Visualising roots of algebraic numbers

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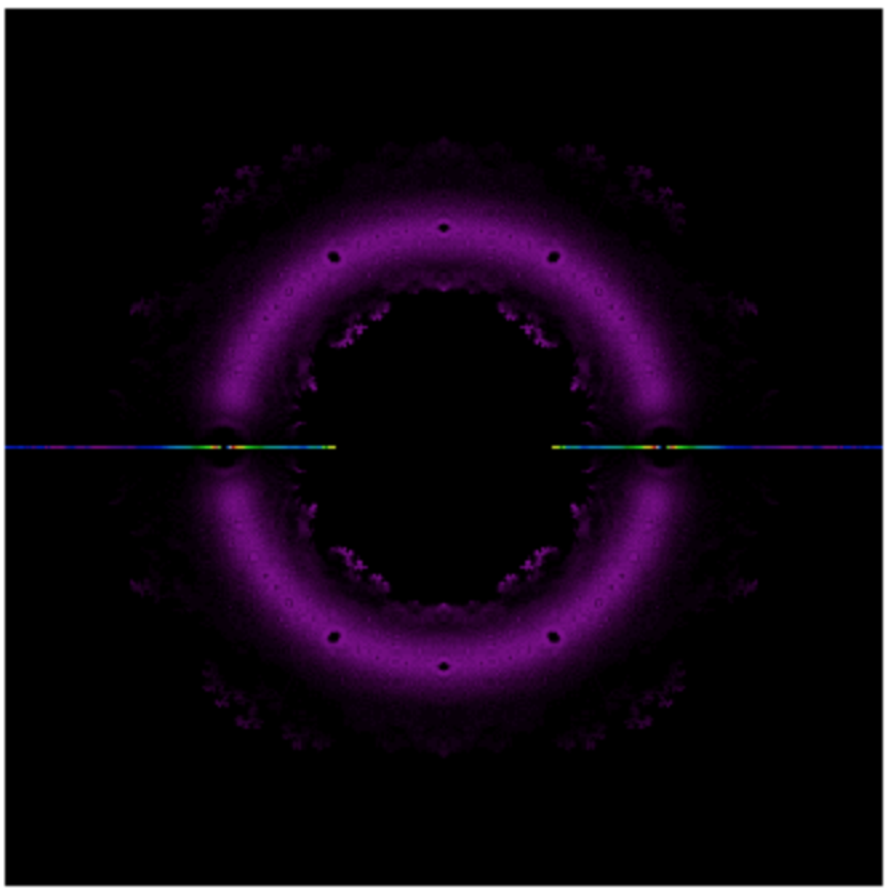
Contents

1 Introduction

Let's briefly explain the mathematical structure of the project. In order to visualise roots of polynomials in the complex plane, one has to calculate them first. For example, for degree 2 one can use the discriminant, for degree 3 the Horner scheme.
After we have obtained the roots of a certain polynomial, we subdivide them into their real part and their imaginary part. This process will be redone for every polynomial of a chosen degree with chosen coefficients. Since we now have the complex and real part of all the roots of the polynomials with the desired variables, we can draw them in the complex plane to obtain their picture.

In this project we will show some of the possible pictures and briefly explain their main characteristics.

To get a little pre-taste, here is a picture of the roots of all polynomials of degree 16 with coefficients -1 and 1:



2 Programs and Explanations

2.1 Code for the list of roots of polynomials of degree d

```
1  from math import*
2  from sage.rings.polynomial.complex_roots import complex_roots
3  from itertools import *
4
5  #Gives the list of polynomials of degree = d and coefficients = [-n, -n+1 , ..., n]\{0}
6  def find_roots(y):
7      x = polygen(CC)
8      list = complex_roots(y)
9      return list
10
11 def part(x):
12     Re=(CC(x[0]).real_part())
13     Im=(CC(x[0]).imag_part())
14     return(Re,Im)
15
16 def list_of_roots(d,n):
17     coeff =[]
18     for i in range(n*2+1):
19         if -n+i !=0:
20             coeff.append(-n+i)
21     permutations = list(product(coeff,repeat=d))
22     roots=[]
23     for element in permutations:
24         poly = x^d
25         for i in range(len(element)):
26             poly = poly + element[i]*x^(d-1-i)
27         for r in find_roots(poly):
28             if CC(r[0]).imag_part() != 0 :
29                 roots.append(part(r))
30     return roots
```

2.2 Explanation of the code for the list of roots of polynomials

In the program above we defined three functions which compute a list of roots that are separated in their real and imaginary part, of all polynomials of degree d, where $d \in \mathbb{N}^*$, with coefficients in {-n,-n+1,...,-1,1,2,...,n}

Lines 1-3: In these three lines we only import some necessary commands which we use later for the functions. The first import is the import of the *math* library so that we can use all mathematical commands. The second import is from the library *sage.rings.polynomial.complex_roots* where we only import the function *complex_roots*, which takes a polynomial and calculates all the roots in the complex plane. The third import is the import of the *itertools* library needed for the import of the function *product*, which takes a list of integers and gives all the possible permutations of those integers.

Lines 6-9: In these four lines we define the function *find_roots*, which takes as parameter a polynomial. We need to write the line "$x = polygen(CC)$", so that the "x" in the code is recognised as a variable, which is in $\mathbb{C}$. Then we define a list, which contains the roots of the polynomial, and then we return this list.

Lines 11-14: Here we define the function *part* which takes as parameter a number which is in $\mathbb{C}$. The application *.real_part()* determines the real part of that number and the application *.imag_part()* determines the imaginary part of the number. The values are stored in the variables *Re* and *Im*, where *Re* is the real part and *Im* is the imaginary part. At the end, the function *part* returns a tuple where the first element is the real part of the number and the second element is the imaginary part.

Lines 16-30: The last function of this first part of the program is the function *list_of_roots* which takes as parameters the degree d of a polynomial and an integer n. First we have a empty list which we fill with the coefficients {-n,-n+1,...,-1,1,2,...,n}. Then we compute the list *permutations*, which contains all permutations of the elements of the list of coefficients, in other words we get a list that contains all possible permutations of the coefficients of the polynomials. After that we construct all possible polynomials with the permutations of the coefficients, compute the roots of those polynomials and finally divide them into their real and imaginary part and construct a list of those roots named *roots*. At the end, our function *list_of_roots* returns the list of roots of all polynomials of degree d and coefficients {-n,-n+1,...,-1,1,2,...,n}.

2.3 Code for the matrix

```
31 from sage.plot.matrix_plot import *
32 # functions to get our matrix
33 def coord_in_matrix(root,x,y,length):
34     Re_w = floor((root[0]-x)/length)
35     Im_w = floor(-(root[1]-y)/length)
36     return (Im_w,Re_w)
37
38 def multiplicity_matrix(window,size,list,center_x,center_y): #the matrix is centered at (c
39     (start_x,start_y) = (center_x - window, center_y + window)
40     mat = matrix(size,size)
41     length = 2*(center_x-start_x)/size
42     for z in list:
43         if start_x<=z[0]<2*center_x-start_x and 2*center_y-start_y<z[1]<=start_y:
44             (Im_w,Re_w) = coord_in_matrix(z,start_x,start_y,length) # coord_in_matrix
45             mat[Im_w,Re_w] += 1
46     return mat
```

2.4 Explanation of the code for the matrix

In the first part of the code, we have programmed a function that computes the roots of all polynomials of degree d with coefficients {-n,-n+1,...,-1,1,2,...,n}. Now we want to be able to use the function *matrix_plot* to plot a picture. Hence why in line 31 of the code, we import *sage.plot.matrix_plot*. In order to achieve this, we need to compute a matrix whose elements represent the number of roots in a given section of the complex plane. In fact, the matrix subdivides a chosen part of the complex plane into small squares of a given length, and counts the number of roots in those squares. The element $a_{i,j}$ of the matrix represents the number of roots in the square located in row i and in column j.

