Faculty of science,Technology and Medecine

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Rational points on quadric varieties

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1 Introduction

In this report we are going to study the rational points of different quadrics. It is an interesting and well studied topic. As we know, quadrics have rational points, in particular the quadrics whose equation is defined with rational coefficients. But is it possible to find all rational points? We know that they are dense, so is it possible to establish some methods to find them systematically? We are going to answer that question. Before we start, we need to recall the definition of the quadric.

For the rest of the report, we assume that we are working over the field of real numbers.

1.1 Quadric

Definition 1.1. We call quadratic polynomial with n variables and coefficients in K, any element of K[X_1,X_2,...,X_n] of the form

$$q(X_1, X_2, \cdots, X_n) = \sum_{1 \leq i \leq j \leq n} a_{i,j} X_i X_j + \sum_{i=1}^{n} a_{0,i} X_i + a_{0,0}$$

where at least one of $a_{i,j}$ for $1 \leq i \leq j \leq n$ is not equal to 0

Example 1.1. Let $n = 1$ with variable X, any quadratic polynomial can be written in the following form

$$a_{1,1}X^2 + a_{0,1}X + a_{0,0} = a_2X^2 + a_1X + a_0$$

where we have indexed all the coefficients in the usual way.

Example 1.2. Let $n = 2$ with variables X and Y, any quadratic polynomial can be written in the following form

$$a_{1,1}X^2 + a_{1,2}XY + a_{2,2}Y^2 + a_{0,1}X + a_{0,2}Y + a_{0,0}$$

In particular, we have the following examples:

1. $\frac{X^2}{a^2} + \frac{Y^2}{b^2} - 1$
2. $\frac{X^2}{a^2} - \frac{Y^2}{b^2} - 1$
3. $\frac{X^2}{a^2} - Y$

2 Rational points of quadrics.

As we know what a quadric is, let's think about its rational points. The rational points of quadrics are the points (x,y) such that x and y are rational. To find such points, we decided to use a program that finds integer solutions for quadric in the interval [-100,100].

For example, we take the following quadric:

$$-2x^2 + y^2 + 3.3xy + 0.4x + 10y + 9 = 0$$

Now we use our program to find its integer solution(s). And we get:

$$\text{integer solutions: } \{(0, -9)\ ;\ (0, -1)\}$$

And we know that we can obtain rational solutions from integer solutions. To do so, we choose a random rational number, for example 7, and we substitute x and y in our initial quadratic equation by 7x and 7y respectively. After manipulations, we obtain:

$$-2(7x)^2 + (7y)^2 + 3.3(7x)(7y) + 0.4 * 7x + 10 * 7y + 9 = 0$$
$$-98x^2 + 49y^2 + 161.7xy + 2.8x + 70y + 9 = 0 \qquad (*)$$

So we know that the points $(0; \frac{-1}{7})$ and $(0; \frac{-9}{7})$ belong to the quadric(*), so there's 2 rational points of the initial quadric.

$$\{(0, -\frac{-9}{7}), (0, \frac{-1}{7})\}$$

Unfortunately, such method does not give a solution of general form. So we had to apply another method to find rational points. We subdivided quadrics in 3 general forms: circle, parabola and ellipsis.

2.1 Rational points of a circle

Let's start with a circle. For the simplicity of computations, we choose the unit circle as it has 4 obvious rational points: (-1,0); (1,0); (0,-1); (0,1). The idea is to choose one of the previously mentioned rational points and a line passing through that point, but the line must have rational slope. So we want to show:

Claim: If the straight line with rational slope intersects the unit circle in 2 points, and one of them is rational, then the second point of intersection is also rational.

Proof: We just need to fix 1 rational point of a circle. We take (-1,0). We illustrated the situation in the following picture.

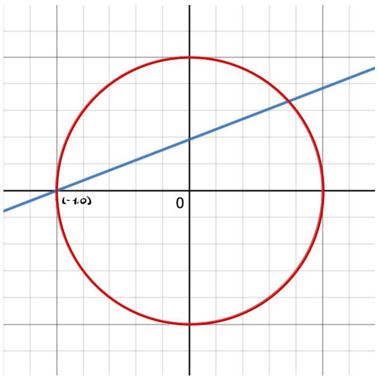

Figure 1: Intersection of a line and a circle with radius 1

Remark: The equation of the line is y = mx + m because of the point (-1,0) belongs to the line and its slope is m. And we have 2 equations, so we have a system of equations for which we want to find rational solutions.

$$\begin{cases} y &= mx + m \qquad \forall m \in \mathbb{Q} \qquad \text{The equation of the line (1)} \\ 1 &= x^2 + y^2 \qquad\qquad\qquad \text{The equation of the unitary circle (2)} \end{cases}$$

(1) in (2):

$$\begin{aligned} & x^2 + (mx + m)^2 &= 1 \\ \Leftrightarrow \quad & x^2 - 1 + [m(x + 1)]^2 &= 0 \\ \Leftrightarrow \quad & (x + 1)[x(m^2 + 1) + (m^2 - 1)] &= 0 \end{aligned}$$

$$\begin{aligned} \text{so } x = -1 \qquad \text{or} \qquad x(m^2+1)+(m^2-1) &= 0 \\ \Leftrightarrow \qquad x(m^2+1) &= 1-m^2 \\ \Leftrightarrow \qquad x &= \frac{1-m^2}{m^2+1} \end{aligned}$$

$$\begin{cases} \text{if } x = -1 & \Rightarrow y = 0 \quad (\text{x , y} \in \mathbb{Q}) \\ \text{if } x = \frac{1-m^2}{m^2+1} & \Rightarrow y = \frac{(1-m^2)m}{m^2+1} + m = \frac{m-m^3+m^3+m}{m^2+1} = \frac{2m}{m^2+1} \qquad (\text{x , y} \in \mathbb{Q}) \end{cases}$$

We know that m is a rational number, so we want to rewrite it in a general form: $m = \frac{a}{b}$, $\forall$ a $\in \mathbb{Z}$ and $\forall$ b $\in \mathbb{Z}^*$.

$$\begin{cases} \text{if } x = -1 & \Rightarrow y = 0 \\ \text{if } x = \frac{1-(\frac{a}{b})^2}{(\frac{a}{b})^2+1} = \frac{\frac{b^2-a^2}{b^2}}{\frac{a+b^2}{b^2}} = \frac{b^2-a^2}{b^2+a^2} & \Rightarrow y = \frac{2\frac{a}{b}}{(\frac{a}{b})^2+1} = \frac{\frac{2a}{b}}{\frac{a^2+b^2}{b^2}} = \frac{2ab}{a^2+b^2} \end{cases}$$

So, there is a unique second point on the intersection of the line and the circle and this second point has rational coordinates. And we can compute now any rational point of the circle. It follows because the line is determined by 2 rational points and the line has a rational slope.

$$\{(-1,0) \quad ; \quad (\frac{b^2-a^2}{b^2+a^2}, \frac{2ab}{a^2+b^2})\}$$

□

Corollary 2.1. *There exists a bijective function:*

$$\begin{aligned} f: \mathbb{Q} \quad &\to \quad \mathbb{Q}^2 \\ \{m \in \mathbb{Q}\} \quad &\mapsto \quad \{ \textit{ rational points } (\frac{1-m^2}{m^2+1}, \frac{2m}{m^2+1}) \textit{ and } (-1,0) \textit{ on the unit circle}\} \end{aligned}$$

So we proved that the 2^{nd} point is also rational. This helps us to parameterize the line by changing its slope in order to find all rational points of the circle. We can simply change the slope of the line.

