FSTM , Bachelor in Mathematics

Experimental Mathematics Lab

Winter Semester 2025-2026

The Fibonacci Word Fractal: Construction and Geometric Properties

Authors :
Lucie BURTON
Auriane NOËL
Victor SOUIRJI

Supervisor :
Dr. Thomas LAMBY

Abstract

The purpose of this paper is to study the properties of the Fibonacci Word Fractal and the Fibonacci Snowflake and explore possible generalizations through the lens of stochastic fractals and complex analysis. We begin with the recursive construction of the Fibonacci word to define the fractal curve using the Odd-Even Drawing rule, implemented using Python's Turtle library. We establish its main geometric properties, including the study of the possible patterns, the symmetry and the self-similarities. We first rigorously compute the fractal dimension of the Fibonacci Word Fractal curve using thermodynamic formalism and proving formally the equality between the Hausdorff dimension and the Box-Counting dimension at $s = 3\ln(\varphi)(\ln(1+\sqrt{2}))^{-1}$, followed by the computation of its boundary dimension, $s = \ln(3)(\ln(1+\sqrt{2}))^{-1}$. The study then extends to Fibonacci Snowflakes, giving their perimeter, $L(n) = 4A_{3n+1}$, and their area, $S(n) = P(2n+1)$, respectively as functions of Fibonacci and Pell numbers. Finally, we generalize the definition of the Fibonacci Word Fractal into a n-dimensional space ($n \geq 3$), introducing a stochastic orientation-switching parameter, analyse the resulting dimensional variance and we conclude with an algebraic study of original fractal in the complex plane.

Keywords: Fibonacci Word, Fibonacci Word Fractal, Combinatoric on Words, Fractal Dimension, Hausdorff Dimension, Box-counting Dimension, Thermodynamic Formalism, Iterated Function System, Fibonacci Snowflake, Golden Ratio, Stochastic Fractals, Generalization of the Fibonacci Word Fractal, Pell Numbers, Python, Turtle, Matplotlib.

Contents

1 Introduction

In 1975, Benoit Mandelbrot (1924–2010) formalized fractal geometry, putting together the work of Felix Hausdorff on fractional dimensions and of his mentor Paul Lévy on Brownian motions [20]. He introduced the notion of a fractal—from the Latin *fractus* (designating shattering or "disrupted," "erratic" objects)—which he characterized by its irregularity, its dimension, and illustrative examples. In this work, we focus on the Fibonacci word fractal: a fractal curve defined on the plane following the Fibonacci word, a Sturmian sequence closely related to the Golden Ratio and the Fibonacci numbers. The Fibonacci Word Fractal serves as a rigorous bridge between combinatorics on words and geometry constructed using a mapping known as the Odd-Even Drawing rule, resulting in a curve with intricate symmetries and a recursive structure.

Following the foundational work of Monnerot-Dumaine [21], the purpose of this paper is to analyse the properties of the Fibonacci Word Fractal, its variants, and the Fibonacci Snowflake, while exploring generalizations through the lens of stochastic fractals. This provides a solid background to understand complex research involving the Fibonacci Word and Fibonacci Word Fractals, such as Fibonacci Quasicrystals.

The work is organized as follows: In Section 2, we define the Fibonacci Word and study its main properties and its fundamental link with Fibonacci numbers, leading to Section 3, where we construct the Fibonacci Word Fractal and detail its geometric properties, covering the similarity of successive patterns and studying their symmetry. Using different proofs and computations, we introduce fractal dimension in Section 4, constituting the core theoretical contribution. Using Measure Theory and Thermodynamic Formalism, we compute the Box-Counting dimension and the Hausdorff dimension of the Fibonacci Word Fractal curve, concluding by showing their equality with the Open Set Condition. Assuming identical conditions, we then compute the Box-Counting dimension of the boundary of the Fibonacci Word Fractal.In the fifth section, we extend our study to the Fibonacci Snowflakes, the construction of which shows that sequences derived from Fibonacci words define closed, non-self-intersecting paths that bound a planar region. Using combinatorial methods presented in the appendix, we show that these curves are characterized as well-defined polyominoes and provide their geometric properties, such as their perimeter and area, respectively determined as functions of Fibonacci and Pell numbers. Section 6 explores an extension of the Fibonacci Word Fractal into n-dimensional space ($n \geq 3$) by introducing a stochastic orientation-switching parameter. We analyse the resulting dimension and the effect the stochastic switching probability has on it, noting its "Jump" and "Plateau" behaviour. Finally, we conclude in Section 7 by representing the Fibonacci Word Fractal in the complex plane, enabling a transition from a geometric analysis to an algebraic and arithmetic approach by associating the endpoint of each path with a complex number.