Lines 33-36: The function *coord_in_matrix* computes in which part of the matrix the root, given as a parameter, is situated. In order to do that, we need to translate the points contained in our "window", so in the region where the matrix is computed. The translation corresponds to the translation of the upper left corner point $z = x - yi$, with x and y parameters of the function, to the $0^{th}-$ row and column. As the squares also have a given length, which is the last parameter of the function, we then divide the translated point by the length of the square and take the floor of the result, in order to classify each point into the right square. Finally, the function returns a tuple that contains the new coordinates.

Lines 38-46: The function *multiplicity_matrix* takes as parameter *center_x* and *center_y*, which correspond to the coordinates in the complex plane of the center of the picture. The parameter *window* fixes the set, on which we compute the matrix, as the set $M := \{z \,|\, center\_x - window \leq Rez < center\_x + window, center\_y - window \leq Imz < center\_y + window\}$. The parameter *size* indicates that the matrix is a matrix $size \times size$ and lastly, the parameter *list* is the list of roots computed in the first part. We then calculate the coordinates of the upper left corner, where we want the matrix to start, create the null square matrix of size *size* and compute the length of the squares by dividing the length of the set, which is equal to $2 \times center\_x$, divided by the parameter *size*, which corresponds to the total number of squares in a row. In the for loop, we go through the list of complex roots, and verify if they lie within the set M. If they satisfy that condition, their position in the matrix is computed using the function *coord_in_matrix*. Lastly, 1 is added to the corresponding element mat_{Im_w,Re_w} that represents the number of roots contained in that square. In the end, the function returns a matrix whose elements are the number of roots in the corresponding square.

2.5 Code for the drawing with colours

```
roots= load("deg1coeff1.sobj")
matr=multiplicity_matrix(2,301,roots,0,0)
m = matrix_plot(matr,colorbar=True,cmap="nipy_spectral")
m.show()
```

2.6 Explanation of the code for the drawing with colours

In the code above we are calling a list of roots of our choice, which is called *roots* with the function *load* which we saved before with the function *save*. Then we use our function *multiplicity_matrix* with the list *roots* and afterwards we define m as the *matrix_plot* of our matrix and with the colormap *nipy_spectral*, where the function *matrix_plot* plots points in the complex plane. We also added the option *colorbar* = *True* to the matrix plot in order to generate a colour bar which indicates what the colours on the plotted image represent. At the end, we just need to use the command *show* for m, that was defined as the plot, to get our picture.

2.7 Code for the normal drawing

```
51 from sage.plot.point import point
52 def draw (list):
53     p = point(list ,size=1)
54     p.show()
55 roots= load("deg11coeff1.sobj")
56 draw(roots)
```

2.8 Explanation of the code for the normal drawing

In the program above we first needed to import the function *point* from the library *sage.plot.point*, which takes coordinates of a point and draws the point in the complex plane. Then we define a function *draw* which takes as parameter a list of points, here a list of roots computed in the first part. Then, the variable p is defined as point with parameters *list* and *size*, which is set as 1 here. In line 54 of the code, the command *show* is used to return the picture containing all of the points of the list. Finally, the last line calls the function *draw* with the parameter *roots*.

3 Interpretation of the pictures

3.1 Degree 3

We will begin with an image of the roots of the polynomials of degree 3 with coefficients 1 and -1.

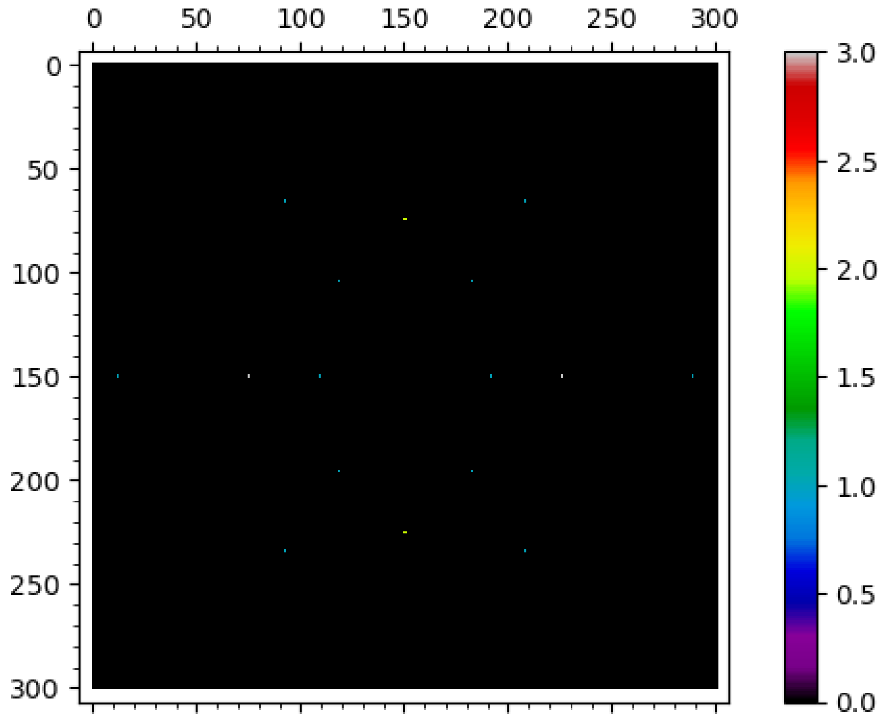


Figure 1: Degree 3

We notice that we do not see much at the first glance, but when we start looking more detailed, we find some coloured dots in the picture. Since the dots have different colours, we notice that even in degree 3 with coefficients 1 an -1 the frequency of the roots is different. Looking at the colour bar on the right side, one can see that the neighbourhoods of the roots have a multiplicity either 1,2 or 3. In fact, only the neighbourhoods around the points -1 or 1 have a multiplicity of 3.
We also can see that those few dots are already situated in a circular form.

Remark. *During the following pages, we will talk about the multiplicity of roots, which refers to the total number of roots in a single pixel of the matrix we use to draw the picture. This means that we do not exactly know if there is a single root with a high multiplicity, or simply many roots in that pixel, we can only determine if the number of roots, counted with multiplicity, in a pixel is a high or low.*

3.2 Degree 4

This is the image generated by our code which plotted the roots of the polynomials of degree 4 with coefficients -1 and 1.

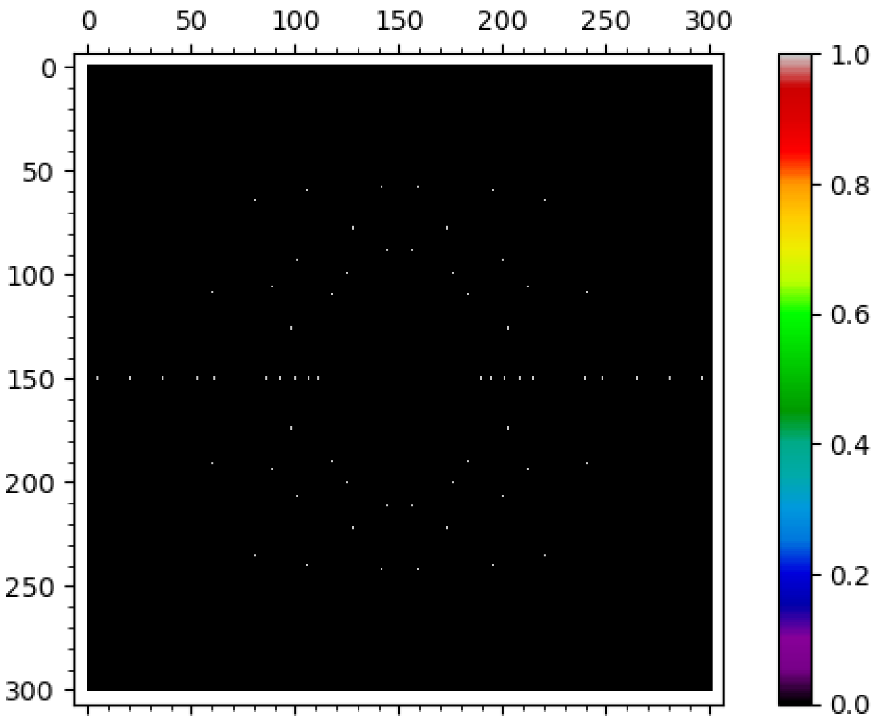


Figure 2: Degree 4

In this picture, we can clearly spot certain symmetries along the x and y axes, which we will prove later on. In addition to that, one can notice that all of the points appear in the same colour. One look at the colour bar proves that every root shown in the picture is of multiplicity 1. We know by Fundamental Theorem of Algebra, that a polynomial of degree n has exactly n roots. So every polynomial of the degree 4 has 4 roots. Moreover, as we consider only the polynomials with coefficients -1 and 1, and we fix the coefficient of the highest degree at 1, we know that there are 2^4 polynomials in our list. This leads us to conclude that there are $4 \cdot 2^4 = 64$ roots in the list of roots. So this, in addition to our observation on the colour bar, indicates that all 64 roots appear only once in the list, thus have a multiplicity of 1.