Here's the precise example: let's choose 2 integers a and b. We say that a = 4 and b = 5. So the slope of a line is $\frac{4}{5}$. After computing the coordinates of the point where the intersection happens, we get the following point: $(\frac{9}{41}; \frac{40}{41})$. In the picture below, you can notice that the intersection happens indeed in a found point. (Graphic calculator rounded the numbers).

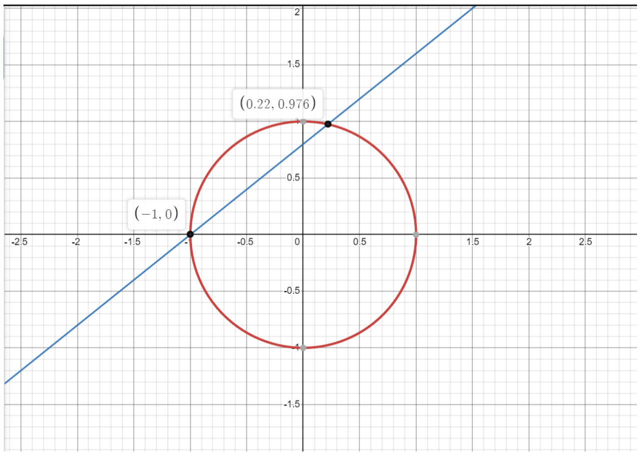

Figure 2: The intersection of a circle and a line with a slope 0,8.

Now we want to generalize the formula for a certain circle with radius r. We are not going to take the complete formula of the circle: $(x-a)^2+(y-b)^2 = r^2$ as we would have 2 more restrictions related to a and b: they should be rational. We assume that a$\in \mathbb{Q}$, b$\in \mathbb{Q}$ and r $\in \mathbb{Q}$. If they are not, we just apply translation of the center to the origin. As we don't have any obvious rational point of the circle, we need to adapt the equation of a straight line (we use its general form). Now we can formulate the claim.

Claim: If a straight line with rational slope intersects the circle with rational radius in 2 points, and one of these points is rational, then the other point must be rational.

Proof:

$$\begin{cases} x^2+y^2 &= r^2 \qquad \forall r \in \mathbb{Q} \qquad \text{General equation of a circle at the origin (1)} \\ mx+n &= y \qquad \forall m,n \in \mathbb{Q} \qquad \text{The equation of the straight line (2)} \end{cases}$$

(2) in (1):

$$\begin{aligned} x^2+(mx+n)^2 &= r^2 \\ \Leftrightarrow \quad x^2+m^2x^2+2mnx+n^2-r^2 &= 0 \\ \Leftrightarrow (1+m^2)x^2+2mnx+(n^2-r^2) &= 0 \end{aligned}$$

$$\begin{aligned} \text{so} \quad x &= \frac{-2mn \pm \sqrt{4m^2n^2-4(1+m^2)(n^2-r^2)}}{2+2m^2} \\ \Leftrightarrow x &= \frac{-2mn \pm \sqrt{4m^2n^2-4(n^2+m^2n^2-r^2-m^2r^2)}}{2+2m^2} \\ \Leftrightarrow x &= \frac{-2mn \pm \sqrt{4(m^2r^2-n^2+r^2)}}{2+2m^2} \\ \Leftrightarrow x &= \frac{-mn \pm \sqrt{m^2r^2-n^2+r^2}}{1+m^2} \end{aligned}$$

$$\begin{cases} \text{if } m^2r^2-n^2+r^2<0 & \Rightarrow \text{no solution in } \mathbb{R} \quad \Rightarrow \text{no solution in } \mathbb{Q} \\ \text{if } m^2r^2-n^2+r^2=0 & \Rightarrow x=\frac{-mn}{1+m^2} \in \mathbb{Q} \\ \text{if } m^2r^2-n^2+r^2>0 & \Rightarrow x=\frac{-mn\pm\sqrt{m^2r^2-n^2+r^2}}{1+m^2} \end{cases}$$

We can conclude that in the first case, the line and a circle have no intersection. In the second case, our line is tangent to the circle. For the last case, we have the situation that we're interested in.

But we should consider a number $\sqrt{m^2r^2 - n^2 + r^2}$ that can be irrational, to avoid such problem, we assume that the line passes through the rational point (-r, 0). We use that information to make a new system of equations:

$$\begin{cases} x^2 + y^2 &= r^2 \qquad \forall r \in \mathbb{Q}^* \qquad \text{General equation of a circle at the origin (1)} \\ \frac{m}{r}x + r &= y \qquad \forall m \in \mathbb{Q} \qquad \text{The equation of the straight line (2)} \end{cases}$$

(2) in (1):

$$\begin{aligned} x^2 + (\frac{m}{r}x + r)^2 &= r^2 \\ \Leftrightarrow \quad x^2 + \frac{m^2}{r^2}x^2 + 2\frac{m^2}{r}x + m^2 - r^2 &= 0 \\ \Leftrightarrow (1 + \frac{m^2}{r^2})x^2 + 2\frac{m^2}{r}x + (m^2 - r^2) &= 0 \\ \text{so} \quad x &= \frac{-2\frac{m^2}{r} \pm \sqrt{(2\frac{m^2}{r})^2 - 4(1 + \frac{m^2}{r^2})(m^2 - r^2)}}{2(1 + \frac{m^2}{r^2})} \\ \Leftrightarrow x &= \frac{-2\frac{m^2}{r} \pm \sqrt{4\frac{m^4}{r^2} - 4(m^2 - r^2 + \frac{m^4}{r^2} - m^2)}}{\frac{2r^2+2m^2}{r^2}} \\ \Leftrightarrow x &= \frac{-2\frac{m^2}{r} \pm \sqrt{4r^2}}{\frac{2r^2+2m^2}{r^2}} \\ \Leftrightarrow x &= \frac{(\frac{-2m^2 \pm 2r^2}{r})r^2}{2r^2 + 2m^2} \\ \Leftrightarrow x &= \frac{-rm^2 \pm r^3}{r^2 + m^2} \end{aligned}$$

$$\begin{cases} \text{if } x = -r & \Rightarrow y = 0 \qquad (\text{x , y} \in \mathbb{Q}) \\ \text{if } x = \frac{-rm^2 \pm r^3}{r^2+m^2} & \Rightarrow y = \frac{m}{r}(\frac{-rm^2 \pm r^3}{r^2+m^2}) + m = m\frac{-m^2 \pm r^2}{r^2+m^2} + m = \frac{-m^3+m^2+r^2 \pm r^2 m}{r^2+m^2} \qquad (\text{x , y} \in \mathbb{Q}) \end{cases}$$

