Frobenius Problem

Lou Meylender
Thomas Thalmaier

31/5/2023

Contents

1 Definition

We consider for all positive integers $p_1, \ldots, p_r$,

$$S(p_1, \ldots, p_r) := \sum_{i=1}^{r} p_i \mathbb{N}$$

$S(p_1, \ldots, p_r)$ represents the set of all the numbers that can be obtained by positive linear combination of $p_1, \ldots, p_r$.
Similary, we define:

$$A(p_1, \ldots, p_r) := \mathbb{N} - \sum_{i=1}^{r} p_i \mathbb{N}$$

$A(p_1, \ldots, p_r)$ represents the set of all integers that are not in the set $S(p_1, \ldots, p_r)$ e. g. of all the integers that cannot be obtained by a positive linear combination of $p_1, \ldots, p_r$.

$$g(p_1, \ldots, p_r) := \sup A(p_1, \ldots, p_r)$$

$g(p_1, \ldots, p_r)$ represents the supremum of $A(p_1, \ldots, p_r)$, so the smallest integer, such that all the integer above of it are in $S(p_1, \ldots, p_r)$.

$$a(p_1, \ldots, p_r) := \# A(p_1, \ldots, p_r)$$

$a(p_1, \ldots, p_r)$ represents the cardinality of $A(p_1, \ldots, p_r)$, e. g. the number of integers that cannot be obtained by a positive linear combination of $p_1, \ldots, p_r$.

2 Coprime

Theorem 2.1. *We have for all* $p_1, \ldots, p_r > 1$*:*

$$a(p_1, \ldots, p_r) < \infty \Leftrightarrow p_1, \ldots, p_r \text{ are coprime.}$$

Proof. If $p_1, \ldots, p_r$ are not coprime, there exists $p \in \mathbb{N}, p \neq 1$, such that :

$$p \mid p_i, \ \forall i \in \{1, \ldots, r\}$$

Hence, all elements of $S(p_1, \ldots, p_r)$ are divisible by p.

$$pr + 1 \in A(p_1, \ldots, p_r), \forall r \in \mathbb{N}$$

$$\Rightarrow \sup A(p_1, \ldots, p_r) = \infty$$

$$\Rightarrow a(p_1, \ldots, p_r) = \#(A(p_1, \ldots, p_r)) = \infty$$

If $p_1, \ldots, p_r$ are coprime, we have by the generalized Bézout identity, that there exists $u_1, \ldots, u_r \in \mathbb{Z}$, such that:

$$u_1 p_1 + u_2 p_2 + \ldots + u_r p_r = 1$$

Assume $p_1 = \min(p_1, \ldots, p_r)$.
Without loss of generality, suppose $u_m < 0$, and $u_i > 0$ for $i \neq m$. We can assume this because the sum equals to 1, so there must be a positive and a

negative u.
We now replace u_m by$-u_m$ to get:

$$u_1p_1 + \ldots - u_mp_m + \ldots + u_rp_r = 1 \qquad | + u_ip_i, \forall 1 \leq i \leq r \tag{1}$$

$$\Rightarrow 2u_1p_1 + \ldots - 0 + \ldots + 2u_rp_r = 1 + (u_1p_1 + \ldots + u_rp_r) \qquad | + p_1u_ip_i, \forall 1 \leq i \leq r$$

$$\Rightarrow (2+p_1)u_1p_1 + \ldots + p_1u_mp_m + \ldots + (2+p_1)u_rp_r = 1 + (p_1+1)(u_1p_1 + \ldots + u_rp_r) \tag{2}$$

Let $q_1 = 1 + (p_1 + 1)(u_1p_1 + \ldots + u_rp_r) \in \mathbb{N}$, can be obtained so:

$$q_1 \in \sum_{i=1}^{r} p_i\mathbb{N}$$

By adding (1) and (2), we get:

$$(3+p_1)u_1p_1 + \ldots + (p_1-1)u_mp_m + \ldots + (3+p_1)u_rp_r = 2 + (p_1+1)(u_1p_1 + \ldots + u_rp_r)$$