2 The Fibonacci Word [21]

This section introduces the Fibonacci Word Fractal. The aim is to define the Fibonacci Word Fractal and prove its main properties using [21].

Definition 2.1 (The Fibonacci Word)**.** The *n-th Fibonacci Word*, denoted by S_n, is defined iteratively by the concatenation of S_{n-1} and S_{n-2} for all $n \geq 3$. We let $S_1 = 1$ and $S_2 = 0$. The *Fibonacci Word* is obtained as $n \to \infty$.

In other words, for all $n \geq 3$, $S_n = S_{n-1}S_{n-2}$, with $S_i, \forall i \in \mathbb{N}$, strings. We can also define the morphism

$$\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases} \tag{$\mathscr{S}$}$$

that maps each character (reading the word from left to right) of S_{n-1} to a defined subword (01 or 0) of the Fibonacci Word. By an abuse of terminology, we also use the name Fibonacci Word to describe the $n-$th Fibonacci Word S_n, with $n < \infty$.

Remark 2.1. Multiple definitions of the Fibonacci Word exist, differing in their initial values S_1 and S_2. We have selected this specific definition to maintain a direct mathematical alignment with the Fibonacci Numbers.

Remark 2.2. The 8 first Fibonacci Words are :

- $S_1 =$ 1
- $S_2 =$ 0
- $S_3 =$ 01
- $S_4 =$ 010
- $S_5 =$ 01001
- $S_6 =$ 01001010
- $S_7 =$ 0100101001001
- $S_8 =$ 010010100100101001010

Definition 2.2. A sequence of letters in a Fibonacci Word is called *subword.*

Remark 2.3. In some literatures, subword may define a possibly non-contiguous subsequence of a word. Contiguous subwords are then called *factors*. In this document, we consider subwords as factors.

Definition 2.3. We define the *length of a finite Fibonacci Word* S_n as the number of letters of S_n, denoted $|S_n|$.

2.1 Some properties of the Fibonacci words

Proposition 2.1. *For $n \geq 3$, the two last letters ab of a Fibonacci Word S_n are 01 or 10 if n is odd or if n is even, respectively.*

Proof. We prove this statement using induction with base cases $S_3 = 01$ and $S_4 = 010$, both verifying Proposition 2.1. We assume n is odd, meaning that $(n-1)$ and $(n-2)$ are respectively even and odd and therefore, that S_{n-1} ends by the subword (10) and S_{n-2} by (01) (*induction hypothesis*). Since $S_n = S_{n-1}S_{n-2}$ and that string concatenation preserves the order of the characters of each substrings, it follows that the two last letters of S_n are the same as the ones of S_{n-2}, i.e. 01. We have therefore shown that for $n \geq 3$ odd, the last two letters of S_n are 01. The proof for n even is analogous. ■