We know that m and r are rational number, so we want to rewrite it in a general form: $m = \frac{a}{b}$ and $r = \frac{c}{d}$, $\forall$ a, c $\in \mathbb{Z}$ and $\forall$ b, d $\in \mathbb{Z}^*$.

$$\begin{cases} \text{if } x = -\frac{c}{d} & \Rightarrow y = 0 \\ \text{if } x = \frac{-\frac{ca^2}{db^2} \pm \frac{c^3}{d^3}}{\frac{c^2}{d^2} + \frac{a^2}{b^2}} = \frac{\frac{ca^2d^2 \pm c^3b^2}{d^3b^2}}{\frac{c^2b^2+a^2d^2}{b^2d^2}} = \frac{-ca^2d^2 \pm c^2b^2}{c^2b^2d+a^2d^3} & \Rightarrow y = \frac{-\frac{a^3}{b^3} + \frac{a^2}{b^2} + \frac{c^2}{d^2} \pm \frac{ac^2}{bd^2}}{\frac{c^2}{d^2} + \frac{a^2}{b^2}} \\ & \quad = \frac{\frac{-a^3d^2+a^2bd^2+c^2b^3 \pm ac^2b^2}{b^3d^2}}{\frac{c^2b^2+a^2d^2}{b^2d^2}} \\ & \quad = \frac{-a^3d^2+a^2bd^2+c^2b^3 \pm ac^2b^2}{c^2b^2d+a^2d^3} \end{cases}$$

Corollary 2.2. *There exists a bijective function f:* $\mathbb{Q} \leftrightarrow \{$*rational points on the circle with radius r* $\}$*, given by m :* $\mapsto \{$*circle*$\} \cap \{$ *line through (-r,0) of slope m* $\}$ *and the coordinates of the second point are* $(\frac{-rm^2 \pm r^3}{r^2+m^2}\ ,\ \frac{-m^3+m^2+r^2 \pm r^2m}{r^2+m^2})$.

□

2.2 Rational points of a parabola.

We want our solution to work also with parabolas. For the simplicity of computations, we take the equation of a non-translated parabola. We can easily find at least one rational point, which is the summit of parabola. So the line will pass through that summit and another rational point of the parabola. The equation of the non-translated and non-homothetized parabola is: y = x^2. As the summit of the parabola is at the origin, we can simplify the equation of the straight line passing through to it y = mx with m $\in \mathbb{Q}$.

Claim: If the straight line with rational slope intersects the parabola in 2 points: parabola's summit at the origin and another arbitrary point, they must be rational.

Proof:

$$\begin{cases} y &= x^2 & & \text{General equation of a parabola at the origin (1)} \\ y &= mx & \forall m \in \mathbb{Q} & \text{The equation of the straight line (2)} \end{cases}$$

(2) in (1):

$$\begin{aligned} & x^2 &= mx \\ \Leftrightarrow \quad & x^2 - mx &= 0 \\ \Leftrightarrow \quad & x(x-m) &= 0 \end{aligned}$$

$$\text{so} \quad x = 0 \quad \text{or} \quad x = m$$

$$\begin{cases} \text{if } x = 0 & \Rightarrow y = 0 \\ \text{if } x = m & \Rightarrow y = m^2 \end{cases}$$

We know that m is a rational number, so we want to rewrite in a general form: $m = \frac{a}{b}$, $\forall$ a $\in \mathbb{Z}$ and $\forall$ b $\in \mathbb{Z}^*$.

$$\begin{cases} \text{if } x = 0 & \Rightarrow y = 0 \\ \text{if } x = \frac{a}{b} & \Rightarrow y = \frac{a^2}{b^2} \end{cases}$$

So we finally have our rational point of a quadric. And we can compute now any rational point of the circle.

$$\{(0,0) \quad ; \quad (\frac{a}{b}, \frac{a^2}{b^2})\}$$

□

Let's take example coefficients: $a = 14$ and $b = 5$, i.e. m = $\frac{14}{5}$ = 2.8

In this case, we find the following intersection:

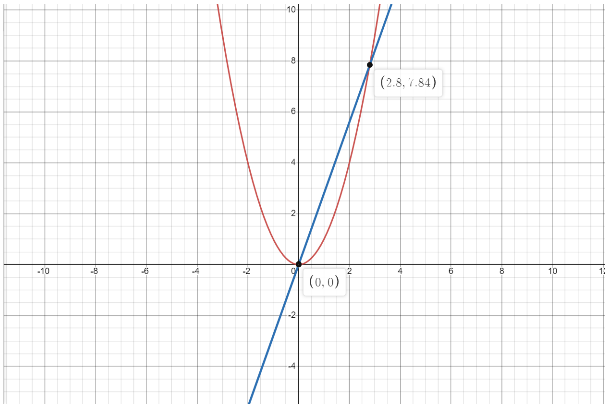

Figure 3: The intersection of a parabola and the line with a slope 2,8 passing through (0,0).

At the end we can identify all the rational points: { m belongs to $\mathbb{Q}$} $\iff$ $\{(m, m^2)$ belongs to the curve of parabola }.

Now we want to take the general equation of translated parabola which is: $y = (x+a)^2 + b$. And we also need the general equation of a line. We rewrite the claim for general case.

Claim: If the straight line with rational slope intersects the parabola in 2 points: its summit and any other point, both must be rational, or none.

Proof:

$$\begin{cases} y &= (x+a)^2 + b \qquad \forall a, b \in \mathbb{Q} \qquad \text{General equation of translated parabola (1)} \\ y &= mx + n \qquad \forall m, n \in \mathbb{Q} \qquad \text{The equation of the straight line (2)} \end{cases}$$

(2) in (1):

$$\begin{aligned} x^2 + 2ax + a^2 + b &= mx + n \\ \Leftrightarrow x^2 + (2a-m)x + (a^2+b-n) &= 0 \\ \text{so } \ x &= \frac{-2a + m \pm \sqrt{(2a-m)^2 - 4(a^2+b-n)}}{2} \\ \Leftrightarrow x &= \frac{m - 2a \pm \sqrt{4a^2 - 4am + m^2 - 4a^2 - 4b + 4n}}{2} \\ \Leftrightarrow x &= \frac{m - 2a \pm \sqrt{m^2 + 4n - 4b - 4am}}{2} \end{aligned}$$

$$\begin{cases} \text{if } m^2 + 4n - 4b - 4am < 0 & \Rightarrow \text{no solution in } \mathbb{R} \quad \Rightarrow \text{no solution in } \mathbb{Q} \\ \text{if } m^2 + 4n - 4b - 4am = 0 & \Rightarrow x = \frac{m-2a}{2} \in \mathbb{Q}, \text{ if } a \in \mathbb{Q} \\ \text{if } m^2 + 4n - 4b - 4am > 0 & \Rightarrow x = \frac{m-2a \pm \sqrt{m^2+4n-4b-4am}}{2} \end{cases}$$