Let $q_2 = 2 + (p_1 + 1)(u_1p_1 + \ldots + u_rp_r) \in \mathbb{N}$, which can be obtained:

$$q_2 \in \sum_{i=1}^{r} p_i\mathbb{N}$$

We procede p_1 times in the same way, and we get in the final step:

$$(1+2p_1)u_1p_1 + \ldots + u_mp_m + \ldots + (1+2p_1)u_rp_r = p_1 + (p_1+1)(u_1p_1 + \ldots + u_rp_r)$$

Let $q_{p_1} = p_1 + (p_1 + 1)(u_1p_1 + \ldots + u_rp_r) \in \mathbb{N}$, which can be obtained:

$$q_{p_1} \in \sum_{i=1}^{r} p_i\mathbb{N}$$

So we have $q_1, q_2, \ldots, q_{p_1} \in \mathbb{N}$, p_1 consecutive numbers, that can be obtained by linear combination of $p_1, \ldots, p_r$. So, every number over q_1 lies in $S(p_1, \ldots, p_r)$. So, we have:

$$g(p_1, \ldots, p_r) < q_1$$
$$\Rightarrow a(p_1, \ldots, p_r) < \infty$$

□

3 Computer program

First of all, by Theorem 2.1, we have to compute if the numbers are coprime. For that, we create a function, which computes the gcd of a list of integers.

```
def pgcd(li):
  pgcd = 1
  minimum = li[0]
  for i in range(1, minimum + 1):
    e = True
    for j in li:
      if j % i != 0:
        e = False
        break
    if e == True:
      pgcd = i
  return pgcd
```

Then, we create a list of all integers different from $p_1, \dots, p_r$ that will later represent the integers in $A(p_1, \dots, p_r)$.

```
def calculate_maximum(li):
  s = 0
  for i in li:
    s += i**2
  result = 1 + (li[0] + 1) * s
  return result

def generate_list(listp, M):
  li = []
  for i in range(1, M):
    if i not in listp:
      li.append(i)
  return li
```

We also create a function which checks if there are p_1 consecutive numbers not in the list, which would mean that all integers over these numbers are in $S(p_1, \dots, p_r)$.

```
def checks(li, p, M):
  if (li[-1] < M):
    M = li[-1]
  i = 0
  a = len(li)
  while (i < a - 1):
    if (li[i] + 1 <= li[i + 1] - p) and li[i] < M:
      M = li[i]
    i += 1
  return M
```

If such a maximum is found, we have a function which deletes all the integers above this maximum.

```
def check_list_max(li, M):
  i = 0
  a = len(li)
  while (i < len(li)):
    if li[i] >= M:
      del li[i]
      i -= 1
    i += 1
  return li
```

The most important part of the program consistes of a function, which deletes from the list all the integers that can be obtained by adding $p_1, \cdots, p_r$.

```
def calculate(listp, li, maximum):
  for i in range(1, maximum):
    if i not in li:
      for j in listp:
        if ((i + j) in li):
          li.remove(i+j)
  return li
```

Finally, we create a loop, which uses all the above functions, to delete integers of $S(p_1, \cdots, p_r)$ and tries to find a maximum with p_1 conscutive numbers in

$S(p_1, \cdots, p_r)$. The loop stops when no element is deleted and the function returns $a(p_1, \cdots, p_r)$.

```
def calculate_a(listp):
  p = listp[0]
  maximum = calculate_maximum(listp)
  li = generate_list(listp, maximum)
  old_len = len(li)
  while True:
    li = calculate(listp, li, maximum)
    maximum = checks(li, p, maximum)
    new_len = len(li)
    if new_len == old_len:
      break
    old_len = new_len
  return len(li)
```

