\frac{p_n + p_{n-1}T^n(x)}{q_n + q_{n-1}T^n(x)} - \frac{p_n}{q_n}\right| \\
&= \left|\frac{q_n p_n + q_n p_{n-1}T^n(x) - p_n q_n + p_n q_{n-1}T^n(x))}{q_n(q_n + q_{n-1}T^n(x))}\right| \\
&= \frac{T^n(x)\,|q_n p_{n-1} - p_n q_{n-1}|}{q_n^2(1 + T^n(x)\frac{q_{n-1}}{q_n})} \\
&< \frac{1}{q_n^2}
\end{aligned}$$

[Ami10]

3 Best approximation

The continued fraction can be used to approach the ratio of the orbital of the planet as did Christiaan Huygens. In this section, we will see if the best approximation possible is the convergents of a continued fraction.

3.1 Ostrowski representation

Definition. (Best approximation) Let $x \in \mathbb{R} \backslash \mathbb{Q}$ and let $\frac{p}{q} \in \mathbb{Q}$ with $\gcd(p,q) = 1$. We say that $\frac{p}{q}$ is a best approximation to x if and only if for all $\frac{r}{s} \in \mathbb{Q}$ such that $\frac{r}{s} \neq \frac{p}{q}$ and $0 < s < q$ we have $|qx - p| < |sx - r|$.

Remark. From this definition we can deduce that:

$$\left|x - \frac{p}{q}\right| = \frac{1}{q}|qx - p| < \frac{1}{q}|sx - r| < \frac{1}{s}|sx - r| = \left|x - \frac{s}{r}\right|$$

Theorem. *(Ostrowski representation). Let $[a_0; a_1, \cdots, a_n]$ be the continued fraction of the irrational number x with convergents $\frac{p_i}{q_i}$. For every natural number m there exists $n \in \mathbb{N}$ such that $q_N \geq m < q_{N+1}$. Then there is an unique finite sequence $(c_{i+1})_{i\in\mathbb{N}}$ such that:*

$$m = \sum_{i=0}^{N} c_{i+1} \cdot q_i$$

where

$$\begin{cases} 0 \geq c_{i+1} \geq a_{i+1}, & \text{if } i > 0 \\ 0 \geq c_{i+1} < a_{i+1}, & \text{if } i = 0 \\ c_i = 0, & \text{if } c_{i+1} = a_{i+1} \end{cases}$$

3.2 Lemma's needed for the theorem on best approximation

Lemma. *Let x be an irrational number and let $[a_0, a_1, \cdots, a_n]$ be its continued fraction. We denote the convergents by $\frac{p_i}{q_i}$ for each $i \in \{1, \cdots, n\}$. Let m be a positive integer. The Ostrowski representation of m is given by $m = \sum_{i=0}^{N} c_{i+1} \cdot q_i$, where $c_{i+1} = 0$ for $0 \geq i < n \geq N$. Then*

$$|(c_{n+1} - 1)D_n - D_{n+1}| < \left| \sum_{i=0}^{N} c_{i+1} \cdot D_i \right| < |c_{n+1} D_n - D_{n+1}|$$

where

$$D_i = q_i x - p_i.$$

Proof. To prove this lemma, there are four cases to be examined:

1. $D_n > 0$ and $D_N > 0$
2. $D_n > 0$ and $D_N < 0$
3. $D_n < 0$ and $D_N > 0$
4. $D_n < 0$ and $D_N < 0$

But this proof is quite long and complicated as it needs theorems we haven't given here. For the complete proof we looked at [Bos+18] section 2.1 Lemma 6. □

Lemma. *Let x be an irrational number and let $[a_0, a_1, \cdots, a_n]$ be its continued fraction. We denote the convergents by $\frac{p_i}{q_i}$ for each $i \in \{1, \cdots, n\}$. Let m be a positive integer. The Ostrowski representation of m is given by $m = \sum_{i=0}^{N} c_{i+1} \cdot q_i$, where $c_{i+1} = 0$ for $0 \geq i < n \geq N$. Then*

1. *if $c_1 = c_2 = 0$ the $\|mt\| = \left|\sum_{i=0}^{N} c_{i+1} \cdot D_i\right|$*
2. *if $c_1 = 0$ and $c_2 > 0$ then*
   (a) *if $\{t\} < \frac{1}{2}$ then $\|mt\| = \left|\sum_{i=0}^{N} c_{i+1} \cdot D_i\right|$*
   (b) *if $\{t\} > \frac{1}{2}$ then*
      i. *if $c_2 > 1$ then $\|mt\| > \|t\|$*

ii. if $c_2 = 1$ then $\|mt\| > D_2$

3. if $c_1 = 0$ then $\|mt\| > |D_1|$

Proof. As for the previous Lemma, we don't have the mathematical level to be able to find a proof by ourselves. For the complete proof we looked at [Bos+18]. □

Remark. Neither of these two Lemmas were proven as we only gave them for the next theorem on the best approximation. We wanted to focuse ourselves on the next proof as it is for the main theorem of this project.