Unfortunately, we have the same problem as before, and we want $\sqrt{m^2 + 4n - 4b - 4am} \in \mathbb{Q}$, so we can make $4n - 4b - 4am = 0$, then we get $\sqrt{m^2} = m$, as m is a rational number, $\sqrt{m^2 + 4n - 4b - 4am}$ is also a rational number. Then we get a new system of equations.

$$\begin{cases} y &= (x+a)^2 + b \qquad \forall a, b \in \mathbb{Q} \qquad \text{General equation of translated parabola (1)} \\ y &= mx + (am + b) \qquad \forall m \in \mathbb{Q} \qquad \text{The equation of the straight line (2)} \end{cases}$$

(2) in (1):

$$\begin{aligned} x^2 + 2ax + a^2 + b &= mx + (am + b) \\ \Leftrightarrow \quad x^2 + (2a - m)x + (a^2 - am) &= 0 \\ \text{so } x &= \frac{-(2a-m) \pm m}{2} \end{aligned}$$

$$\begin{cases} \text{if } x = \frac{-2a+m-m}{2} = -a & \Rightarrow y = -am + am + b = b \qquad (\text{x , y} \in \mathbb{Q}) \\ \text{if } x = \frac{-2a+m+m}{2} = -a + m & \Rightarrow y = m(-a+m) + am + b = m^2 + b \qquad (\text{x , y} \in \mathbb{Q}) \end{cases}$$

We know that a,b and m are rational number, so we want to rewrite it in a general form: $a = \frac{c}{d}$, $b = \frac{f}{g}$ and $m = \frac{h}{j}$, $\forall$ c, f, h $\in \mathbb{Z}$ and $\forall$ d, g, j $\in \mathbb{Z}^*$.

$$\begin{cases} \text{if } x = -\frac{c}{d} & \Rightarrow y = \frac{f}{g} \\ \text{if } x = -\frac{c}{d} + \frac{h}{j} = \frac{-cj+dh}{dj} & \Rightarrow y = \frac{h^2}{j^2} + \frac{f}{g} = \frac{h^2 g + f j^2}{j^2 g} \end{cases}$$

□

Corollary 2.3. *There exists a bijective function f: $\mathbb{Q} \leftrightarrow$ {rational points on the parabola }, given by m : $\mapsto$ {parabola}$\cap${ line through (-a,b) of slope m } and the coordinates of the second point are $(-a + m \ , \ m^2 + b)$.*

Example: Let's take example coefficients: $a = \frac{5}{2}; b = \frac{4}{5}$ and $m = \frac{13}{5}$

After choosing these coefficients, we compute and find the following intersection:

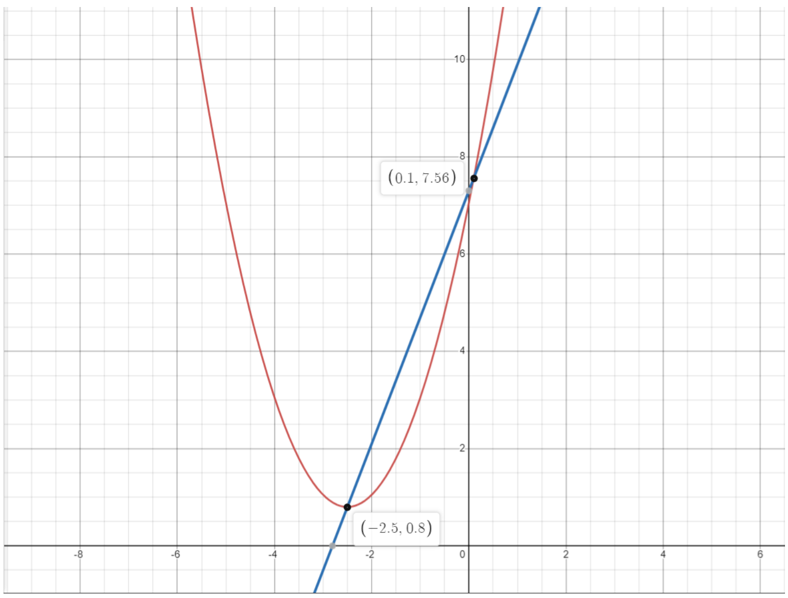

Figure 4: intersection of a parabola with equation : $y = (x + \frac{5}{2})^2 - \frac{4}{5}$ and a line with equation: $y = \frac{13}{5}(x + \frac{5}{2}) + \frac{4}{5}$

2.3 Rational points of an ellipsis.

Now we want to find rational points of an ellipsis. First, as example, we take the non-translated ellipsis with the equation:$3x^2 + 5y^2 = 17$. We choose a point of that ellipsis (2,1). So we want our line to pass through that point. We find the following equation: $y = (\frac{1-b}{2})x + b$. Such equation will always pass through the point (2,1) and it doesn't depend on parameter b. We thought the following: $y = ax + b$ which pass through the point (2,1), then $1 = 2a + b \Leftrightarrow a = \frac{1-b}{2}$. We assume that b is rational, so the slope of a chosen line is also rational. Based on that information, we can write the following theorem.

Claim: If a straight line passing through (2,1) with rational slope intersects the ellipsis with equation $3x^2 + 5y^2 = 17$ in 2 points, then second point is also rational.

Proof:

$$\begin{cases} 17 &= 3x^2 + 5y^2 & & \text{Example of equation of ellipsis (1)} \\ y &= (\frac{1-b}{2})x + b & \forall b \in \mathbb{Q} & \text{The equation of the straight line (2)} \end{cases}$$