3.3 Degree 5

Let's take a look at the image of the roots of polynomials of degree 5 with coefficients 1 and -1.

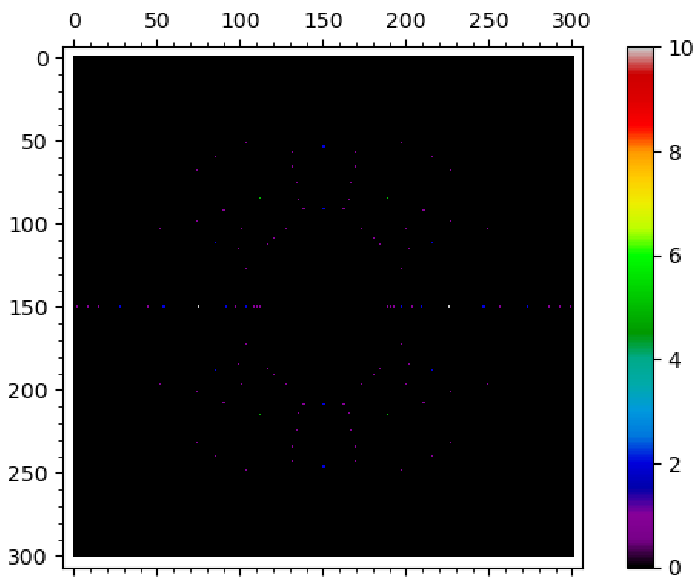


Figure 3: Degree 5

First, we notice, by comparing this image to the picture of the roots of degree 3, that we have many more roots in degree 5, which is clear since the number of polynomials increases with the degree. We now see that the complex roots appear around the circle of radius 1. Moreover, we notice that the real roots start to become more dense on the real axis.

Finally, we start to see multiple holes on the circle, that are situated around the sixth roots of unity, whose signification we will discuss in a following section. By taking a more precise look, we see that the sixth roots of unity, which correspond to the green points, actually are roots of the polynomials of degree 5, but that there are no other roots in their neighbourhood.

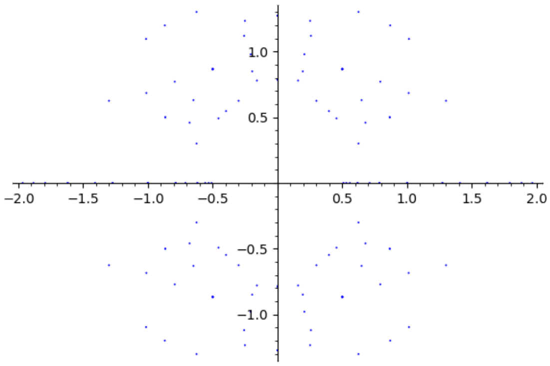


Figure 4: Degree 5

By comparing the two pictures above, one can conclude that the two white points on the first picture are the points -1 and 1. Those are the points with the highest multiplicity on this picture, with each multiplicity being 10. The sixth roots of unity, at the exception of -1 and 1, are coloured green in the first picture. This means that their multiplicity is 6. The other roots do not seem to stand out much, as they all have a multiplicity of either 1 or 2.

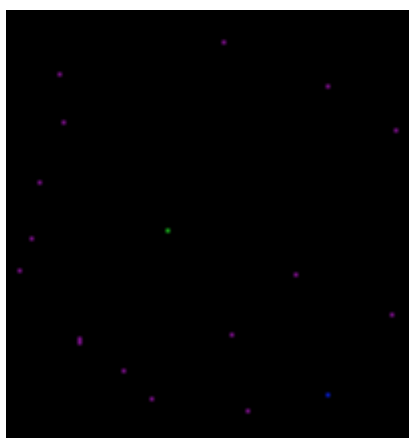


Figure 5: Sixth root of unity

3.4 Degree 6

Next, we will look at an image of the roots of all the polynomials of degree 6 with the same coefficients, 1 and -1.

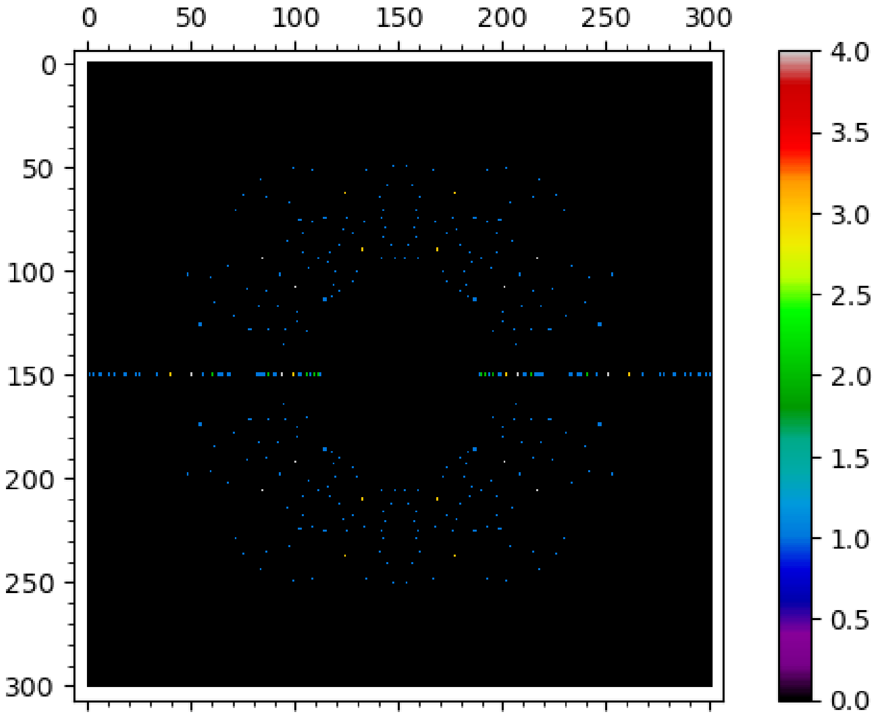
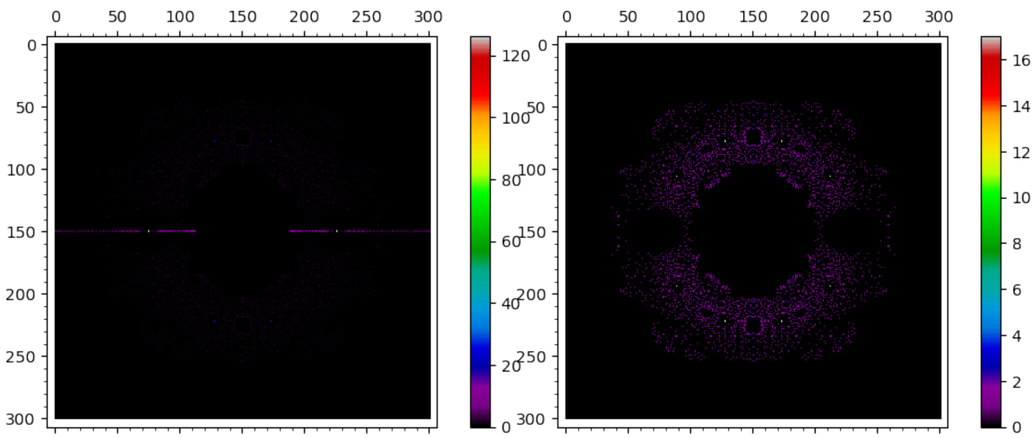


Figure 6: Degree 6

In comparison to the picture of roots of degree 5, we first notice that for the polynomials of degree 6 we have again more different roots which are shown by the increased number of dots in the picture.
We now see that the real roots start to paint the real axis entirely, except for the point 1 and -1. However, one can also notice that the modulus of the roots seems to be bounded, as there are no roots that appear past a certain limit on the inside of the picture. This observation will be discussed and proven later on.
We also clearly see that the complex roots are located around the circle of radius 1.
Furthermore, we observe that we have a symmetry to the x-axis and to the y-axis.