3.3 Theorem

Theorem. *(Best approximation). Let x be an irrational number and p, $q \in \mathbb{Z}$ with $(p,q) = 1$ and $q > 0$. Then $\frac{p}{q}$ is a best approximation if and only if it is a convergent of x.*

Proof. Let the Ostrowski representation of q be $\sum_{i=0}^{N} c_{i+1}q_i$. Suppose there exists $n \in \mathbb{N}$ such that $c_{i+1} = 0$ for all $0 \geq i < n \geq N$ and $c_{n+} > 0$. We have to check every possibility for c_1 and c_2.

Case 1: $c_1 = c_2 = 0$
We need to examined four cases:

1. $D_n > 0$ and $D_N > 0$

$$\left|\sum_{i=0}^{N} c_{i+1} \cdot D_i\right| \geq (c_{n+1} - 1)D_n - D_{n+1} + D_N > D_N = |D_N| = |q_N x - p_N|$$

2. $D_n > 0$ and $D_N < 0$

$$\left|\sum_{i=0}^{N} c_{i+1} \cdot D_i\right| \geq (c_{n+1} - 1)D_n - D_{n+1} + D_{N+1} > -D_{n+1} = |D_{n+1}| \geq |D_N| = |q_N x - p_N|$$

Remark. $n \neq N$ as N_n and D_N have different signs. Thus we have that $N \geq n+1$.

3. $D_n < 0$ and $D_N > 0$

$$\left|\sum_{i=0}^{N} c_{i+1} \cdot D_i\right| \geq (1 - c_{n+1})D_n + D_{n+1} - D_{N+1} > D_{n+1} = |D_{n+1}| > |D_N| = |q_N x - p_N|$$

Remark. In this case we also have $N \geq n+1$.

4. $D_n < 0$ and $D_N < 0$

$$\left|\sum_{i=0}^{N} c_{i+1} \cdot D_i\right| \geq (1 - c_{n+1})D_n + D_{n+1} - D_N > -D_N = |D_N| = |q_N x - p_N|$$

So $|q_N x - p_N|$ minimizes $|qx - p|$ in all possible cases. let's prove the theorem for this case now. Assume that $\frac{p}{q}$ is a best approximation. By definition of a best approximation we have that $|qx - p| < |q_N x - p_N|$ for all $\frac{p}{q} \neq \frac{p_N}{q_N}$ such that $0 < q_N < q$. But

$$|q_N x - p_N| = \|qt\| = \left| \sum_{i=0}^{N} c_{i+1} \cdot D_i \right|$$

so we have $|qx - p| > |q_N x - p_N|$ for all $\frac{p}{q} \neq \frac{p_N}{q_N}$. And thus we deduce $p_N = p$ and $q_N = q$. Assume that $\frac{p}{q}$ is a convergent. Then there exists an integer m such that $|qx - p| = |q_m x - p_m| = |D_m|$. We have that $q_N \leq q_m < q_{n+1}$. So we deduce that $q_N = q$, so $N = m$ and conclude that $\frac{p_N}{q_N}$ is a best approximation.

Case 2 (a): where $c_1 = 0$, $c_2 > 1$, and $\{x\} < \frac{1}{2}$
This case is similar to Case 1.

Case 2 (b) (i) : $c_1 = 0, c_2 > 1, \{t\} > \frac{1}{2}$.
Assume that $q = q_0 = q_1 = 1$. We have $|D_1| < \frac{1}{2}$, so $\|q_1 x\| = \|q_1 x - p_1\| = |D_1|$. Thus we have $|q_1 x - p_1| > |q_1 x - p_1|$. So $q > q_1$ and from

$$|qx - p| \geq \|qx\| > 1 - \{t\} = -D_1 = |D_1| = |q_1 x - p_1|$$

we have that $|qx - p| > |q_1 x - p_1|$. And by definition we conclude that $\frac{p}{q}$ is not a best approximation.
Now assume that $\frac{p}{q}$ is a convergent, there exists an integer m such that $|qx - p| = |q_m x - p_m| = |D_m|$. We know that $|D_m| = |q_m x - p_m| = |qx - p| > \|qx\| > |D_1|$. We find $|D_m| > |D_k|$ for all k. Since $q > q_1$ we deduce that $\frac{p}{q}$ is not a convergent.