Proposition 2.2. *The subwords (11) and (000) can't appear in a Fibonacci Word*

Proof. We proceed by induction. Let us first prove that (11) is not a subword of S_n $\forall n \in \mathbb{N}$. Given as base cases S_1, S_2, S_3, S_4 (all verifying our property), we assume that S_{n-1} and S_{n-2} do not contain the subword (11) and prove our statement for S_n. Since $S_n = S_{n-1}S_{n-2}$ but neither S_{n-1} nor S_{n-2} contain (11), we have to show that (11) can't be formed by crossing the boundary of S_{n-1} and S_{n-2}. From Proposition 2.1, we have that only for n odd, S_n ends by the letter 1 and moreover the morphism σ ($\mathscr{S}$) shows that any letter is mapped to either 0, or 01, i.e. two subwords starting with 0, meaning that S_{n-2} can't start by the letter 1. Therefore, all possible subwords of length 2 formed by crossing the boundary of S_{n-1} and S_{n-2} are (00) or (10). Hence we have shown that (11) is not a subword of S_n $\forall n \in \mathbb{N}$.
We apply the same principle for the subword (000), using as base cases S_i with $i \in \{1, ..., 5\}$ and assuming neither S_{n-1} nor S_{n-2} contains (000). We can reformulate the problem as proving that no Fibonacci Word starts or ends by the substring (00), as it is the only way that (000) is formed by crossing the boundary of S_{n-1} and S_{n-2}. By Proposition 2.1, we already know that no Fibonacci word ends by the letters (00) and we have already discussed that $\forall n \geq 2, S_n$ starts by (0), therefore, ($\mathscr{S}$), $0 \mapsto 01$ and hence $\forall n \geq 2, S_{n+1}$ must start by (01). In conclusion, (000) is not a subword of the Fibonacci word sequence. ■

Before stating the next propositions, we define a new function that takes as an input a finite word of any alphabet and returns that word but reversed, in other words:

Definition 2.4. Let Σ^* be the set of all sequences of letters from a non-empty finite set of letters Σ, called an alphabet. We define the application

$$^{\leftarrow} : \Sigma^* \to \Sigma^*$$

such that for all $s \in \Sigma^*, s = a_1a_2...a_{n-1}a_n$, $s^{\leftarrow} = a_na_{n-1}...a_2a_1 \in \Sigma^*$, where $a_1, ..., a_n \in \Sigma$.

Let $s = ab \in \Sigma^*$, where $a = a_1...a_n$ and $b = b_1...b_m$ with $a_1, ..., a_n, b_1, ..., b_m \in \Sigma$. We can verify that $s^{\leftarrow} = b^{\leftarrow}a^{\leftarrow}$. Indeed, $s^{\leftarrow} = b_m...b_1a_n...a_1 = (b_m...b_1)(a_n...a_1) = b^{\leftarrow}a^{\leftarrow}$. Moreover, since, $\{0, 1\}$ is an alphabet, any Fibonacci word belongs to $\{0, 1\}^*$, where $^{\leftarrow}$ is defined.

Proposition 2.3. *If $S_n = p_nab$ with $n \geq 4$, where the suffix ab represents the last two letters of S_n , then p_n is a palindrome.*

Proof. Proving by induction on n, we have for $n = 4, 5, 6$, $S_{n=4} = 010, S_5 = 01001, S_6 = 01001010$, as base cases and $p_4 = 0, p_5 = 010, p_6 = 010010$ are palindromes. Our induction hypothesis states that S_{n-2} and S_{n-3} without their two last letters forms respectively p_{n-2} and p_{n-3}, both palindromes. We therefore need to show that $p_n = S_{n-1}p_{n-2}$ is a palindrome, meaning that $S_{n-1}p_{n-2} = (S_{n-1}p_{n-2})^{\leftarrow}$. By Proposition 2.1 and since $\{0, 1\}$ is our alphabet if S_{n-2} ends by ab, then S_{n-3} must end by ba. Therefore, $S_{n-1} =$

$S_{n-2}S_{n-3} = p_{n-2}abp_{n-3}ba$ and hence, $S_n = S_{n-1}S_{n-2} = p_{n-2}abp_{n-3}bap_{n-2}ab \implies p_n = p_{n-2}abp_{n-3}bap_{n-2} = p_n^{\leftarrow}$. This all implies that p_n is a palindrome. ■

Proposition 2.4. *$p'_n = baS_n = bap_nab$ is a palindrome.*

Proof. Applying Definition 2.4, we have that : $(p'_n)^{\leftarrow} = S_n^{\leftarrow}(ba)^{\leftarrow} = (ab)^{\leftarrow}p_n^{\leftarrow}b^{\leftarrow}a^{\leftarrow} = bap_nba = p'_n$, showing that p'_n is a palindrome. ■

The two following statements will be very useful to prove geometric properties of the Fibonacci Word Fractal.