(2) in (1):

$$\begin{aligned} & 3x^2 + 5((\frac{1-b}{2})x + b)^2 = 17 \\ \Leftrightarrow \quad & 3x^2 + 5(\frac{(1-b)^2}{4}x^2 + (1-b)bx + b^2) = 17 \\ \Leftrightarrow \quad & 3x^2 + \frac{5(1-b)^2}{4}x^2 + (5b - 5b^2)x + 5b^2 - 17 = 0 \\ \Leftrightarrow \quad & (3 + \frac{5}{4}(1-b)^2)x^2 + (5b - 5b^2)x + (5b^2 - 17) = 0 \end{aligned}$$

$$\begin{aligned} \text{so } x &= \frac{5b^2 - 5b \pm \sqrt{(5b - 5b^2)^2 - 4(3 + \frac{5}{4}(1-b)^2(5b^2 - 17))}}{6 + \frac{5}{2}(1-b)^2} \\ \Leftrightarrow x &= \frac{5b^2 - 5b \pm \sqrt{25b^2 - 50b^3 + 25b^4 - 4(15b^2 + \frac{25}{4}b^2(1 - 2b + b^2) - 51 - \frac{85}{4}(1 - 2b + b^2))}}{6 + \frac{5}{2}(1-b)^2} \\ \Leftrightarrow x &= \frac{10b^2 - 10b \pm 2\sqrt{25b^2 - 50b^3 + 25b^4 - (60b^2 - 25b^2(1 - 2b + b^2) - 204 - 85(1 - 2b + b^2))}}{12 + 5(1-b)^2} \\ \Leftrightarrow x &= \frac{10b^2 - 10b \pm 2\sqrt{25b^2 - 50b^3 + 25b^4 - (60b^2 - 25b^2 + 50b^3 - 25b^4 - 204 - 85 + 170b - 85b^2)}}{12 + 5(1-b)^2} \\ \Leftrightarrow x &= \frac{10b^2 - 10b \pm 2\sqrt{25b^2 - 170b + 289}}{12 + 5(1-b)^2} \\ \Leftrightarrow x &= \frac{10b^2 - 10b \pm (10b - 34)}{12 + 5(1-b)^2} \end{aligned}$$

$$\begin{cases} \text{if } x = \frac{10b^2 - 34}{12 + 5(1-b)^2} & \Rightarrow y = (\frac{1-b}{2})(\frac{10b^2 - 34}{12 + 5(1-b)^2}) + b = \frac{(1-b)(5b-17) + 12b + 5b(1-b)^2}{12 + 5(1-b)^2} \\ \text{if } x = \frac{10b^2 - 20b + 34}{12 + 5(1-b)^2} = \frac{10b^2 - 20b + 34}{5b^2 - 10b + 17} = 2 & \Rightarrow y = 2(\frac{1-b}{2}) + b = 1 \end{cases}$$

$$\{(2,1)\ ;\ (\frac{10b^2 - 34}{12 + 5(1-b)^2}\ ,\ \frac{(1-b)(5b-17) + 12b + 5b(1-b)^2}{12 + 5(1-b)^2})\}$$

We proved that the 2^{nd} intersecting point of the ellipsis will be rational and its coordinates can be calculated above.

□

And we choose b=-1, so we compute the intersection points. And we obtain: $(\frac{-3}{4}; \frac{-7}{4})$

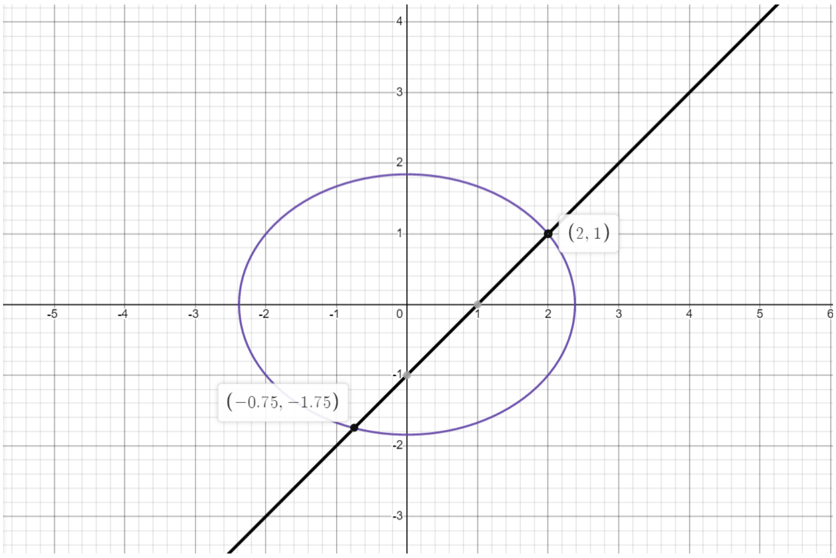

Figure 5: intersection of a ellipsis $3x^2 + 5y^2 = 17$ and a line with equation: $y = x - 1$

For an arbitrary ellipsis, we use the general equation for ellipsis which is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$

For the following case, we identify the rational points as the intersection of the ellipsis with lines of rational slope.

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} & = r^2 \qquad \forall a, b, r \in \mathbb{Q} \qquad \text{General equation of ellipsis (1)} \\ mx + n & = y \qquad \forall m, n \in \mathbb{Q} \qquad \text{The equation of the straight line (2)} \end{cases}$$

(2) in (1):

$$\begin{aligned} \frac{x^2}{a^2} + \frac{(mx+n)^2}{b^2} &= r^2 \\ \Leftrightarrow \qquad \frac{b^2x^2 + a^2(mx+n)^2 - a^2b^2r^2}{a^2b^2} &= 0 \\ \Leftrightarrow \qquad b^2x^2 + a^2(mx+n)^2 - a^2b^2r^2 &= 0 \\ \Leftrightarrow \qquad b^2x^2 + a^2(m^2x^2 + 2mnx + n^2) - a^2b^2r^2 &= 0 \\ \Leftrightarrow \qquad (b^2 + a^2m^2)x^2 + 2mna^2x + a^2n^2 - a^2b^2r^2 &= 0 \end{aligned}$$

$$\text{so } x = \frac{-2mna^2 \pm \sqrt{(2mna^2)^2 - 4(b^2+a^2m^2)(a^2n^2 - a^2b^2r^2)}}{2(b^2+a^2m^2)}$$
$$\Leftrightarrow x = \frac{-2mna^2 \pm \sqrt{4m^2n^2a^4 - 4(b^2a^2n^2 + a^4m^2n^2 - a^2b^4r^2 - a^4b^2m^2r^2)}}{2(b^2+a^2m^2)}$$
$$\Leftrightarrow x = \frac{-2mna^2 \pm \sqrt{4(a^2b^4r^2 + a^4b^2m^2r^2 - b^2a^2n^2)}}{2(b^2+a^2m^2)}$$
$$\Leftrightarrow x = \frac{-2mna^2 \pm 2ab\sqrt{(b^2 + a^2m^2r^2 - n^2)}}{2(b^2+a^2m^2)}$$
$$\Leftrightarrow x = \frac{-mna^2 \pm ab\sqrt{b^2 + a^2m^2r^2 - n^2}}{b^2+a^2m^2}$$

$$\begin{cases} \text{if } b^2 + a^2m^2r^2 - n^2 < 0 & \Rightarrow \text{no solution in } \mathbb{R} \\ \text{if } b^2 + a^2m^2r^2 - n^2 = 0 & \Rightarrow x = \frac{-mna^2}{b^2+a^2m^2} \\ \text{if } b^2 + a^2m^2r^2 - n^2 > 0 & \Rightarrow x = \frac{-mna^2 \pm ab\sqrt{b^2+a^2m^2r^2-n^2}}{b^2+a^2m^2} \end{cases}$$

$$\begin{cases} \text{if } x = \frac{-mna^2}{b^2+a^2m^2} & \Rightarrow y = m \cdot \frac{-mna^2}{b^2+a^2m^2} + n = \frac{-nm^2a^2+nb^2+na^2m^2}{b^2+a^2m^2} \\ \text{if } x = \frac{-mna^2 \pm ab\sqrt{b^2+a^2m^2-n^2}}{b^2+a^2m^2} & \Rightarrow y = m \cdot \frac{-mna^2 \pm ab\sqrt{b^2+a^2m^2r^2-n^2}}{b^2+a^2m^2} + n \\ & = \frac{nb^2 \pm abm\sqrt{b^2+a^2m^2r^2-n^2}}{b^2+a^2m^2} \end{cases}$$

Again, we have 3 possibilities: 2 intersecting rational points, 1 intersecting point and no intersecting points.