Similarly, we can also create a function, which computes and returns $g(p_1, \cdots, p_r)$ and $A(p_1, \cdots, p_r)$.

```
def calculate_g(listp):
  p = listp[0]
  maximum = calculate_maximum(listp)
  li = generate_list(listp, maximum)
  old_len = len(li)
  while True:
    li = calculate(listp, li, maximum)
    maximum = checks(li, p, maximum)
    new_len = len(li)
    if new_len == old_len:
      break
    old_len = new_len
  return li[-1]

def calculate_A(listp):
  p = listp[0]
  maximum = calculate_maximum(listp)
  li = generate_list(listp, maximum)
  old_len = len(li)
  while True:
    li = calculate(listp, li, maximum)
    maximum = checks(li, p, maximum)
    new_len = len(li)
    if new_len == old_len:
      break
    old_len = new_len
  return li
```

We will use this computer program in order to find conjectures for the formulas in part 5.

4 Independent

Definition 1. *We call $p_1, p_2, ..., p_r$ independent, if:*

(i) $0 < p_1 < p_2 < ... < p_r$ *are integers;*

(ii) the p_i are coprime;

(iii) none of the p_i is a sum of the others.

Lemma 4.1. *If $p, q_1, q_2, ..., q_r$ are independent, then $r < p$.*

Proof. Assume $p, q_1, q_2, ..., q_r$ are independent, so $p < q_1 < q_2 < ... < q_r \in \mathbb{N}$. By euclidean division we get for every i in $\{1, 2, ..., r\}$:

$$q_i = k_i \times p + m_i \quad (k_i, m_i \in \mathbb{N}),$$

so $1 \leq m_i \leq p - 1$
Show that the m_i are unique.
By contradiction, assume $m_i = m_j$, we have $(i < j \Rightarrow q_i < q_j)$:

$$q_i = k_i \times p + m_i$$
$$q_j = k_j \times p + m_j = k_j \times p + m_i$$

So, we get:

$$q_i - k_i \times p = q_j - k_j \times p$$
$$\Rightarrow q_j = (k_j - k_i) \times p + q_i \quad \lightning$$

Contradiction, because none of the q_i is a sum of the others.

$$\Rightarrow \text{ all } m_i \text{ are unique.}$$

So, we have that $\{m_1, m_2, ..., m_r\} \subseteq \{1, 2, ..., p - 1\}$

$$\Rightarrow \#\{m_1, m_2, ..., m_r\} \leq \#\{1, 2, ..., p - 1\}$$
$$\Rightarrow r \leq p - 1$$
$$\Rightarrow r < p$$

□

4.1 Computer program

We have the following code in order to determine if the $p_1, p_2, ..., p_r$ in the list li are independent or not. In the examples below we always assume that the integers are independent.

```
def are_independent(li):
  li.sort()
  li.reverse()
  if pgcd(li) == 1:
    li2 = li.copy()
    while li2 != []:
      i = li2[0]
      del li2[0]
      if len(li2) >= 1:
        if i not in calculate_A(li2.copy()):
          return False
        else:
          if i % li[1] == 0 or li[1] % i == 0:
            return False
        li.reverse()
      return True
    return False
```

5 Conjecture of formulas

5.1 For 2 integers

We are searching by conjecture a formula for two integers.
For $a(2,i)$, we have:

$$a(2,3) = 1$$
$$a(2,5) = 2$$
$$a(2,7) = 3$$

We realise computationally that for every i, such that a and i are indepedent, we have:

$$i = 2a(2,i) + i$$

$$\Rightarrow a(2,i) = \frac{i-1}{2}$$

Similarly for $a(3,i)$, we have that:

$$a(3,i) \times \frac{2}{i-1} = 2$$

And, for $a(4,i)$:

$$a(4,i) \times \frac{2}{i-1} = 3$$

So, we seem to have the following formula:

$$\boxed{a(p,q) = \frac{(p-1)(q-1)}{2}}$$

5.2 For 3 integers

We are now searching by conjecture for a formula for three integers:
For $a(3,10,q)$, we get:

$$a(3,10,11) = 6$$
$$a(3,10,14) = 7$$
$$a(3,10,17) = 8$$

$\frac{1}{3} \times 11 + s = 6 \Rightarrow s = 6 - \frac{11}{3} = \frac{7}{3} = \frac{10-3}{3} = \frac{10}{3} - 1$
So the slope is $\frac{7}{3}$.