Case 2 (b) (ii) : $c_1 = 0$, $c_2 = 1$, $\{t\} > \frac{1}{2}$.
There are three possibilities for N.

1. $N = 0$
   We know that $c_1 = 0$ hence $q = c_1 q_0 = 0$, which is a contradiction.

2. $N = 1$
   The theorem about the three basic properties of continued fractions and the fact that $a_1 = 1$ enable us to deduce that

   $$q = c_1 q_0 + c_2 q_1 = q_1$$
   $$p = c_1 p_0 + c_2 p_1 = p_1.$$

   Assume now that $\frac{p}{q}$ is a convergent. We have $q = q_1 = 1$, hence we can't find a $s \in \mathbb{N}$: $0 < s < q$. Therefore $\frac{p}{q}$ is a best approximation.

3. $N > 1$
   Remember that $c_1 = 0$, $c_2 = 1$, $q_1 = a_1 q_0 + q_{-1} = 1$. Thus

   $$q = c_1 q_0 + c_2 q_1 + c_3 q_2 + \cdots + c_{N+1} q_N = 1 + c_3 q_2 + \cdots + c_{N+1} q_N.$$

But $N > 1$ which means there must exist $z \in \{2,3,\ldots,N\}$ such that $c_{z+1}q_z \neq 0$. We also know that $q > q_z \geq q_2$ and that

$$|q_t - p| \geq \|qt\| > D_2 = |D_2|.$$

Now since $q > q_2$ and $|q_t - p| > |q_2 t - p_2|$, we deduce that $\frac{p}{q}$ isn't a best approximation. Also supposing it would be a convergent, we'd be able to find a $m \in \mathbb{N}$ such that $|qt - p| = |q_m x - p_m| = D_m|$. But $|q_m x - p_m| = |D_m| > |D_2|$ and actually, $|D_m| > |D_i| \forall i \in \mathbb{N}_{\geq 2}$.

Case 3: $c_1 \geq 0$
We have two cases to examined.

1. $c_1 = 1$ and $N = 0$
   We find that $q = c_1 q_0 = q_0$ and $p = c_1 p_0 = p_0$.
   $\frac{p}{q}$ is obviously a convergent. As $q = q_0 = 1$ there are no $s \in \mathbb{N} : 0 < s < q$. Thus $\frac{p}{q}$ is a best approximation.

2. $N \neq 0$
   So $q > 1$. Assuming that $\frac{p}{q}$ is a best approximation, we have:

$$q = c_1 q_0 + c_2 q_1 + \cdots + c_{N+1} q_N \geq c_1 + c_{z+1} q_z \geq q_z.$$

   We know that $q_z \geq q_1$, so $q \geq q_1$ and $|q_t - p| > |q_1 t - p_1|$. And so by definition $\frac{p}{q}$ is not a best approximation and there is a contradiction.
   Suppose $\frac{p}{q}$ is a convergent, there exists an integer m such that $|qx - p| = |q_m x - p_m| = |D_m|$. We know that $|D_m| = |q_m x - p_m| = |qx - p| > \|qx\| > |D_1|$. We find $|D_m| > |D_k|$ for all k. Since $q > 1$, $q_m \neq q_0$ we deduce that $\frac{p}{q}$ is not a convergent.

□

[Bos+18]

4 Planetarium

4.1 Definition of a Planetarium

A planetarium is a representation, a model of the solar system. In our case, we attempt to print our planets and gears with a 3D printer which we will have programmed using the computations we made using the method of the best approximations, continued fractions and algorithms. We then assemble those pieces in the most realistic possible way in order to recreate a mini version of the solar system.
A planetarium though can also be defined as a building in which moving images are shown at night with a star projector.

4.2 Huygens Planetarium

Christiaan Huyguens was a Dutch scientist specialised in many disciplines such as mathematics, engineering, physics or astronomy during the XVII-th century. Some of his work involves the invention of the pendulum clock, the magic lantern, the centrifugal governor, and in our case, his planetarium.