3.5 Degree 9

We will just have a quick look at the image of the roots of polynomials of degree 9 with coefficients 1 and -1.



(a) with real axis

(b) without real axis

Figure 7: Degree 9

When we compare both images, we first notice that on the left picture, we cannot see the circle we see on the right picture. This indicates that most roots of the polynomials of degree 9 are real roots. In addition to that, if we take a glance at the colour bar, we notice that the roots -1 and 1 have the highest multiplicity of 120, while the other roots seem to have a multiplicity around the mark of 30 or lower.
On the left picture, we have removed all the real roots. This allows us to be able to have a look at the complex roots. We can observe the points of the 10th roots of unity, except for -1 and 1, since they shine bright in white. One can notice that the 10th roots of unity have a multiplicity of 17, while the other complex roots only have a multiplicity of 1 or 2.
Moreover, in the second picture, we can see the feather-like shapes very clearly.

3.6 Degree 10

Next we will take a look at the image of the roots of polynomials of degree 10 with coefficients 1 and -1.

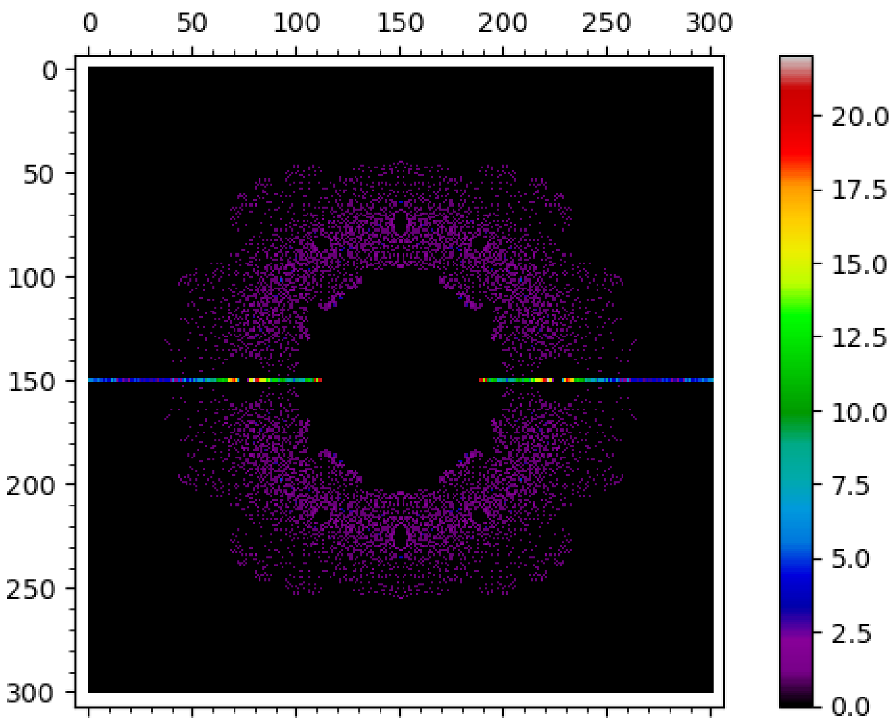
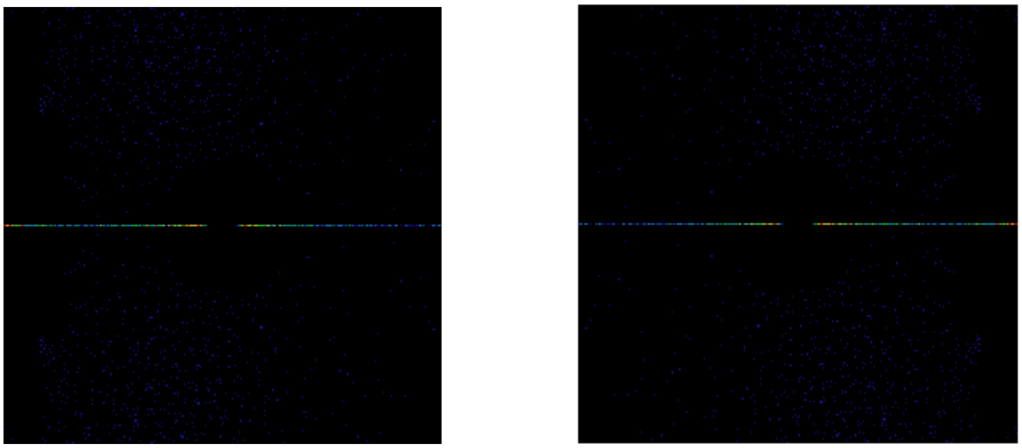


Figure 8: Degree 10 with colour bar

In this picture we can observe perfectly that most roots of the polynomials of degree 10 lie around the circle of radius 1. Moreover we see that the polynomials have many real roots on the x-axis as the colours there are brighter than around the unit circle. If we now take a look at the colour bar, one can notice that the neighbourhoods of roots that lie on the x-axis have a much higher multiplicity, with some being up to 20, while all of the other neighbourhoods of roots seem to have a multiplicity of 1 or 2.

In addition to that, the roots do not take the value 1 and -1 as one sees in the following pictures:



(a) Picture at 1 (b) Picture at -1

Figure 9: Comparison

In these two pictures we can clearly see that there is a hole at 1, respectively -1. One can also see that there are only few complex roots around those points since we can only find a small number of purple points around 1 and -1. We can also see that the multiplicity in a neighbourhood of real roots varies a lot as the colour on the real axis changes every few millimetres, although we can see that many spots are either red or green.
In the main picture wee also see holes at certain points around the circle, those holes represent the roots of unity.

Definition. *A root of unity is a complex number z satisfying the equation*

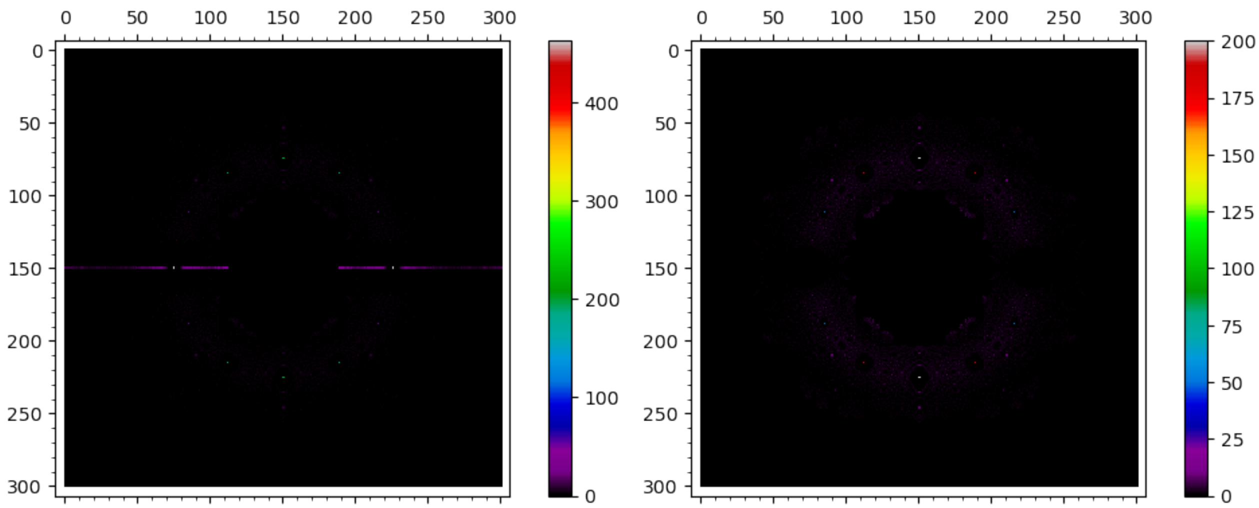
$$z^n = 1,$$

for some $n \in \mathbb{Z}_{\geq 1}$

We will take a more precise look at these points in an higher degree.