$$\text{So, } \boxed{a(3,10,q) = \frac{1}{3}q + \frac{7}{3}}$$

For $a(3,10,q)$, we get:

$$a(3,11,13) = 7$$
$$a(3,11,16) = 8$$
$$a(3,11,19) = 9$$

$\frac{1}{3} \times 13 + s = 7 \Rightarrow s = 7 - \frac{13}{3} = \frac{8}{3} = \frac{11-3}{3} = \frac{11}{3} - 1$
So the slope is $\frac{8}{3}$.

$$\text{So, } \boxed{a(3,11,q) = \frac{1}{3}q + \frac{8}{3}}$$

For $a(3,13,q)$, we get:

$$a(3,13,14) = 8$$
$$a(3,13,17) = 9$$
$$a(3,13,20) = 10$$

$\frac{1}{3} \times 14 + s = 8 \Rightarrow s = 8 - \frac{14}{3} = \frac{10}{3} = \frac{13-3}{3} = \frac{13}{3} - 1$
So the slope is $\frac{10}{3}$.

$$\text{So, } \boxed{a(3,13,q) = \frac{1}{3}q + \frac{10}{3}}$$

So, we finally get:

$$\boxed{a(3,p,q) = \frac{p+q}{3} - 1}$$

5.3 for 4 numbers and more

We are now searching by conjecture for a formula with four integers:
For $a(4,9,11,p_4)$, we get:

$$a(4,9,11,6) = 5$$
$$a(4,9,11,10) = 6$$
$$a(4,9,11,14) = 7$$

$\frac{1}{4} \times 6 + s = 5 \Rightarrow s = 5 - \frac{3}{2} = \frac{7}{2} = \frac{14}{4} = \frac{11+3}{4}$
So the slope is $\frac{7}{2}$.

$$\text{So, } \boxed{a(4,9,11,p_4) = \frac{1}{4}p_4 + \frac{7}{2}}$$

For $a(4,9,14,p_4)$, we get:

$$a(4,9,14,11) = 7$$
$$a(4,9,14,15) = 8$$
$$a(4,9,14,19) = 9$$

$\frac{1}{4} \times 11 + s = 7 \Rightarrow s = 7 - \frac{11}{4} = \frac{17}{4} = \frac{14+3}{4}$
So the slope is $\frac{17}{4}$.

$$\text{So, } \boxed{a(4,9,14,p_4) = \frac{1}{4}p_4 + \frac{17}{4}}$$

For $a(4,9,10,p_4)$, we get:

$$a(4,9,10,7) = 5$$
$$a(4,9,10,11) = 6$$
$$a(4,9,10,15) = 7$$

$\frac{1}{4} \times 7 + s = 5 \Rightarrow s = 5 - \frac{7}{4} = \frac{13}{4} = \frac{10+3}{4}$
So the slope is $\frac{13}{4}$.

$$\text{So, } \boxed{a(4,9,10,p_4) = \frac{1}{4}p_4 + \frac{13}{4}}$$

So we have:

$$\begin{aligned} a(4,9,p_3,p_4) &= \frac{1}{4}p_4 + \frac{1}{4}p_3 + \frac{3}{4} \\ &= \frac{p_3+p_4}{4} + \frac{3}{4} \end{aligned}$$

Since $\frac{3}{4} = \frac{9-6}{4} = \frac{9}{4} - \frac{3}{2}$, we find:

$$\boxed{a(4,p_2,p_3,p_4) = \frac{p_2+p_3+p_4}{4} - \frac{3}{2}}$$

Let's check this formula:
$a(4,21,22,23) = \frac{21+22+23}{4} - \frac{3}{2} = 15$
$a(4,6,7,9) = \frac{6+7+9}{4} - \frac{3}{2} = 4$
$a(4,15,17,18) = \frac{15+17+18}{4} - \frac{3}{2} = 11$
Since we find the same results with the computer program, we assume the formula is true.