Theorem 2.1. *Let $S_n = p_nab$ and define $t_n = p_nba$. Then for every $n > 6$,*

$$S_n = S_{n-3}S_{n-3}S_{n-6}t_{n-3}t_{n-3}$$

Proof.

$$\begin{aligned} S_n &= S_{n-1}S_{n-2} \\ &= S_{n-2}S_{n-3}S_{n-3}S_{n-4} \\ &= S_{n-3}S_{n-4}S_{n-5}S_{n-6}S_{n-5}S_{n-4}S_{n-5}S_{n-4} \end{aligned}$$

One can show using induction that $S_n = S_{n-2}t_{n-1}$ and $t_n = S_{n-2}S_{n-1}$. One can proof it using as base case $n = 5$ and then assuming that $S_n = S_{n-2}t_{n-1}$ and $t_n = S_{n-2}S_{n-1}$ as induction step. Indeed, we get that $t_{n+1} = p_{n+1}ab = S_np_{n-1}ab = S_nt_{n-1} = S_{n-1}(S_{n-2}t_{n-1}) = S_{n-1}S_n$. Therefore,

$$\begin{aligned} S_n &= (S_{n-3})(S_{n-4}S_{n-5})(S_{n-6})(S_{n-5}S_{n-4})(S_{n-5}S_{n-4}) \\ &= S_{n-3}S_{n-3}S_{n-6}t_{n-3}t_{n-3} \end{aligned}$$

■

Corollary 2.1. *For every $n > 6$ $p_n = p_{n-3}\,ab\,p_{n-3}\,p_{n-6}\,p_{n-3}\,ba\,p_{n-3}$.*

Proof. Applying Proposition 2.3 to the result shown in Theorem 2.1, one gets

$$\begin{aligned} S_n &= S_{n-3}S_{n-3}S_{n-6}t_{n-3}t_{n-3} \\ &= (p_{n-3}ab)(p_{n-3}ab)S_{n-6}(p_{n-3}ba)(p_{n-3}ba) \\ &= (p_{n-3}ab)p_{n-3}(abS_{n-6})p_{n-3}ba(p_{n-3}ba) \\ &= p_{n-3}abp_{n-3}p'_{n-6}p_{n-3}bap_{n-3}ba \qquad (1) \end{aligned}$$

(1) since $S_{n-6} = p_{n-6}ba$ by Proposition 2.1.

■

Also, note that the Fibonacci Word is a classical example of Sturmian words. We will not prove this property, but it helps understanding how the ratio computed in Proposition 2.7 appears in the Fibonacci Word Sequence and the construction of the Fibonacci Word Fractal.

Definition 2.5 (Sturmian Words)**.** Let $\Sigma = \{a, b\}$ be an alphabet. An infinite word on Σ is called Sturmian if its complexity (measured by the subword complexity function $p : \mathbb{N} \to \mathbb{N}$) is $p(n) = n + 1$. That is, $s \in \Sigma^*$ is Sturmian if for all $n \geq 1$, s has $n + 1$ subwords (factors) of length n, denoted $|s'|$ where s' is a subword of s. Moreover, all finite subwords of s are Sturmian.

2.2 Relation with the Fibonacci Numbers

Definition 2.6. The n-th Fibonacci Number, denoted as A_n, is defined by $A_n = A_{n-1} + A_{n-2}$ for all $n \geq 3$. We let $A_1 = 1$ and $A_2 = 1$.