3.7 Degree 11

The following image represents the roots of the polynomials of degree 11 with coefficients -1 and 1. When



(a) with real axis

(b) without real axis

Figure 10: Degree 11

we compare both picture we clearly see that many roots of the polynomials of degree 11 with coefficients 1 and -1 are real roots, since we can hardly see the roots around the unit circle. Moreover, we clearly see that the points 1 and -1 are the roots with the highest density since those points shine very bright, which is confirmed by the colour bar.

Next, we take a look at the picture without the real axis, where we now see the unit circle, but more interestingly, we see that the density of the 12-th roots of unity is much higher than the density of any other root in the complex plane. Furthermore, we see that the colours of the 12-th roots of unity vary. For example, as mentioned before -1 and 1 have a high multiplicity of more than 400, while the other 12-th roots of unity have a multiplicity of around 200.

3.8 Degree 12

Let's now take a look at the roots of polynomials of degree 12 with coefficients -1 and 1.

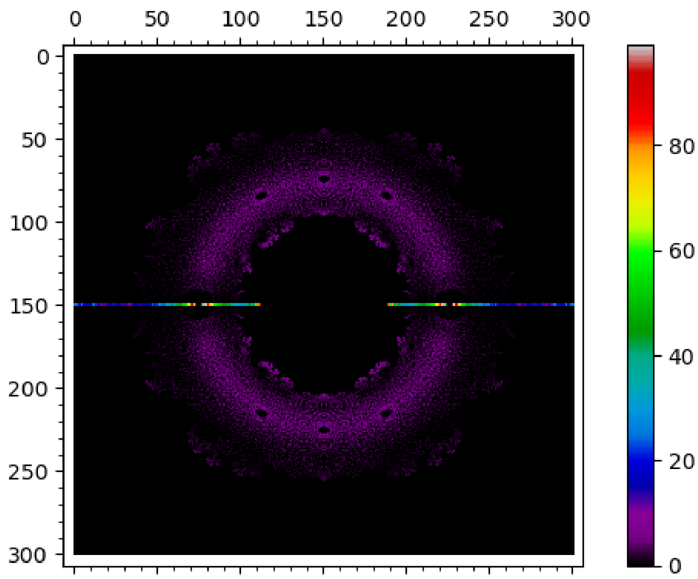


Figure 11: Degree 12

Let's compare these picture with the one of degree 10.
First, we immediately see that the colour is much denser here than it has been in the previous picture which implies that we have many more roots for the polynomials of degree 12 with the same coefficients.
Moreover, the roots of unity can now be observed in a much clearer way.

These observations are due to the difference in the multiplicity, here we have the highest multiplicity over 80, where the multiplicity in degree 10 was bounded slightly above 10.
However, the biggest difference we see are the "feather-like patterns" around the circle of roots which we see especially on the inside of the circle.

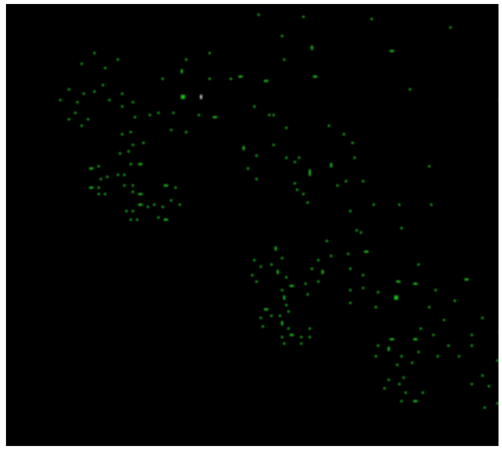


Figure 12: Feather-like patterns

In this zoomed picture of the feather patterns, we observe that we only have few roots which form those patterns in degree 12, later on we will see that there will be more roots contained in these regions and that the points representing them will be brighter, so that most roots appear more often than in degree 12.

3.9 Degree 14

Let's next take a look at the image of the roots of the polynomials of degree 14 with coefficients -1 and 1.

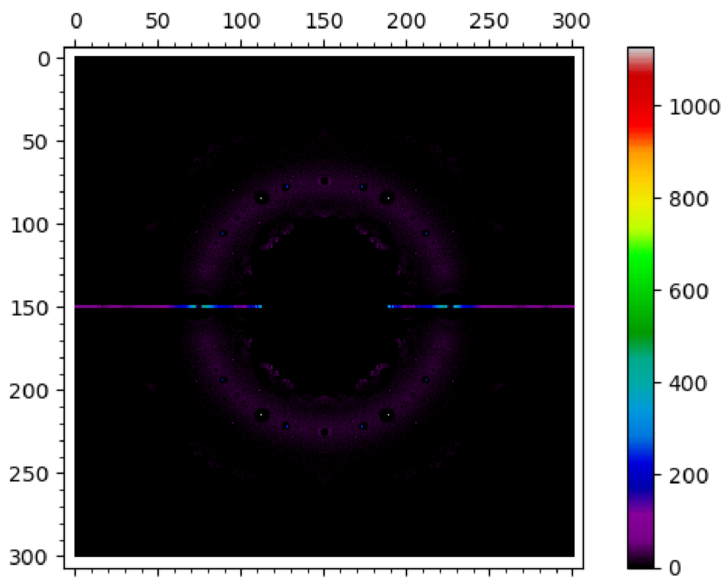


Figure 13: Degree 14

When we compare this picture to the picture of the visualisation of the roots of polynomials of degree 12, we see that the multiplicity of roots increased drastically since the highest one is now over 1000. We also see that the multiplicity of the roots increases when we approach the points 1 and -1 since the colour changes from violet (multiplicity around 100) to green (multiplicity around 700). Moreover, we can again see some roots of unity, here the 6-th roots of unity except for -1 and 1, which are represented by the bright points on the circle, clearer than in degree 12. On the other side, the holes around the roots of unity cannot be seen as well as before.

Moreover, we can see the roots of unity very clearly in this picture, but on the other hand the circle of imaginary roots is quite obscure. This means that there are a lot of roots of unity when we look at the polynomials of degree 14, so that their multiplicity is much higher than for all the other roots. Since the roots of unity are present more often, their points in the image outshine all the other roots so that one cannot see them very well.

3.10 Degree 16

Let's take a look at the image of the roots of the polynomials of degree 16 with coefficients -1 and 1.