Christiaan Huygens

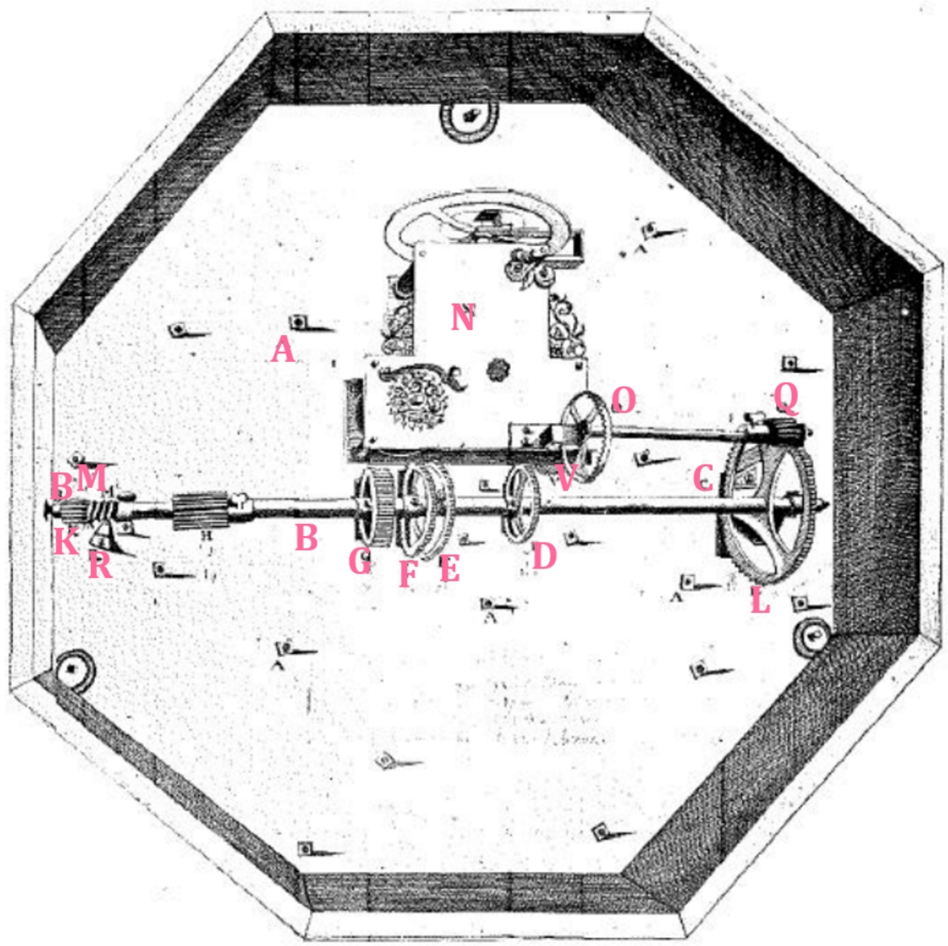
Huygens Planetarium

4.3 Mechanism

Below are two pictures of the inside of Huygens planetarium. We have annotated a sketch of the interior of Huygens planetarium after with the explanation of what are each letters.

Interior of Huygens planetarium

Interior of Huygens planetarium



Huygens planetarium - interior

1. A : square bases, end of pillars that are screwed into
2. B-C : 61cm long iron rod, the shaft of the planetarium
3. D : gear with 121 teeth, responsible for Mercury's movement
4. E : gear with 52 teeth, responsible for Venus movement
5. F : gear with 60 teeth, responsible for Earth's movement
6. G : gear with 84 teeth, responsible for Mars movement
7. H : gear with 14 teeth, responsible for Jupiter's movement
8. K : gear with 7 teeth, responsible for Saturn's movement
9. L : gear with 73 teeth, responsible for the movement of the circle with days and months
10. M : piece of metal spiral, responsible with the aid of 2 gears attached to an axis for the cycle of 300 years
11. N : clock mechanism

12. V : gear responsible for the movement of the 61cm long iron rod. Its revolution period is of 96hours
13. P : 4 teeth of gear V
14. O : gear with 45 teeth moved by teeth P
15. Q : gear attached to axis of O with 9 teeth responsible for the movement of L, which ends up turning B-C

[Ami10]

The force driving the planets around the planetarium is generated by a clock-mechanism which was also invented by Huygens. This mechanism is connected to a shaft which is connected to 6 gears each for a different planet. The gears each have a different number of teeth which was calculated using continued fraction. The planetarium is composed of 6 rings with teeth each representing the orbit of a planet. When the planetarium is closed, the rings are connected to the gears through their teeth. Another gear connects the shaft to the clock-mechanism and is responsible for the movement of the planets.
This planetarium (and the one we aim to build) are 'ideals" planetariums as the orbits of the planets have no inclinations and are all on a same plane. We also need to add that the orbits describe by the planets are circles while it is not the case in reality.

5 Calculation for a planetarium

5.1 Algorithm

The following algorithm is given in its entirety and in python language in the annex. Below is just the most important part which has to do with the loop computing each a_i of the continued fraction. The value for the earth orbital period is $365,256$ days which can be written as $365 + \frac{32}{125}$ days or $\frac{45657}{125}$ days. Thus each orbital period is multiplied by 125 so that there are no problems for the calculations afterward.