Remark 2.4. The 8 first Fibonacci numbers are :

- $A_1 = 1$
- $A_2 = 1$
- $A_3 = 2$
- $A_4 = 3$
- $A_5 = 5$
- $A_6 = 8$
- $A_7 = 13$
- $A_8 = 21$

Proposition 2.5. *The length of the Fibonacci Word $|S_n|$ is the Fibonacci number A_n, $\forall n \geq 1$.*

Proof. We can prove this by induction. The base cases for $n = 1$ and $n = 2$ are done, as shown above. Let's now assume that $|S_{n-1}| = A_{n-1}$ and $|S_{n-2}| = A_{n-2}$. We have to show that $|S_n| = A_n$, which can directly be shown by adding the two last equalities. ■

Proposition 2.6. *The ratio $\frac{A_{n+1}}{A_n}$ tends to φ, the Golden ratio, as $n \to +\infty$.*

Proof. Let $r_n = \frac{A_{n+1}}{A_n}$ and $L = \lim_{n\to\infty} r_n$
By definition of the Fibonacci number, we get :

$$r_n = \frac{A_{n+1}}{A_n} = \frac{A_n + A_{n-1}}{A_n} = 1 + \frac{A_{n-1}}{A_n} = 1 + \frac{1}{r_{n-1}}$$

Taking the limit, we get :

$$L = 1 + \frac{1}{L}$$

Solving for L, we obtain $L = \dfrac{1 \pm \sqrt{5}}{2}$

Since A_n , $A_{n-1} > 0 \ \forall n$ and $r_n > 0 \ \forall n$

We get $L > 0$

Hence $L = \dfrac{1+\sqrt{5}}{2}$

We have successfully proven that $\frac{A_{n++1}}{A_n}$ tends to $\varphi = \frac{1+\sqrt{5}}{2}$, the Golden ratio. ■

Proposition 2.7. *The n-th Fibonacci number A_n is even* $\Leftrightarrow n \equiv 0 \mod 3, \forall n \geq 1$

Proof. It suffices to prove that :

$$\begin{cases} A_{3k+1} \text{ is odd,} \\ A_{3k+2} \text{ is odd,} \\ A_{3k+3} \text{ is even.} \end{cases}$$

Let's proceed by induction.

Base case : $k = 0$

For $k = 0$, we have

$$A_1 = 1 \text{ (odd)}, \qquad A_2 = 1 \text{ (odd)}, \qquad A_3 = 2 \text{ (even)},$$

Inductive step

Assume that, for $k \geq 0$,

$$\begin{cases} A_{3k+1} \text{ is odd,} \\ A_{3k+2} \text{ is odd,} \\ A_{3k+3} \text{ is even,} \end{cases}$$

Using definition 2.6, we obtain :

- $A_{3k+4} = \underbrace{A_{3k+3}}_{\text{even}} + \underbrace{A_{3k+2}}_{\text{odd}} \Rightarrow A_{3k+4}$ is odd.
- $A_{3k+5} = \underbrace{A_{3k+4}}_{\text{odd}} + \underbrace{A_{3k+3}}_{\text{even}} \Rightarrow A_{3k+5}$ is odd.
- $A_{3k+6} = \underbrace{A_{3k+5}}_{\text{odd}} + \underbrace{A_{3k+4}}_{\text{odd}} \Rightarrow A_{3k+6}$ is even.

This proves that :

A_n is even $\iff n \equiv 0 \pmod 3$.

∎

3 Construction and geometrical properties of the Fibonacci Word Fractal

3.1 The construction of the Fibonacci Word Fractal

We now introduce the Fibonacci Word Fractal (sometimes noted FWF) itself. By a slight abuse of terminology, we refer to the finite curves as "fractals", we call n-th Fibonacci Word Fractal F_n, obtained from the Fibonacci word S_n. **The actual Fibonacci Word Fractal is obtained only in the limit, as** $n \to \infty$. To emphasize the finiteness of F_n for instance in the context of Iterated Function System (a method of constructing fractal), we later use the notion of pre-fractal for F_n with $n < \infty$.