Let $\sqrt{b^2 + a^2m^2r^2 - n^2} \in \mathbb{Q}$, $\sqrt{b^2 + a^2m^2r^2 - n^2} = \frac{c}{d}$, for some c $\in \mathbb{Z}$ and for some d $\in \mathbb{Z}^*$

$$\sqrt{b^2 + a^2m^2r^2 - n^2} = \frac{c}{d}$$
$$\Leftrightarrow \quad b^2 + a^2m^2r^2 - n^2 = (\frac{c}{d})^2$$
$$\Leftrightarrow \quad r^2 = \frac{(\frac{c}{d})^2 - b^2 + n^2}{a^2m^2}$$
$$\Leftrightarrow \quad r^2 = \frac{c^2 - b^2d^2 + n^2d^2}{a^2d^2m^2}$$

We know a,b,m and n $\in \mathbb{Q}$, so $a = \frac{f}{g}$, $b = \frac{h}{j}$, $m = \frac{k}{l}$ and $n = \frac{o}{p} \in \mathbb{Z}$, f, h, k, o $\in \mathbb{Z}$ and $\forall$ g, j, l, p $\in \mathbb{Z}^*$

$$x = \frac{-\frac{kof^2}{lpg^2} \pm \frac{fhc}{gjd}}{\frac{h^2}{j^2} + \frac{f^2k^2}{g^2l^2}} = \frac{\frac{-kof^2jd \pm fhclpg}{lpg^2jd}}{\frac{h^2g^2l^2+f^2k^2j^2}{j^2g^2l^2}} = \frac{(kofjd \pm hclpg)jlf}{pd(h^2g^2l^2 + f^2k^2j^2)}$$

$$y = \frac{\frac{oh^2}{pj^2} \pm \frac{fhkc}{gjld}}{\frac{h^2}{j^2} + \frac{f^2k^2}{g^2l^2}} = \frac{\frac{ohgld \pm pjfhkc}{pj^2gld}}{\frac{h^2g^2l^2+f^2k^2j^2}{j^2g^2l^2}} = \frac{glh(ogld \pm pjfkc)}{pd(h^2g^2l^2 + f^2k^2j^2)}$$

So we have our rational point of a quadric. And we can compute any rational point of the ellipsis:

$$\{(\frac{(kofjd + hclpg)jlf}{pd(h^2g^2l^2 + f^2k^2j^2)} , \frac{glh(ogld + pjfkc)}{pd(h^2g^2l^2 + f^2k^2j^2)}) ; (\frac{(kofjd - hclpg)jlf}{pd(h^2g^2l^2 + f^2k^2j^2)} , \frac{glh(ogld - pjfkc)}{pd(h^2g^2l^2 + f^2k^2j^2)})\}$$

□

After all the computations and proofs, we can conclude that there exists a 1 to 1 correspondence between rational numbers and rational points on the quadric. And that works for each general type of quadric: circle, ellipsis and parabola.

And now we can pass to observations.

2.4 Observations

While we were working with quadrics, we noticed some interesting things. For some quadrics, it is very difficult to find rational points. For example, $x^2 + y^2 = \sqrt{2}$. We know that $\sqrt{2}$ is irrational. Consequently we conclude that x^2 must be irrational and or y^2 must be irrational. It follows from the fact that if we add 2 rational numbers, their sum is also rational. We can conclude that if 2 arbitrary quadrics intersect each other in some points, we do not have a guarantee of their rationality. Also, the general formula of the quadrics contains the information about its origin (for a circle), summit (if we work with parabola), size and direction (in case of parabola) or focal point (in case of ellipsis) etc.That's why we want to study the behaviour of quadrics if we apply a transformation to the quadrics.

We take 2 transformations: translation and homothety. We rewrite what we need in a theorem, that we want to prove.

Theorem 2.4. *The information about rational points of a quadric is preserved under translation by a rational vector and under homothety by a rational coefficient. In other words: a point on a quadric is rational if and only if it is still rational after a rational translation or a rational homothety.*

This allows us to reduce the problem of finding rational points on an arbitrary quadrics to certain, normalized ones. We will prove that theorem for 3 types of quadrics: circle, ellipsis and parabola.

2.5 Observations on a circle.

So, let's get back to our circle of the form $(x - a)^2 + (y - b)^2 = r^2$ ($\forall a, b \in \mathbb{Q}$ and $\forall r \in \mathbb{Q}^*$). We can divide the whole formula by r^2 , so we obtain $\frac{(x-a)^2}{r^2} + \frac{(y-b)^2}{r^2} = 1 \Leftrightarrow (\frac{x-a}{r})^2 + (\frac{y-b}{r})^2 = 1$.

And we can conclude that if r is rational, then the previously found rational points are of the form :

∃ a ∈ $\mathbb{Q}$, b ∈ $\mathbb{Q}$

$$\frac{x - a'}{r} = \frac{b^2 - a^2}{b^2 + a^2} \quad \text{and} \quad \frac{y - b'}{r} = \frac{2ab}{a^2 + b^2} \quad \Leftrightarrow \quad x = r\frac{b^2 - a^2}{b^2 + a^2} + a' \quad \text{and} \quad y = r\frac{2ab}{a^2 + b^2} + b'$$

So our point is rational if and only if a',b' are rational. Because $x = r\frac{b^2-a^2}{b^2+a^2} + a'$, a,b,r are rational, for x to belong to $\mathbb{Q}$, a' needs to be rational , same reason for b'. If it is not the case, then we we have irrational point. However, we can choose irrational a and b to get rid of r. We just need $a^2 + b^2$ be a rational multiple of r and such that the corresponding fractions will be a rational multiple of a' and b'.

2.5.1 Remark

There's a very famous examples of Pythagorian triples (for example: 3,4,5 or 5,12,13), i.e. for $x = 3$ and y = 4, $x^2 + y^2 = 5^2$. Consequently we can divide the equation by 5 and we obtain rational point on the unit circle. Like this that any Pythagorian triple gives a rational point on the unit circle.

2.6 Observations on a parabola.

As we established the general formula for parabola, we can say that the information contained in that general formula is the coordinates of the summit and its width. $y = a(x+b)^2+c$, $\forall a, b, c \in \mathbb{Q}$. To work easily with parabola, we can apply a translation to the origin (0,0).