Similarly, we also find by conjecture that:

$$\boxed{a(5,p_2,p_3,p_4,p_5) = \frac{p_2+p_3+p_4+p_5}{5} - 2}$$

and

$$\boxed{a(6,p_2,p_3,p_4,p_5,p_6) = \frac{p_2+p_3+p_4+p_5+p_6}{6} - \frac{5}{2}}$$

So it seems to be, that the formula is:

$$\boxed{a(p,q_2,q_3,...,q_p) = \frac{\sum\limits_{k=1}^{p} p_k}{p} - \frac{p-1}{2}}$$

5.4 For the maximum with 2 integers

We now want to find the formula for $g(p,q)$, where $p,g > 1$ coprime integers.
For $g(2,i)$:

$$\begin{gathered} g(2,3) = 1 \\ g(2,5) = 3 \\ g(2,7) = 5 \\ g(2,9) = 7 \\ \text{So, } \boxed{g(2,i) = (i-1)-1} \end{gathered}$$

For $g(3,i)$:

$$\begin{gathered} g(3,4) = 5 \\ g(3,5) = 7 \\ g(3,7) = 11 \\ g(3,8) = 13 \end{gathered}$$

$i+1 \Rightarrow g+2$, so the slope is 2.
$g(3,i) = 2i - 3 = 2(i-1) - 1 = (3-1)(i-1) - 1$

$$\text{So, } \boxed{g(3,i) = (3-1)(i-1) - 1}$$

For $g(4,i)$

$$g(4,5) = 11$$
$$g(4,7) = 17$$
$$g(4,9) = 23$$
$$g(4,11) = 29$$

$i+2 \Rightarrow g+6$, so the slope is 3.
$g(4,i) = 3i - 4 = 3(i-1) - 1 = (4-1)(i-1) - 1$

$$\text{So, } \boxed{g(4,i) = (4-1)(i-1) - 1}$$

So we find the following formula:

$$\boxed{g(p,q) = (p-1)(q-1) - 1}$$

6 Formula for the general case

Let $p, q_1, \ldots, q_r$ be independent
Then,

$$a(p, q_1, \ldots, q_r) = \frac{m_1 + \ldots + m_{p-1}}{p} - \frac{p-1}{2} \tag{3}$$

with $m_i = \min \{m \in S(p, q_1, \ldots, q_r) \mid m \equiv i \mod p\}$

7 Special cases

7.1 Special case for 2 numbers

For two independent numbers $p, q > 1$, we have found by conjecture, in section 5, the following formula:

$$a(p,q) = \frac{(p-1)(q-1)}{2}$$

We will now prove this formula.

Proof. We have that:

$$m_i = i \mod p$$
$$\Rightarrow m_i = i + kp,\ k \in \mathbb{N}$$

We also know that $m_i \in$ S(p,q), so there exists $a_1, a_2 \in \mathbb{N}$, such that:

$$a_1 p + a_2 q = i + kp$$

Suppose $a_1 \neq 0$, this would mean that:

$$(a_1 - 1)p + a_2 q \in S(p,q) \quad \lightning$$

Contradiction, because this would mean that m_i is not the minimum. So,

$$m_i = a_2 q \Rightarrow q | m_i, \forall 1 \leq i \leq p-1$$

Suppose $m_i = m_j$

$$\Rightarrow i \mod p = i \mod q$$
$$\Rightarrow i = j$$

So, because $q|m_i$, all m_i are distinct and $m_i < pq$ (by minimality of m_i), we have that:

$$\{m_i \mid 1 \leq i \leq p-1\} = \{iq \mid 1 \leq i \leq p-1\} \tag{4}$$

So by (4) we can say that:

$$\{m_i \mid 1 \leq i \leq p-1\} = \{q, 2q, 3q, \cdots, (p-1)q\}$$

And so by applying (3), we get:

$$\begin{aligned} a(p,q) &= \frac{q + 2q + \cdots + (p-1)q}{p} - \frac{p-1}{2} \\ &= \frac{q}{p} \sum_{i=1}^{p-1} i - \frac{p-1}{2} \\ &= \frac{q}{p} \frac{p(p-1)}{2} - \frac{p-1}{2} \\ &= \frac{q(p-1)}{2} - \frac{p-1}{2} \\ &= \frac{q(p-1) - (p-1)}{2} \\ &= \frac{(q-1)(p-1)}{2} \end{aligned}$$