Definition 3.1 (The n-th Fibonacci Word Fractal)**.** The n-th Fibonacci Word Fractal is a geometric representation of the n-th Fibonacci word. It is generated by interpreting each letter of the sequence S_n as a specific movement command in the plane. This construction follows the Odd-Even Drawing Rule (OEDR):
$\forall$ letter in the k-th position:

1. If the letter is 1, draw a line segment in the previous direction of length 1.
2. If the letter is 0, draw a line segment of length 1 after rotating 90 $^\circ$
   (a) to the right if k is even
   (b) to the left if k is odd

Throughout this document, we will use Python to build Fibonacci Word Fractals and variants, as changing the angle of rotation whenever the letter 0 occurs, or considering for instance a new dimension to our Fractal, see Definition 6.1 . We will mainly use Turtle for the classical definition and classical angle of rotation changes and Matplotlib for more complex programs.
The following code generate a Fibonacci Word Fractal for any given n.

```
from turtle import *
from math import *

def fibonacci_word(n):
    if n == 1:
        return "1"
    elif n == 2:
        return "0"
    else: return fibonacci_word(n-1)+fibonacci_word(n-2)

def draw_fibonacci_word_fractal(word, step):

    speed(0)
    penup()
    setposition(-200,0) #Initial position of the turtle
    pendown()
    setheading(90) #Initial orientation of the turtle (in degrees)

    #The characters of f_n and their index are stored in a list:
    fwinlist = list(enumerate(word))
```

```

    #The fractal is constructed using OEDR:

    for i, c in fwinlist: #Where i, the index and c, the character
        forward(step)
        if c == "0":
            if (len(fwinlist) - i) % 2 == 0:
                left(90)
            else:
                right(90)

    done()


n = int(input("Please give a positive non-zero number: "))
s = float(input("Give a step measure: "))
w = fibonacci_word(n)

print(w) #To print the Fibonacci Word
draw_fibonacci_word_fractal(w,s,a) #Draws the Fractal
```

Remark 3.1. A more elaborate version of the code can be found in the document `The Fibonacci Word Fractal - Construction and Geometric Properties - Final > Codes`. In this version, we provide a solution to:

1. Center the fractal and compute the optimal step size to draw the fractal in a given window. Given a step measure, we can determine the size of the window the fractal will be centered in by simulating the OEDR with the given step size. We keep trace of the "bounds", meaning that, choosing the initial position of the turtle to be the origin of the Cartesian plane with x-axis and y-axis, we store the maximum and minimum values of x and y the curve goes through and use them to compute the width and height of the wanted window using the formulas width $= (\max_x - \min_x + 100)$ and height $= (\max_y - \min_y + 100)$, adding a margin of 100 pixels.
   Given the window, we use the same technique as described above, computing the "bounds" and the necessary window measures simulating a Fibonacci Word Fractal with an initial step of size 1. The optimal step size is given by the minimal value found computing the ratio between the width of the preferred window and the newly computed width and between the height of the preferred window and the newly computed one.

2. Get directly the drawn fractal using `tracer()` and `update()` methods.

3. Enable generating variants of the Fibonacci Word Fractal, where we replace 90 degrees turns by other angle turns (usual examples are the 60° and the 72° variants). We define outside of `draw_fibonacci_word_fractal()` the following function:

```
def alternating_turn(i, word_len, chosen_angle):
    return chosen_angle if (word_len - i) % 2 == 0 else -
  chosen_angle
```

We obtain the variants passing `alternating_angle()` as a new argument of `draw_fibonacci_word_fractal()`:

```
def draw_fibonacci_word(word, step, chosen_angle, turn_rule):

    speed(0)
    penup()
    setposition(-200,0)
    pendown()
    setheading(90)

    for i, c in enumerate(word):
        forward(step)
        if c == "0":
            angle_change = turn_rule(i, len(word), base_angle)
            left(angle_change)
```

We therefore can generalize Definition 3.1:

Definition 3.2. We consider $\alpha^\circ \in [0, 2\pi]$.
The *α°-variant of n-th Fibonacci Word Fractal* is constructed using the Odd-Even Drawing Rule (OEDR), consisting in :
$\forall$ letter in the k-th position:

1. If the letter is 1, draw a line segment in the previous direction of length 1.
2. If the letter is 0, draw a line segment of length 1 rotate with an angle of
   (a) $+\alpha^\circ$ if k is even
   (b) $-\alpha^\circ$ if k is odd

If $\alpha = 90^\circ$, we call the variant the $n-$th Fibonacci Word Fractal.