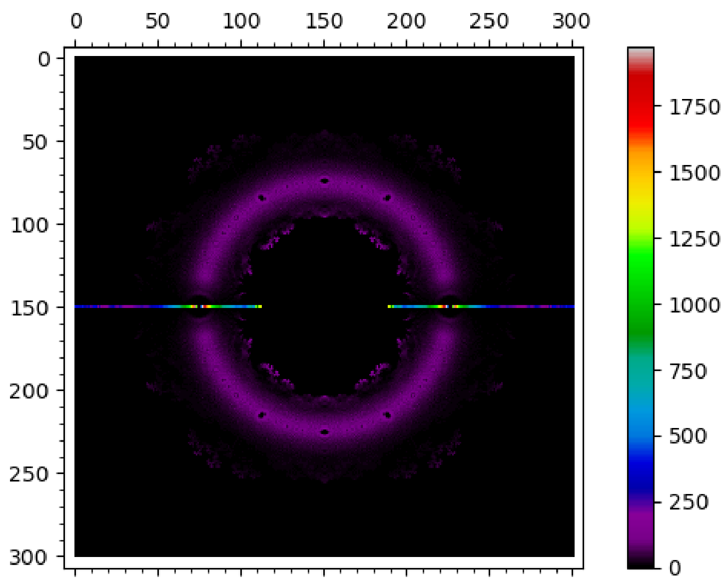


Figure 14: Degree 16

When we compare the picture to the one of degree 14, we can clearly see that the circle is again much clearer and that we do not see the roots of unity. So the roots are differently distributed in degree 16. Again, by interpreting the image, we can say that most imaginary roots are around the unit circle and that the number of roots per point declines when the radius increases or decreases, which can be said by looking at the colour bar.
We also see that the highest multiplicity nearly doubled from degree 14 to degree 16.
Since the real line is build of many different colours, we can say that the number of roots per point changes partially very drastically. Furthermore, since the colours on the real axis are brighter than on the imaginary circle, we have more real roots in a neighbourhood than there are imaginary roots in every different neighbourhood of the imaginary plane.

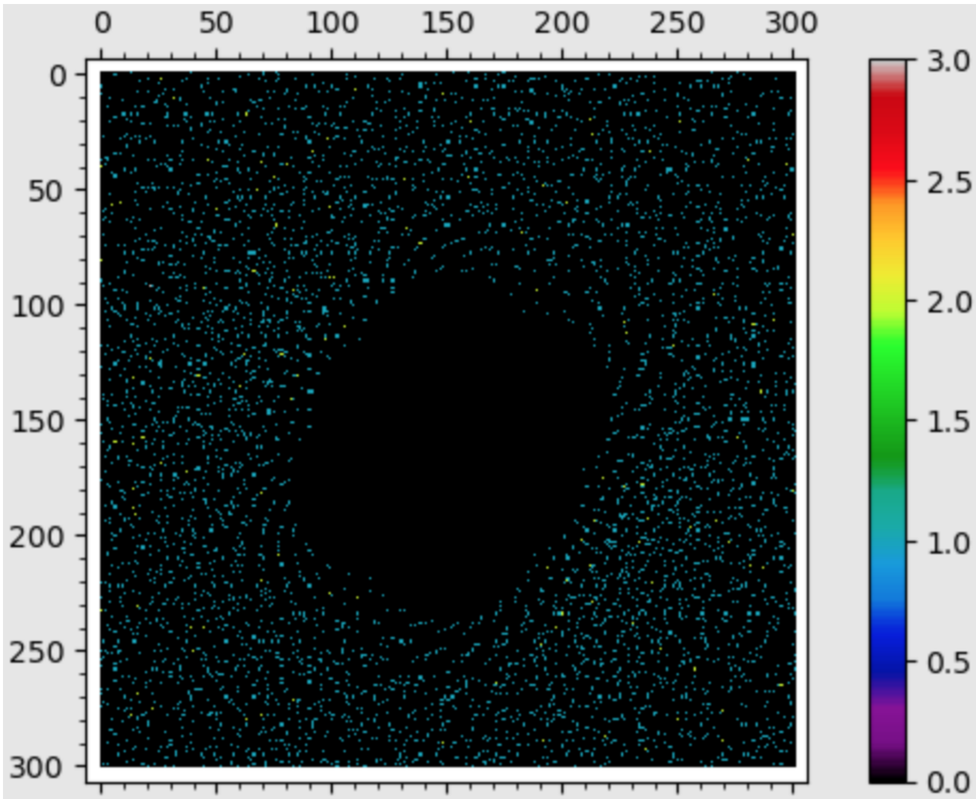


Figure 15: 3rd 12-th Root of unity

Next, we look more closely at a root of unity. We know that the (n+1)th roots of unity are always present since they are at least roots of the polynomial

$$\sum_{k=0}^{16} z^k = \frac{1 - z^{17}}{1 - z}.$$

Furthermore, we see that most roots around the 3rd 12-th root of unity are coloured blue, so that they have multiplicity 1. Moreover, we see that around the root of unity we have a hole without roots.
Let's now look at the more interesting part, the visualised roots, we can observe that even if the colours seem pretty equal, most adjacent points have a slightly different colour, which indicates that the multiplicity of the roots is different for every possible root of the polynomials with degree 16. Since the colour scheme is similar, the difference in the multiplicity of the roots is not very high, so we can conclude that most roots are present in approximately the same order.
We also note that between the roots we can see some black space, so we can say that even if we have many roots on the circle of radius 1, there still is free space on the unit circle, which is clear since for any degree, we will always have a finite number of roots.

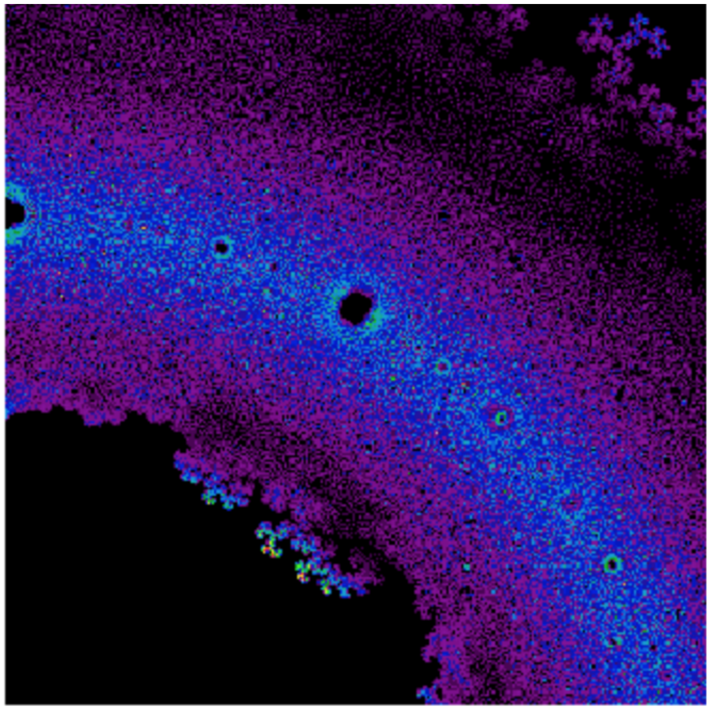


Figure 16: Zoomed picture

Above, we have a picture with a different scale, so that we can take a more precise look at the unit circle. We can see that the colour of the points changes when the modulus of the point increases or decreases. This means that the density of the roots declines when we move away from the unit circle.
We also observe that the extremities of the the feather-like patterns have again a higher density of the separate roots since the colour becomes again bluer and more likely than around the unit circle.
Finally we can see that around the roots of unity, the density of the roots is very high.

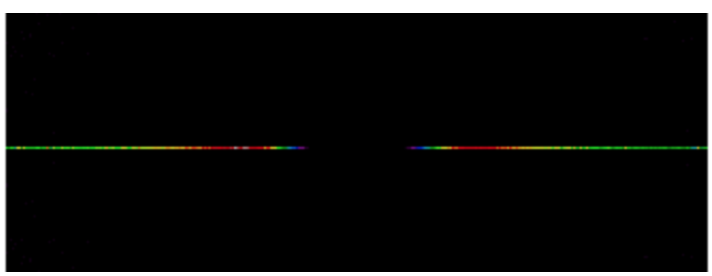
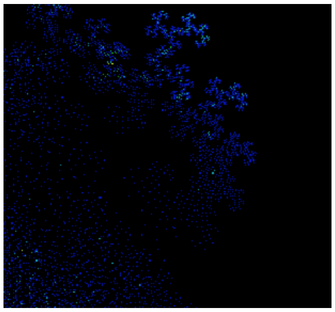


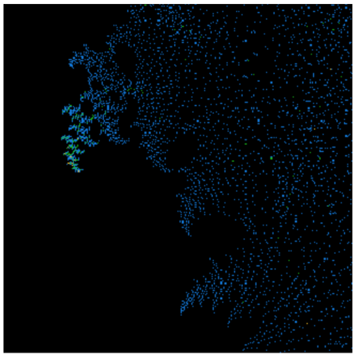
Figure 17: Point 1

Let's also take a quick look at a picture centered at the point 1, we can see that the density of the roots increases as we come closer to the point 1 by the colour scheme. Although, there are no roots in the neighbourhood of 1 itself, shown by the black colouring in the middle of the picture.