Finally, we get the following equation of parabola:

$$\begin{cases} y &= ax^2 \qquad \forall a \in \mathbb{Q} \\ y &= mx \qquad \forall m \in \mathbb{Q} \end{cases}$$

(2) in (1):

$$\begin{aligned} & & ax^2 &= mx \\ \Leftrightarrow & & ax^2 - mx &= 0 \\ \Leftrightarrow & & x(ax-m) &= 0 \end{aligned}$$

$$\text{so} \quad x = 0 \quad \text{or} \quad ax = m$$

$$\begin{cases} \text{if } x = 0 & \Rightarrow y = 0 \\ \text{if } x = \frac{m}{a} & \Rightarrow y = (\frac{m}{a})^2 \end{cases}$$

And we finally obtain that if a is rational ($\Rightarrow$ we apply rational homothety), then the intersection is rational. Otherwise m must be a rational multiple of a. If we don't want to change m and we want to keep it rational, then x must be of the form $x = \frac{x'}{a}$ where x' is a rational.

2.7 Observations on an ellipsis.

For the ellipsis, we take the following formula: $(\frac{x}{a})^2 + (\frac{y}{b})^2 = r^2$. As before, we can divide the equation by r^2.

$(\frac{x}{ar})^2 + (\frac{y}{br})^2 = 1$

And we can substitute each squared term by x" and y". So we have again the famous equation of the unitary circle. After necessary computations, we obtain:

$$\frac{x}{ra} = \frac{b^2 - a^2}{b^2 + a^2} \quad \text{and} \quad \frac{y}{rb} = \frac{2ab}{a^2 + b^2} \quad \Leftrightarrow \quad x = ra\frac{b^2 - a^2}{b^2 + a^2} \quad \text{and} \quad y = rb\frac{2ab}{a^2 + b^2}$$

So we must be sure, that $r \cdot a$ and $r \cdot b$ are rational.

3 Visualization of the intersection of tri-dimensional quadrics.

We were also interested in visualizing 2 quadrics in R^3 and their intersection. To do it a little bit more interactive, we can modify its coefficients. To do so, we used Matlab. Further, we show some of the results:

Example 1:

The blue quadric has the formula

$$f_{\text{blue}} := x^2 + 2y^2 + 0.5z^2 + 4xy + 5xz + 6yz + 7x + 8y + 9z - 100$$

The red quadric has the formula

$$f_{\text{red}} := 1.8x^2 + 2.9y^2 + 0.6z^2 + 7xy + 13xz + 3yz + x + 2y - 25$$

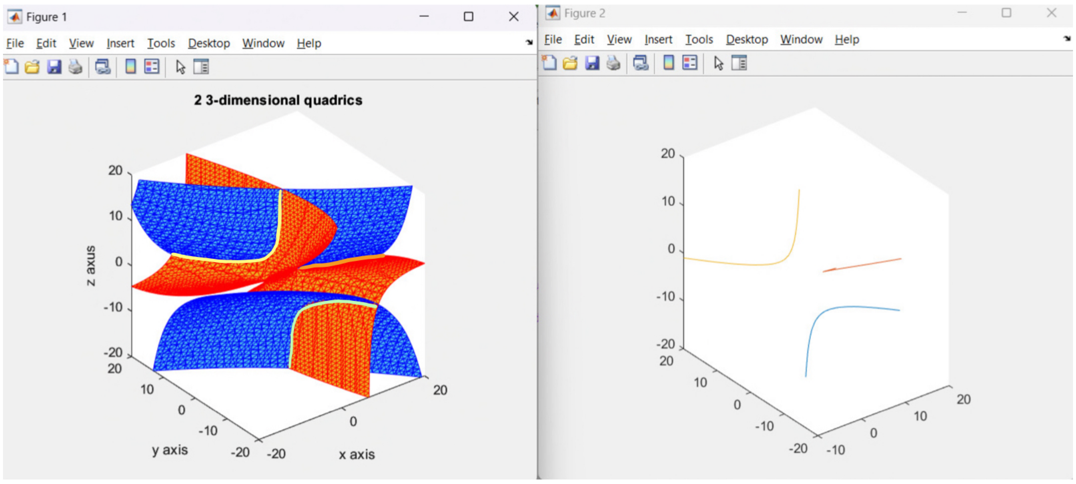

Figure 6: 1^{st} example of the intersection

We colored the intersection with corresponding colour to make it more visible.

Example 2:

The blue quadric has the formula

$$f_{\text{blue}} := x^2 + y^2 + z^2 - 150$$

The red quadric has the formula

$$f_{\text{red}} := 9x^2 + 18y^2 + 25z^2 - 3000$$

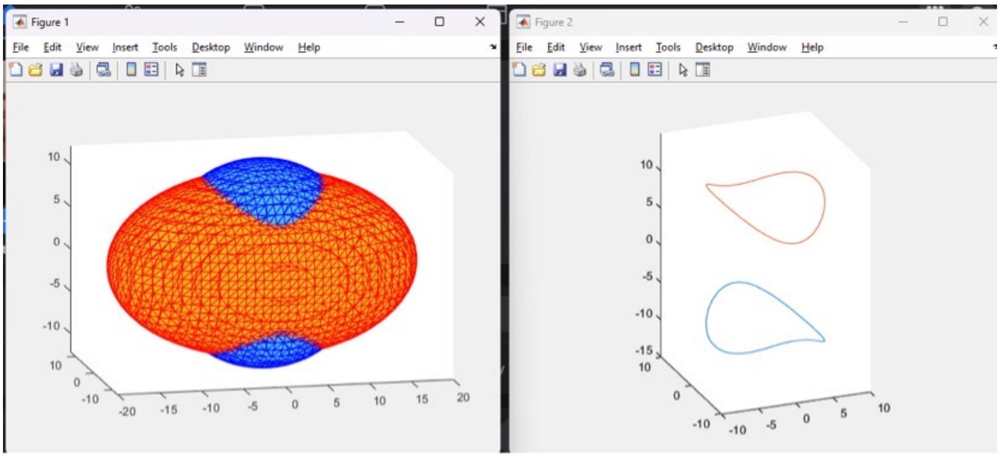

Figure 7: 2^{nd} Example of the intersection

Remark 1: For this example, we obtain as the intersection 2 curves because we work only with the contour of the quadrics. In other words, they are empty inside, otherwise the result would be voluminous figure.

Remark 2: Sometimes the numerical intersection does not look smooth because of the equations of quadrics. This means, that the volumes of quadrics that we are working with are small, and our program is not so precise to draw them on small intervals. Consequently, the intersection of 2 tridimensional quadrics has low resolution.