The above algorithms output the following curves:

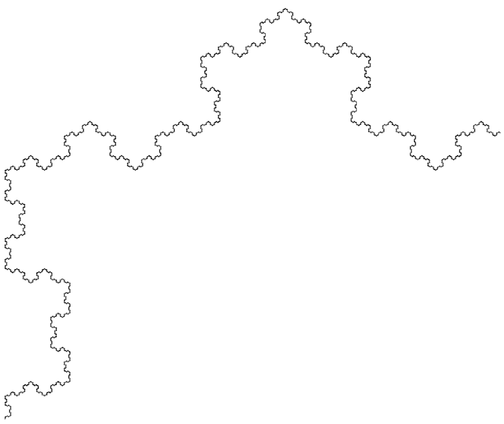


Figure 2: F_{23} with $\alpha = 60^\circ$

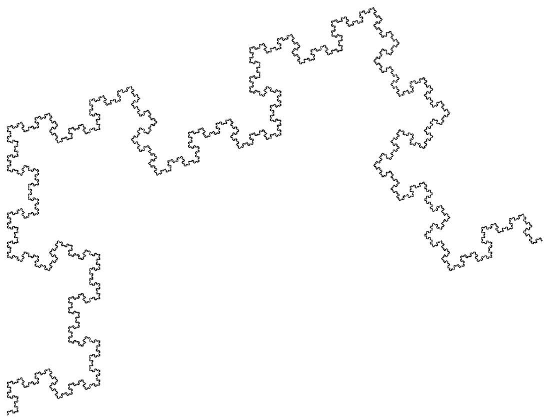


Figure 3: F_{23} with $\alpha = 72^\circ$

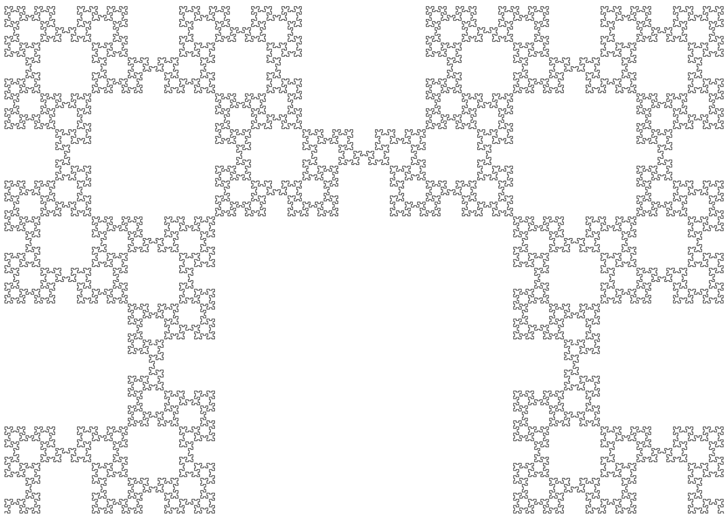


Figure 1: F_{23} with $\alpha = 90°$

Note that not all $\alpha°$-variant of n-th Fibonacci Word Fractal is a self-similar set or satisfy the property of not overlapping or intersecting with itself, as in Figures 4 and 5. Later, when studying the fractal dimension of the curve of the $\alpha°$-variant of the Fibonacci Word Fractal, we will only consider $\alpha \in [0, \frac{\pi}{2}]$ as we assume these variants don't overlap, see Theorem 3.3.