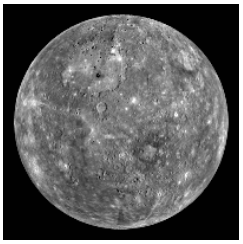
- *orb.p* represents the orbital period of the chosen planet given in days $\times 125$
- *orb.e* represents the earth orbital period
- c represents the number of digit after the decimal point in *orb.p*

$n \leftarrow orb.p \times 10^c$
$d \leftarrow orb.e \times 10^c$
$r = 1$
while $r \neq 0$ **do**
    $n' \leftarrow n$
    $x \leftarrow \lfloor \frac{n}{d} \rfloor$
    $n \leftarrow d$
    $d \leftarrow n' - r \times d$
end while

The result x is a list with all the component of the continued fraction.

5.2 Application

In this section, we will calculate the convergents for the ratio of the orbital period of each planet with the earth's orbital period, using the algorithm presented in the previous subsection. With those calculations, we will compute the each gear ratio for the construction of our planetarium.



Mercury

Mercury

Orbital period [days]: 87.969.

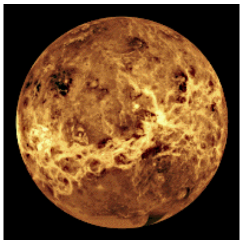
[Dav21a]

The continued fraction is of the ratio between Mercury orbital period and the Earth orbital period is:

$$\frac{10996125}{45675000} = \frac{29323}{121752} = [0;4,6,1,1,2,1,5,1,1,2,6,3]$$

The table of convergent with the margin of error is:

Order of convergent	Convergent Ratio	Margin of error in %
0	0	2.4×10^{1}
1	$\frac{1}{4}$	9.2×10^{-1}
2	$\frac{6}{25}$	8.4×10^{-2}
3	$\frac{7}{29}$	5.3×10^{-2}
4	$\frac{13}{54}$	1.0×10^{-2}
5	$\frac{33}{137}$	3.4×10^{-3}
6	$\frac{46}{191}$	4.3×10^{-4}
7	$\frac{263}{1092}$	4.5×10^{-5}
8	$\frac{309}{1283}$	2.6×10^{-5}
9	$\frac{572}{2375}$	6.6×10^{-6}
10	$\frac{1453}{6033}$	4.1×10^{-7}
11	$\frac{9290}{38573}$	2.1×10^{-8}
12	$\frac{29323}{121752}$	0



Venus

Venus

Orbital period [days]: 224.701.

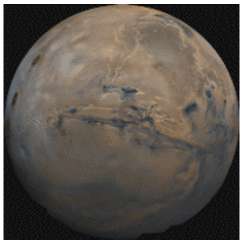
[Dav21b]

The continued fraction is of the ratio between Venus orbital period and the Earth orbital period is:

$$\frac{28087625}{45675000} = \frac{224701}{365256} = [0;1,1,1,1,2,29,1,1,1,77,4]$$

The table of convergent with the margin of error is:

Order of convergent	Convergent Ratio	Margin of error in %
0	0	6.2×10^{1}
1	$\frac{1}{1}$	3.8×10^{1}
2	$\frac{1}{2}$	1.2×10^{1}
3	$\frac{2}{3}$	5.1×10^{0}
4	$\frac{3}{5}$	1.5×10^{0}
5	$\frac{8}{13}$	2.0×10^{-2}
6	$\frac{235}{382}$	4.5×10^{4}
7	$\frac{243}{395}$	2.2×10^{-4}
8	$\frac{478}{777}$	1.1×10^{-4}
9	$\frac{721}{1172}$	9.3×10^{-7}
10	$\frac{55995}{91021}$	3.0×10^{-9}
11	$\frac{224701}{365256}$	0



Mars

Mars

Orbital period [days]: 686.980.

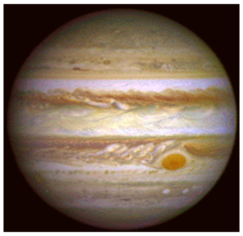
[Dav21c]

The continued fraction is of the ratio between Mars orbital period and the Earth orbital period is:

$$\frac{858725}{456750} = \frac{171745}{91314} = [1;1,7,2,1,1,3,1,1,1,1,1,1,1,2,9]$$

The table of convergent with the margin of error is:

Order of convergent	Convergent Ratio	Margin of error in %
0	1	8.8×10^{1}
1	$\frac{2}{1}$	1.2×10^{1}
2	$\frac{15}{8}$	5.8×10^{-1}
3	$\frac{32}{17}$	1.5×10^{-1}
4	$\frac{47}{25}$	8.1×10^{-2}
5	$\frac{79}{42}$	1.3×10^{-2}
6	$\frac{284}{151}$	2.3×10^{-3}
7	$\frac{363}{193}$	1.1×10^{-3}
8	$\frac{647}{344}$	3.9×10^{-4}
9	$\frac{1010}{537}$	1.5×10^{-4}
10	$\frac{1657}{881}$	5.8×10^{-5}
11	$\frac{2667}{1418}$	2.1×10^{-5}
12	$\frac{4324}{2299}$	9.0×10^{-6}
13	$\frac{6991}{3717}$	2.7×10^{-6}
14	$\frac{18306}{9733}$	1.1×10^{-7}
15	$\frac{171745}{91314}$	0



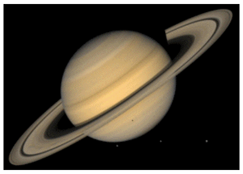
Jupiter

Jupiter
Orbital period [days]: 4330.595.
[Dav21d]
The continued fraction is of the ratio between Jupiter orbital period and the Earth orbital period is:

$$\frac{541324375}{45675000} = \frac{4330595}{365256} = [11; 1, 5, 1, 24, 5, 5, 1, 1, 36]$$

The table of convergent with the margin of error is:

Order of convergent	Convergent Ratio	Margin of error in %
0	11	8.6×10^{1}
1	$\frac{12}{1}$	1.4×10^{1}
2	$\frac{71}{6}$	2.3×10^{0}
3	$\frac{83}{7}$	8.1×10^{-2}
4	$\frac{2063}{174}$	6.3×10^{-4}
5	$\frac{10398}{877}$	2.3×10^{-5}
6	$\frac{54053}{4559}$	2.2×10^{-6}
7	$\frac{64451}{5436}$	1.8×10^{-6}
8	$\frac{118504}{9995}$	2.7×10^{-8}
9	$\frac{4330595}{365256}$	0



Saturn

Saturn

Orbital period [days]: 10759.22.

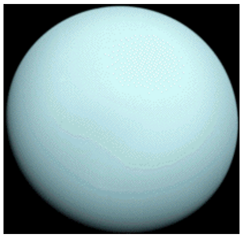
[Dav21e]

The continued fraction is of the ratio between Saturn orbital period and the Earth orbital period is:

$$\frac{13449025}{456750} = \frac{2689805}{91314} = [29; 2, 5, 3, 1, 2, 1, 3, 1, 1, 1, 2, 4, 1, 2]$$

The table of convergent with the margin of error is:

Order of convergent	Convergent Ratio	Margin of error in %
0	29	4.6×10^{1}
1	$\frac{59}{2}$	4.3×10^{0}
2	$\frac{324}{11}$	2.1×10^{-1}
3	$\frac{1031}{35}$	4.9×10^{-2}
4	$\frac{1355}{46}$	1.3×10^{-2}
5	$\frac{3741}{127}$	3.8×10^{-3}
6	$\frac{5096}{173}$	7.7×10^{-4}
7	$\frac{19029}{646}$	1.3×10^{-4}
8	$\frac{24125}{819}$	6.0×10^{-5}
9	$\frac{43154}{1465}$	2.3×10^{-5}
10	$\frac{67279}{2284}$	6.7×10^{-6}
11	$\frac{1777212}{6033}$	5.4×10^{-7}
12	$\frac{778127}{26416}$	8.3×10^{-8}
13	$\frac{955839}{39449}$	3.4×10^{-8}
14	$\frac{2689805}{91314}$	0



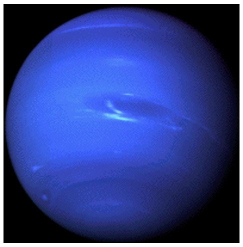
Uranus

Uranus
Orbital period [days]: 30685.4.
[Dav21f]
The continued fraction is of the ratio between Uranus orbital period and the Earth orbital period is:

$$\frac{38356750}{456750} = \frac{3835675}{45657} = [84; 93, 1, 3, 40, 3]$$

The table of convergent with the margin of error is:

Order of convergent	Convergent Ratio	Margin of error in %
0	84	1.1×10^{0}
1	$\frac{7813}{93}$	8.6×10^{-3}
2	$\frac{7897}{94}$	2.8×10^{-3}
3	$\frac{31504}{375}$	1.8×10^{-5}
4	$\frac{1268057}{15094}$	1.5×10^{-7}
5	$\frac{3835675}{45657}$	0



Neptune

Neptune
Orbital period [days]: 60189.