Moreover, since the picture is of the polynomials of degree 16 with coefficients 1 and -1, the point 1 itself is not a root.



(a) Outside



(b) Inside

Figure 18: Feather-like patterns

Let's take a look at the feather-like patterns and compare the patterns on the inside and the outside of the circle, we can immediately notice that they have different shapes. On the outside, we see "snowflake" like shapes, where on the inside the patterns rather have a "mountain" shape, since we can see that they have peaks and are formed like mountains.
Moreover, the colours tell us that we have more roots on the patterns inside the circle than on the outside since the colours are brighter on the inside. We can also see that most roots are placed on the peak of the "mountains" as the colours become brighter and brighter from the inside to the outside of the feather-like patterns on the inside of the circle.

3.11 Degree 17

The figure below is the image representing the roots of polynomials of degree 17 with coefficients -1 and 1.

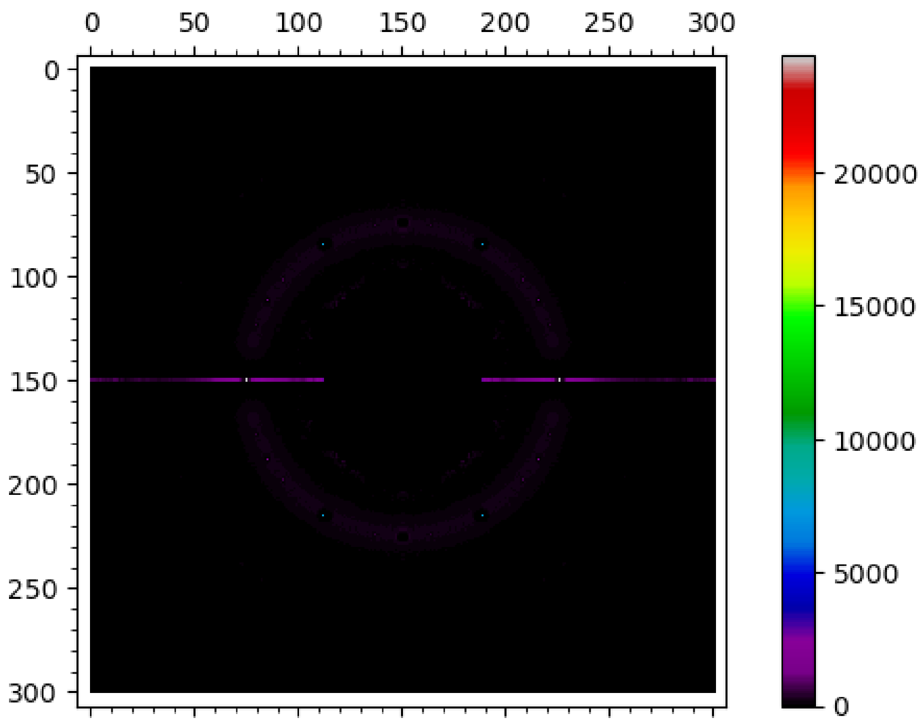


Figure 19: Degree 17

In this picture we see that the real roots around the points 1 and -1 surpass the imaginary roots with a multiplicity of more than 20000 except for the other 6-th roots of unity, which also shine brightly. Those 6-th roots of unity have a multiplicity of 10000 to 15000.

Based on our observations until now, we can say that for odd degrees, the pictures of polynomials with coefficients 1 and -1, are represented mostly by the real roots and the roots of unity. We still can see the unit circle, however in most cases the circle is not identifiable at a first look. This means that in the neighbourhood of all complex points, at the exception of the roots of unity, the multiplicity of the points is lower for odd degrees than for even degrees of polynomials.

We also observed that neighbourhoods of the points $1, -1$ and the roots of unity contain a high number of roots for polynomials of an odd degree.

3.12 Degree 18

Next, we take a look at the visualisation of the roots of the polynomials of degree 18 with coefficients -1 and 1 and compare it to the image obtained in degree 16.

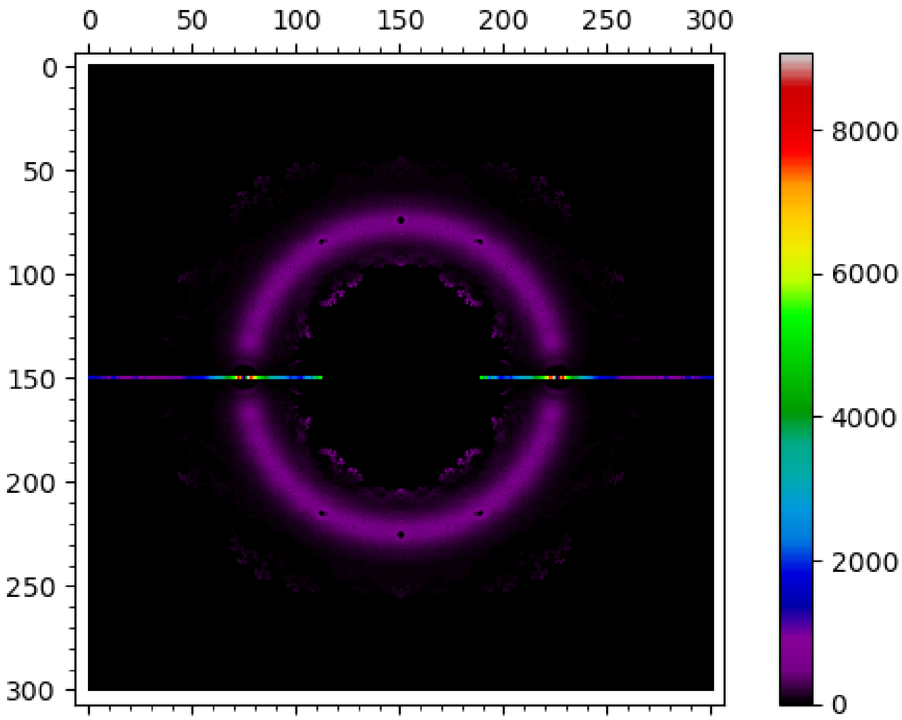


Figure 20: Degree 18

When we compare this image to the one obtained in degree 16, we do not see big differences. The main observation we make is the brightness of the circle and the feather-like patterns on the inside and outside of it. We see that the colours are definitely brighter in this case which indicates that the density of the roots is higher in degree 18, which is logical since we take the roots of more polynomials than in degree 16.

3.13 Assumptions and proofs

By analysing the pictures, one can see that there seems to be a certain symmetry in the pictures. For example, in the picture generated by our program where one can see the roots of polynomials of degree 4 with coefficients -1 an 1, one can clearly see that for every root, there is another root that is symmetric to it with respect to the y-axis and also to the x-axis.

3.13.1 Symmetry to the origin

In fact, let us denote by $S := \{P(X) \quad | \quad P(X) = \sum_{k=0}^{n} a_k \cdot X^k, a_n = 1, a_k \in \{-1; 1\} \text{ for k} \in \{0, 1, .., n-1\}\}$

Let $\alpha := x + yi$ with $x, y \in \mathbb{R}$ be a root of a polynomial in S, i.e $P(\alpha) = \sum_{k=0}^{n} a_k \cdot \alpha^k = 0$ for $P(X) \in S$.