□

7.2 Special case when r = p - 1

For $p, q_1, \ldots, q_{p-1}$ independent numbers, we have found the following conjecture in section 5:

$$a(p, q_1, \ldots, q_{p-1}) = \frac{p + q_1 + \ldots + q_{p-1}}{p} - \frac{p-1}{2}$$

We will now prove this formula.

Proof. By Lemma 4.2, we know that for all $1 \leq i \leq p-1$, there is $1 \leq j \leq p-1$, such that:

$$q_{j_i} = k_i p + i,\ k_i \in \mathbb{N}$$
$$\Rightarrow q_{j_i} \equiv i \mod p$$

So, $q_{j_i} \in \{m \in S(p, q_1, \cdots, q_{p-1}) \mid m \equiv i \mod p\}$
Show that it is the minimum.

Suppose $m_i < q_{j_i}$
So, we have:

$$m_i \equiv i \mod p$$

$$\Rightarrow m_i = i + l_i p, \quad l_i \in \mathbb{N}$$

We know that $l_i < k_i$, because $m_i < q_{j_i}$, so there exists an $\epsilon \in \mathbb{N}$, such that $k_i = l_i + \epsilon$, $\epsilon > 0$, so we get:

$$m_i + \epsilon_p = i + k_i p = q_{j_i}$$

So, there is a linear combination in $S(p, q_1, \cdots, q_{j_i - 1})$ to optain q_i. So q_i is not independent.
So, we can conclude that q_{j_i} is the minimum of the set and that $m_i = q_{j_i}$
So,

$$a(p, q_1, \cdots, q_{p-1}) = \frac{q_1 + \cdots + q_{p-1}}{p} - \frac{p-1}{2}$$

□

7.3 Special case with 5

Suppose $5, p, q$ are independent, with:

$$p \equiv 1 \mod 5 \qquad \text{and} \qquad q \equiv 2 \mod 5$$

We have $m_1 = p$, because $p \equiv 1 \mod 5$.
For m_2, we would have the minimum between q and $2p$.
By contradiction, suppose $q > 2p$ Because $q \equiv 2 \mod 5$ and $2p \equiv 2 \mod 5$, there must exist a $k \in \mathbb{N}$, such that:

$$q = 2p + 5k \quad \lightning$$

Contradiction, because q is independent and cannot be obtained by a positive linear combination of p and 5.
So, $m_2 = q$.
Similary, we have:

$$m_3 = q + p \qquad \text{and} \qquad m_4 = 2q$$

So, we get by applying (3):

$$\begin{aligned} a(5, p, q) &= \frac{m_1 + m_2 + m_3 + m_4}{5} - \frac{5-1}{2} \\ &= \frac{p + q + p + q + 2q}{5} - 2 \\ &= \frac{2p + 4q}{5} - 2 \\ &= \frac{2p + 4q - 10}{5} \end{aligned}$$