The continued fraction is of the ratio between Neptune orbital period and the Earth orbital period is:

$$\frac{75236250}{456750} = \frac{2507875}{15219} = [164; 1, 3, 1, 2, 63, 1, 1, 2, 3]$$

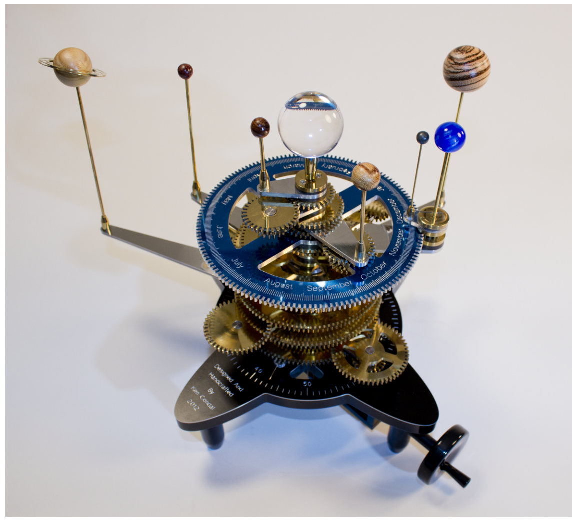
The table of convergent with the margin of error is:

Order of convergent	Convergent Ratio	Margin of error in %
0	164	7.9×10^{1}
1	165	2.1×10^{1}
2	$\frac{659}{4}$	3.6×10^{0}
3	$\frac{824}{5}$	1.4×10^{0}
4	$\frac{2307}{14}$	8.0×10^{-3}
5	$\frac{146165}{887}$	7.4×10^{-5}
6	$\frac{148472}{901}$	5.1×10^{-5}
7	$\frac{294637}{1788}$	1.1×10^{-5}
8	$\frac{737746}{4477}$	1.5×10^{-6}
9	$\frac{2507875}{15219}$	0

6 Printable planetarium

The aim is now to choose a convergent for each ratio which will enable us to have on the one side the smallest margin of error possible and on the other side a printable gear or train of gear. Indeed the gear ratio is limited by the number of teeth a gear can have. It is possible to attached multiple gear together in order to have a more precise ratio but we do not want to have a planetarium that is too big.
We first had to choose what type of planetarium, i.e., how will the gears be placed. While

looking for ideas and possibilities, we found a planetarium by Zeamon which seems feasible. [Zea22]