Show that $\beta := -x - yi$ with $x, y \in \mathbb{R}$ is also a root of a polynomial in S. Set $P'(X) := \sum_{k=0}^{n} (-1)^k a_k \cdot X^k$.

$$
\begin{aligned}
P'(\beta) &= \sum_{k=0}^{n} (-1)^k a_k \cdot \beta^k \\
&= \sum_{k=0}^{n} (-1)^k a_k \cdot (-x - yi)^k \\
&= \sum_{k=0}^{n} (-1)^k a_k \cdot (-1)^k (x + yi)^k \\
&= \sum_{k=0}^{n} (-1)^{2k} a_k \cdot (x + yi)^k \\
&= \sum_{k=0}^{n} a_k \cdot (x + yi)^k \\
&= \sum_{k=0}^{n} a_k \cdot \alpha^k \\
&= 0
\end{aligned}
$$

3.13.2 Symmetry to the the y axis

With the same notations as before, show that $\gamma := \overline{\alpha} = x - yi$ with $x, y \in \mathbb{R}$ is also a root of a polynomial in S. Set $P'(X) := \sum_{k=0}^{n} \overline{a_k} \cdot X^k$.

$$\begin{aligned} P'(\gamma) &= \sum_{k=0}^{n} \overline{a_k} \cdot \gamma^k \\ &= \sum_{k=0}^{n} \overline{a_k} \cdot \overline{\alpha}^k \\ &= \sum_{k=0}^{n} \overline{a_k \cdot \alpha}^k \\ &= \overline{\sum_{k=0}^{n} a_k \cdot \alpha^k} \\ &= 0 \end{aligned}$$

3.13.3 Boundedness of the roots

In all of the pictures, it seems that the roots z of the polynomials in S are contained in an annulus. Most of the roots are either contained along the unit circle or on the real axis up until $|z| < 2$.
Using the theorem stated below, show that for every z such that $P(z) = 0$ for $P(X) \in S$, i.e for every root of a polynomial in S, we have $|z| < 2$.

Theorem. *Fundamental Theorem of Algebra* [1]
Let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ for $n \geq 1, a_n \neq 0$. Then p has exactly n zeros inside $|z| < R$ where $R = 1 + \max\left\{\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}\right\}$

As $S := \left\{ P(X) \,|\, P(X) = \sum_{k=0}^{n} a_k \cdot X^k, a_n = 1, a_k \in \{-1; 1\} \text{ for k} \in \{0, 1, .., n-1\} \right\}$
we know that $\max\left\{\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}\right\} = 1$, so

$$\begin{aligned} R &= 1 + \max\left\{\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}\right\} \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

So by the Fundamental Theorem of Algebra, every polynomial $P(x) \in S$, P has exactly n zeros inside $|z| < 2$, so all of its roots are contained inside $|z| < 2$

Now, let's show that the roots are also bounded on the interior so that for every z that is a root of a polynomial $P(X) \in S$, we have that $\frac{1}{2} < |z|$.

[1] The theorem was extracted from our Complex Analysis course.

Take $P(X) = \sum_{k=0}^{n} a_k \cdot X^k, a_n = 1, a_k \in \{-1; 1\}$ for k $\in \{0, 1, .., n-1\}\} \in S$ and
$P'(X) = \sum_{k=0}^{n} a_{n-k} \cdot X^k, a_0 = 1, a_{n-k} \in \{-1; 1\}$ for k $\in \{0, 1, .., n-1\}\} \in S$
As we know that the modulus of the roots of the polynomial $P(X)$ have as upper bound $M := 2$, the lower bound of the modulus of the roots of the polynomial $P'(X)$ is given by $\frac{1}{M} = \frac{1}{2}$ as the roots of both polynomials are multiplicative inverses.

In fact, let us denote by α a root of the polynomial $P(X)$. So

$$P(\alpha) = \sum_{k=0}^{n} a_k \cdot \alpha^k = a_0 + a_1 \cdot \alpha + a_2 \cdot \alpha^2 + \cdots + a_{n-1} \cdot \alpha^{n-1} + a_n \cdot \alpha^n = 0$$

We have to show that $P'(\frac{1}{\alpha}) = 0$.

$$\begin{aligned}
P'(\frac{1}{\alpha}) &= \sum_{k=0}^{n} a_{n-k} \cdot \left(\frac{1}{\alpha}\right)^k \cdot \frac{\alpha^n}{\alpha^n} \quad \text{as} \quad \alpha \neq 0 \\
&= \sum_{k=0}^{n} \frac{a_{n-k} \cdot \alpha^{n-k}}{\alpha^n} \\
&= \frac{1}{\alpha^n} \cdot \sum_{k=0}^{n} a_{n-k} \cdot \alpha^{n-k} \\
&= \frac{1}{\alpha^n} \cdot a_0 + a_1 \cdot \alpha + a_2 \cdot \alpha^2 + \cdots + a_{n-1} \cdot \alpha^{n-1} + a_n \cdot \alpha^n \\
&= 0
\end{aligned}$$

As $P(X)$ was chosen arbitrarily and $P'(X)$ has the same coefficients in opposite order, this is enough to show that for every polynomial in S, the roots are bounded by $\frac{1}{2} < |z|$.

If we put both results together, we get that all of the roots are contained in the open annulus $\{z : \frac{1}{2} < |z| < 2\}$

3.13.4 -1, 1 and roots of unity

By analysing the pictures, one can see that there seem to be "holes" at certain points. Some of these holes appear at 1 and -1 in certain pictures.
In fact, let us denote by $S := \{P(X) \quad | \quad P(X) = \sum_{k=0}^{n} a_k \cdot X^k, a_n = 1, a_k \in \{-1; 1\}$ for k $\in \{0, 1, .., n-1\}\}$
There are two cases : if $P(X) \in S$ has $deg \equiv 1 \bmod (2)$ and if $P(X) \in S$ has $deg \equiv 0 \bmod (2)$.
In the first case, the degree of the polynomials in S is odd, and some of the polynomials take the value 0 for 1 and -1. For example the polynomial $P'(X) = \sum_{k=0}^{n} (-1)^{k+1} \cdot X^k$ is equal to 0 when evaluated in 1 and the

polynomial $P''(X) = \sum_{k=0}^{n} X^k$ is equal to 0 when evaluated in -1.

$$\begin{aligned} P'(1) &= \sum_{k=0}^{n} (-1)^{k+1} \cdot 1^k \\ &= 1 - 1 + \cdots 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} P''(-1) &= \sum_{k=0}^{n} (-1)^k \\ &= -1 + 1 + \cdots - 1 + 1 \\ &= 0 \end{aligned}$$

However, in the second case, when the degree of the polynomials in S is even, none of the polynomials take the value 0 at the points -1 and 1. Take $P(X) \in S$, so $P(X)$ is of the form $P(X) = \sum_{k=0}^{n} a_k \cdot X^k$ with $n \equiv 0 \bmod (2)$. As the coefficients of the polynomials are either -1 or 1, if we insert either -1 or 1 into the polynomials, we can never obtain the value 0.

$$\begin{aligned} P(1) &= \sum_{k=0}^{n} a_k \cdot 1^k \\ &= \text{"}(n+1) \cdot (\pm 1)\text{"} \\ &\equiv 1 \operatorname{mod}(2) \\ &\neq 0 \end{aligned}$$

$$\begin{aligned} P(-1) &= \sum_{k=0}^{n} a_k \cdot (-1)^k \\ &= \text{"}(n+1) \cdot (\pm 1)\text{"} \\ &\equiv 1 \operatorname{mod}(2) \\ &\neq 0 \end{aligned}$$

Other points that seem to be more special are the roots of unity.
Let us consider the polynomial $P(X) = \sum_{k=0}^{n} X^k = \frac{1 - X^{n+1}}{1 - X}$. This shows that for the $(n+1) - th$ roots are roots of that polynomial, except for the first $(n+1) - th$ root, so except for 1.

4 Sources

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