Example 3:

The blue quadric has the formula

$$f_{\text{blue}} := 12x^2 + 10y^2 + 3.1xy + 6.8xz + 5yz + 0.5x + 7y + 10.2z - 11$$

The red quadric has the formula

$$f_{\text{red}} := 9x^2 + 18y^2 + 25z^2 + 2.5yz - 4.6x - 1y + 55z - 38$$

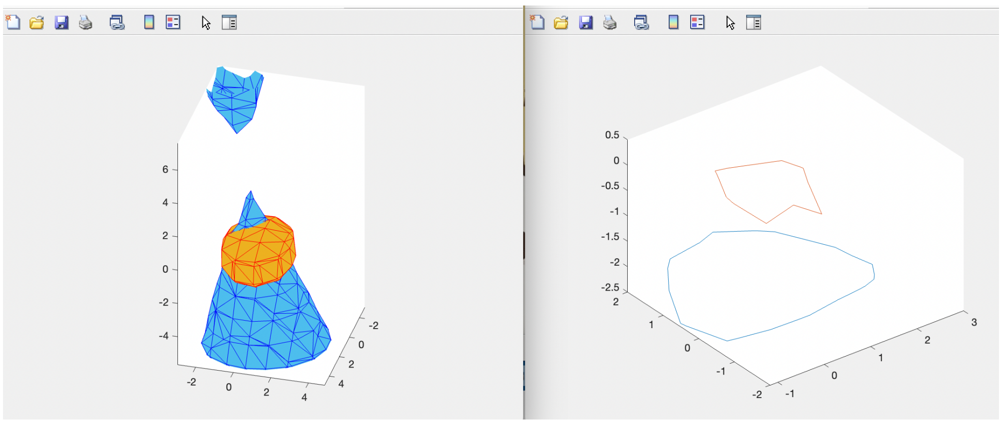

Figure 8: 3^{rd} Example of the low poly intersection.

Remark 3: The term low poly is used in both a technical and a descriptive sense; the number of polygons in a mesh is an important factor to optimize for performance but can give an undesirable appearance to the resulting graphics.

The blue quadric has the formula

$$f_{\text{blue}} := 120x^2 + 100y^2 + 31xy + 68xz + 50yz + 5x + 70y + 102z - 1500$$

The red quadric has the formula

$$f_{\text{red}} := 90x^2 + 180y^2 + 250z^2 + 25yz - 46x - 10y + 550z - 38000$$

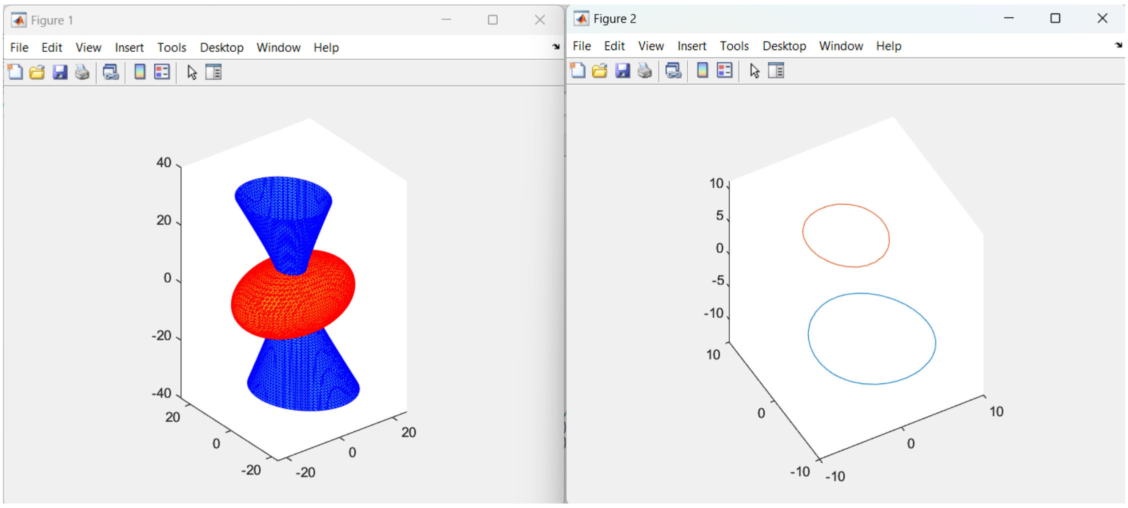

Figure 9: 4^{nd} Example of the intersection high on polygons

As you can see, we changed the last coefficient. That changed the size of the quadrics without changing its classifications (the red quadric is ellipsoid and the blue one is paraboloid).

4 Intersection of a quadric implementing magic figures

Now we want to find a particular solution to some magic figures. The figures are: square 3x3, tetrahedron and square 4x4.

4.1 Magic square of squares 3x3

For that magic figure, we want every number be a square of some other number. The most difficult part of the task is to find a combination such that all the entries are different form each other. After trying different methods and failing all of them to find such combination. We found out that this problem was studied already and apparently there's no solution for such square yet (if we want to use small numbers). Otherwise, there might be a solution with big integers. But such research demands huge computational power.

a^2	b^2	c^2
d^2	e^2	f^2
g^2	h^2	i^2

$$a^2 + b^2 + c^2 = d^2 + e^2 + f^2 = g^2 + h^2 + i^2 = a^2 + d^2 + g^2$$
$$= b^2 + e^2 + h^2 = c^2 + f^2 + i^2 = a^2 + e^2 + i^2 = c^2 + e^2 + g^2 = 3e^2$$

However, we have found the following solution:

By the property of magic square of size 3x3, we know that for any column, line or diagonal the sum will be 3 times the middle. In particular: $a^2 + b^2 + c^2 = 3e^2$ (1). Also we noticed that $(x + y)^2 + (x - y)^2 = 2x^2 + 2y^2$ (2). This equation is almost similar to $a^2 + b^2 + c^2 = 3e^2$. We can add $x^2 + y^2$ to both sides of the equation (2). So we obtain the equation matching the equation (1).

then the magic square becomes

$(x+y)^2$	$(x-y)^2$	x^2+y^2
$(x-y)^2$	x^2+y^2	$(x+y)^2$
x^2+y^2	$(x+y)^2$	$(x-y)^2$

for example: $x = 4$ and $y = 3$

7^2	1^2	5^2
1^2	5^2	7^2
5^2	7^2	1^2

$$7^2 + 1^2 + 5^2 = 1^2 + 5^2 + 7^2 = 5^2 + 7^2 + 1^2 = 7^2 + 1^2 + 5^2$$
$$= 1^2 + 5^2 + 7^2 = 5^2 + 7^2 + 1^2 = 7^2 + 5^2 + 1^2 = 5^2 + 5^2 + 5^2 = 3 * 5^2 = 75$$

However, we want to add another result of similar research. It is an example of semi-magic square of squares.

- *LS1. Three rows, three columns and one diagonal have the same magic sum S2=21609. But unfortunately the other diagonal has a different magic sum S2=38307:*

127^2	46^2	58^2
2^2	113^2	94^2
74^2	82^2	97^2

Figure 10: semi-magic square of squares

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