7.4 Special case with modulo 1 and 2

Let $p < q < r$ be three independent numbers, with p odd, such that:

$$q \equiv 1 \mod p \qquad \text{and} \qquad r \equiv 2 \mod p$$

We want to show that:

$$m_{2k} = kr \tag{5}$$

$$m_{2k+1} = kr + q \tag{6}$$

We first want to show (5). First, we can say that there exists $a, b, c \geq 0$, such that:

$$m_{2k} = ap + bq + cr$$

Suppose $a \neq 0$. We know that $m_{2k} \equiv 2k \mod p$, so we would get:

$$m_{2k} - p \equiv 2k \mod p \qquad \lightning$$

Contradition, because m_{2k} must be the minimum. So, $a = 0$ and

$$m_{2k} = bq + cr$$

By contradiction, assume now that b is odd, so there exists $l \in \mathbb{N}$, such that $b = 2l + 1$. So, we get:

$$\begin{aligned} m_{2k} &= (2l+1)q + rc \\ &= 2ql + q + rc \end{aligned}$$

We know that $2q \equiv 2 \mod p$ and $r \equiv 2 \mod p$, so $(2ql + rc) \equiv 2m \mod p$, with $m \in \mathbb{N}$.
But, $q \equiv 1 \mod p$, so $m_{2k} \equiv (2m + 1) \mod p$. $\lightning$
This is a contradiction, because $2k$ is even and $2m+1$ is odd. So, we know that b is even, we get for some $l \in \mathbb{N}$:

$$m_{2k} = 2lq + cr$$

Moreover we can say that $r < 2q$, because otherwise p, q, r would not be independent (the proof is identical as for the case 5 in subsection 7.5).
So, by assuming that $l \neq 0$, we know that:

$$2lq + cr \equiv 2k \mod p$$

But, because $r < 2q$ and $r \equiv 2 \mod p$ as well as $2q \equiv 2 \mod p$, we get that:

$$2(l-1)q + (c+1)r \equiv 2k \mod p \quad \text{and} \quad 2(l-1)q + (c+1)r < 2lq + cr$$

This is a contradiction, because this would mean that m_{2k} is not the minimum.
So, we get that $l = 0$ and:

$$m_{2k} = cr$$

Finally, we know that $r \equiv 2 \mod p$, so $cr \equiv 2c \mod p$, so the smallest c such that $cr \equiv 2k \mod p$ is $c = k$.
So, we get:

$$m_{2k} = kr$$

One proceeds similarly to prove (6).

We now want to find a formula for any even integer p.
So, we have:

$$\begin{aligned} m_{2k-1} + m_{2k} &= kr + (k-1)r + q \\ &= (2k-1)r + q \end{aligned}$$

So, by considering u an odd number, we get:

$$\begin{aligned} \sum_{k=1}^{u} m_k &= \sum_{k=1}^{\frac{u-1}{2}} ((2k-1)r + q) \\ &= \sum_{k=1}^{\frac{u-1}{2}} 2kr - \sum_{k=1}^{\frac{u-1}{2}} r + \sum_{k=1}^{\frac{u-1}{2}} q \\ &= 2r \sum_{k=1}^{\frac{u-1}{2}} k - r\frac{u-1}{2} + q\frac{p-1}{2} \\ &= 2r\frac{(\frac{u-1}{2})(\frac{u-1}{2}+1)}{2} - r\frac{u-1}{2} + q\frac{u-1}{2} \\ &= (\frac{u-1}{2})(r\frac{u+1}{2} - r + q) \\ &= (\frac{u-1}{4})(2q + ru - r) \end{aligned}$$

For every p even, we have that $p-1$ odd. And, we know that:

$$m_{p-1} = m_{p-2+1} = m_{2(\frac{p-2}{2})+1} = (\frac{p-2}{2})r + q$$

So, by using the above formula by replacing u by $p-1$ (in order to remove the two last terms), and by adding the last element, we get:

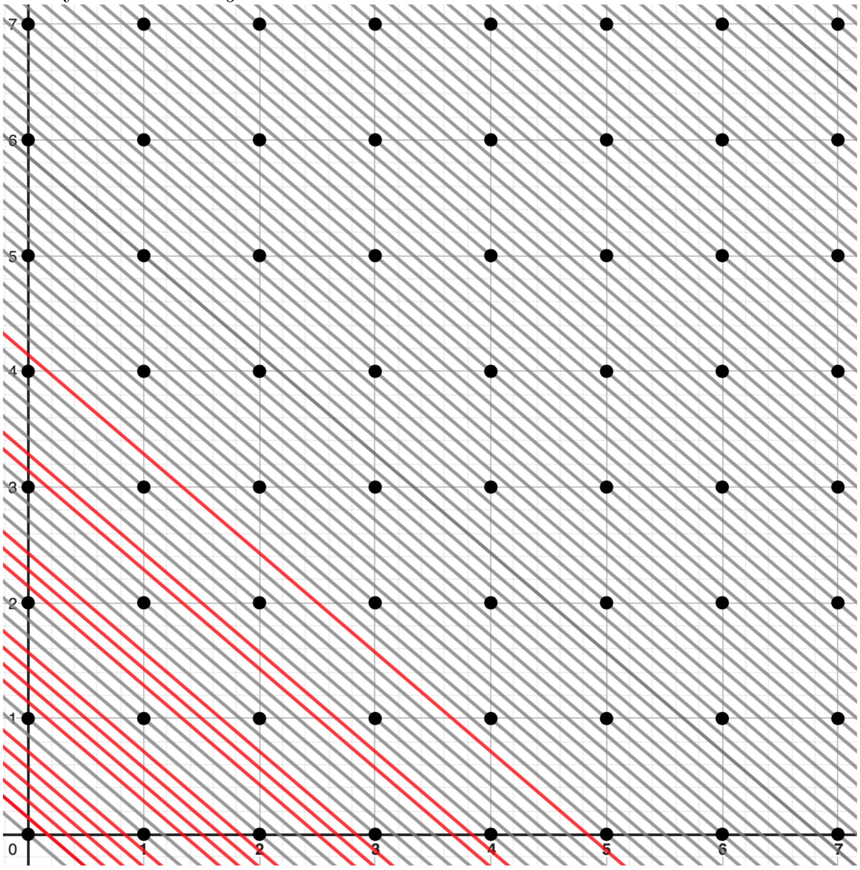
$$\begin{aligned} \sum_{i=1}^{p-1} m_i &= (\frac{p-2}{4})(2q + r(p-1) - r) + (\frac{p-2}{2})r + q \\ &= (\frac{p-2}{4})(2q + rp - r - r + 2r) + q \\ &= \frac{(p-2)(2q+rp) + 4q}{4} \\ &= \frac{(2pq + rp^2 - 2rp + 4q}{4} \\ &= \frac{2pq + rp^2 - 2rp}{4} \\ &= \frac{p(2q + rp - 2r)}{4} \end{aligned}$$

So, we get:

$$a(p,q,r) = \frac{p(2q+rp-2r)}{4p} - \frac{p-1}{2}$$
$$= \frac{2q+rp-2r-2p+2}{4}$$

8 Graphical interpretation of an example

Let's consider the example, where $p_1 = 6$ and $p_2 = 7$. We have the following graphical interpretation, where every line represents the set of points which satisfy $6 \times x + 7 \times y = i$ for i in $\mathbb{N}$.



When the line crosses one of the points which have non negative integers as coordinates, i is in $S(6,7)$ and the line is grey. Otherwise, i is in $A(6,7)$ and the line is red.
From the graph, we can see:

$$S(6,7) = \{0,6,7,12,13,14,18,19,20,21,24,25,26,27,28,30,31,\ldots\}$$

$$A(6,7) = \{1,2,3,4,5,8,9,10,11,15,16,17,22,23,29\}$$

$$g(6,7) = 29$$

$$a(6,7) = 15$$

If we check with the formulas:

$$g(6,7) = (6-1)(7-1) - 1 = 5 \times 6 - 1 = 29$$

$$a(6,7) = \frac{(6-1)(7-1)}{2} = \frac{5 \times 6}{2} = 15$$

We find the same results as before.

References

[1] S. J. E. Brauer, Alfred. On a problem of frobenius. *Journal für die reine und angewandte Mathematik*, 211:215–220, 1962.

[2] J. J. Sylvester. On subvariants, ie semi-invariants to binary quantics of an unlimited order. *American Journal of Mathematics*, 5(1):79–136, 1882.