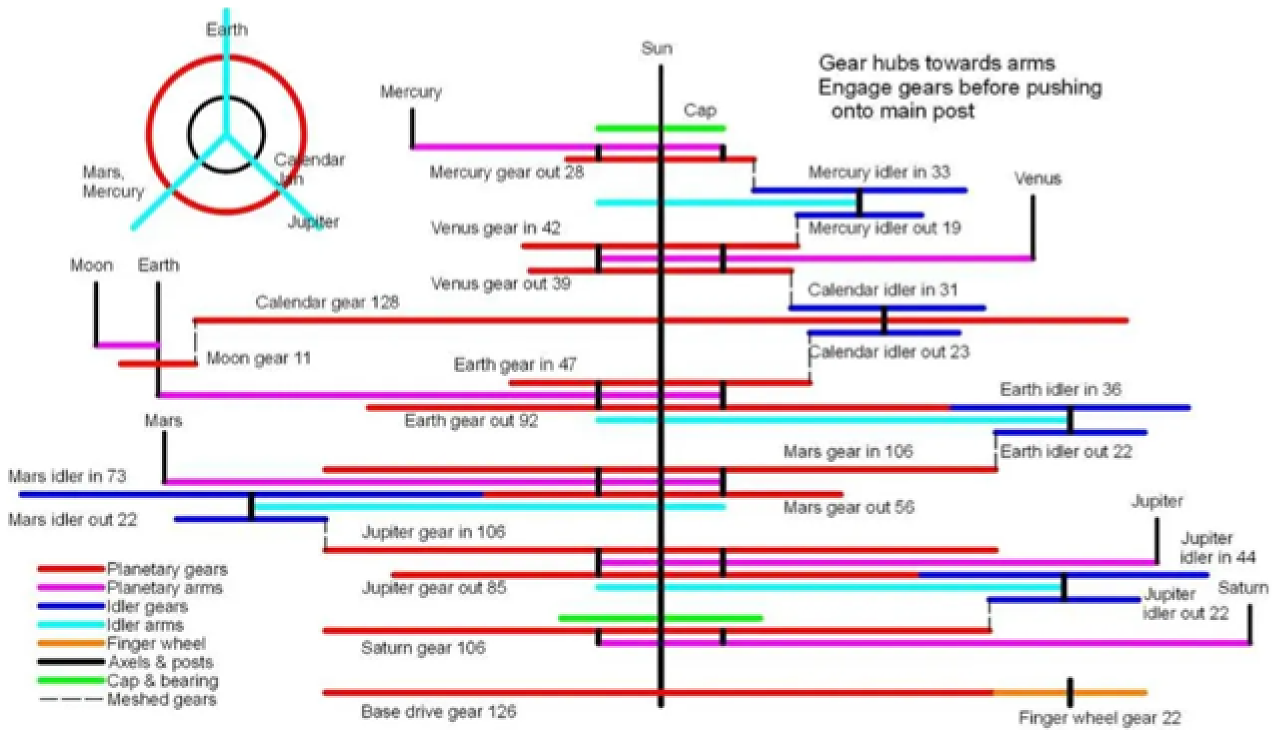


Planetarium by Zeamon

We also found a planetarium based on this model that was printed by a 3D printer. The ratio were different from ours, but we could at least follow the model.
[Zip20]

Planetarium by Zippitybamba

The aim is now to find the right number of teeth for each gear in order to have the right ratio for each orbital period. Here is a layout of the planetarium we would like to build but with ratios we found online. We started computing our ratio but it is quite complicated as some numbers we found are prime number but are also too big to be a number of teeth on a gear. We would like for at least each gears to have less than 100 teeth which is still a big number and would probably be complicated to print and thus not really precise.



Sketch of the planetarium gears

As we now have the layout, we need to adapt this to our calculations and then draw to pieces of the planetarium with the right software to be able to print it with a 3D printer.

7 References

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[Dav21b] Dave Williams. *Venus fact sheet*. 23 December 2021. URL: `https://nssdc.gsfc.nasa.gov/planetary/factsheet/venusfact.html`.

[Dav21c] Dave Williams. *Venus fact sheet*. 23 December 2021. URL: `https://nssdc.gsfc.nasa.gov/planetary/factsheet/marsfact.html`.

[Dav21d] Dave Williams. *Venus fact sheet*. 23 December 2021. URL: `https://nssdc.gsfc.nasa.gov/planetary/factsheet/jupiterfact.html`.

[Dav21e] Dave Williams. *Venus fact sheet*. 23 December 2021. URL: `https://nssdc.gsfc.nasa.gov/planetary/factsheet/saturnfact.html`.

[Dav21f] Dave Williams. *Venus fact sheet*. 23 December 2021. URL: `https://nssdc.gsfc.nasa.gov/planetary/factsheet/uranusfact.html`.

8 Annex

Bellow is an algorithm to compute the continued fraction of the ratio between the orbital period of a planet of our choice and the earth orbital period. We just need to enter the orbital period of our chosen planet in days and the algorithm compute the continued fraction of the ratio, its convergent and the margin of error for each convergent.

```
from math import modf, sqrt, pi
from decimal import Decimal

#Write in continued fraction

planet_orbital = float(input(
    "Enter the orbital period of the planet in days"))

#As the the earth orbital period is 365,256 days, it can be written as
# 365+(32/125) and thus it is 45657/125, so we have to multiply by 125.
planet_orbital = float(planet_orbital * 125)
earth_orbital = float(45657)

#Fix the result i.e. the conitnued fraction
result = []
print ("Planet Orbital: ",float(planet_orbital),
       "Earth Orbital: ",int(earth_orbital))

#Fix the rest that will be used to compute the continued fraction
rest = 1

#Find how many digit after the decimal point are there
# in order to get rid of them to do the computations
nb_of_decimal= str(planet_orbital)[::-1].find('.')

#Mutiply both orbital period for there to be no decimals
n0 = n = int(planet_orbital * 10**nb_of_decimal)
d0 = d = int(earth_orbital * 10**nb_of_decimal)

print (n,d)

#Loop to compute the continued fraction
while rest != 0:
    #Add the quotient of the euclidean division of n by d to the resut
    result.append(int(divmod(n , d)[0]))
    rest = int(divmod(n , d)[1])
    #n becomes d = turn the fraction (it becomes 1 over the fraction)
    n = d
    #d becomes the rest of the euclidean division of n by d
    d = rest
```

```python
print (result)

#Table of convergent

#Define the position in the result
pos = 0

#Define the p_i for the computation afterward
p0 = 0
p1 = 1
q0 = 1
q1 = 0

#Define the margin of error
err = 0

#Loop to compute the convergent to each order
while pos < len(result):
    p = p0 + result[pos] * p1
    q = q0 + result[pos] * q1
    err = abs((p/q)-(n0/d0)) * 100
    #Print the first line of the table of convergent with
    #the margin of error
    print(pos, "--", p, ":", q, "---", "margin of error = ", err)
    #Jump to the next line of the table
    pos = pos + 1
    p0 = p1
    p1 = p
    q0 = q1
    q1 = q
```