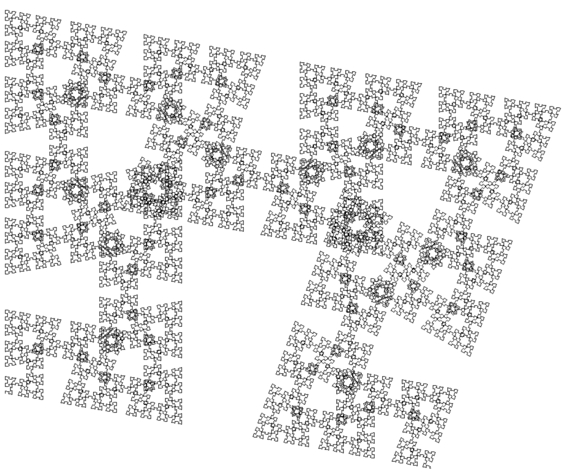


Figure 4: F_{23} with $\alpha = 100°$

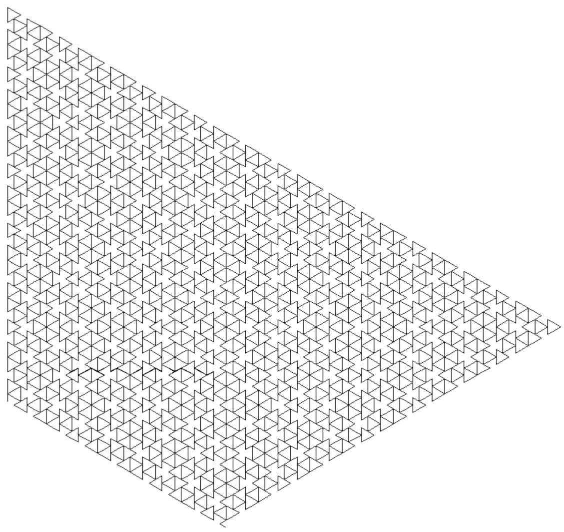


Figure 5: F_{23} with $\alpha = 120°$

Remark 3.2. The overlapping and self intersection properties of Figures 4 and 5 can be emphasized using `Manim` Python script and switching from the set of points representing $F_{23}, \alpha = 90°$ to the variants with $\alpha = 100°$ and $\alpha = 120$ with the function `ReplacementTransform(old_fractal, new_fractal)`. The code structure and the main new functions are provided in Appendix 8.4.

Let us now define the following terms, useful to study the properties of the fractal.

Definition 3.3. The *resulting angle* of a word ω is defined as the change in angle from the first to the last angle of the curve generated by the word ω.

Definition 3.4. The *pattern* P_n describes the global drawing of the n-th FWF.

Definition 3.5. The *length of the fractal* F_n, denoted by L_n, is defined by the length of the straight line from first to last point drawn.

Definition 3.6. The *height of the fractal* F_n, denoted by H_n, is defined by the maximal distance separating an arbitrary point of the fractal and the straight line from first to last point drawn.

3.2 Similarity of the patterns P_n and P_{n-3}

The following proposition is needed for the proof of theorem 3.1.

Proposition 3.1. *If a morphism ϑ preserves the parity of the length, then it preserves the parity of the position.*

Proof.
ϑ preserves the parity of length $:\leftrightarrow |\vartheta(x)| = |x| \mod 2$ for all words x.
We want to show that letters that are in odd (resp. even) positions before the morphism remain in odd (resp. even) positions under the morphism ϑ.
Here are our assumptions :

1. $x = x_1 x_2 \ldots x_n$ is the n-th Fibonacci Word where the x_i are its letters $\forall i \in \{1, \ldots, n\}$
2. ϑ is such that $|\vartheta(x)| = |x| \mod 2$
3. $p_k :=$ position of the first letter of $\vartheta(x_k)$ in $\vartheta(x)$

Let's now prove by induction that $p_k \equiv k \mod 2$

Base case :
$p_1 \equiv 1$ because $p_1 = 1$, it's the first letter of the word.

Inductive step :
Assume that $p_k \equiv k \mod 2$

$$
\begin{aligned}
&p_{k+1} = p_k + |\vartheta(x_k)| \\
&\Rightarrow p_{k+1} \equiv p_k + |\vartheta(x_k)| \mod 2 \\
&\Rightarrow p_{k+1} \equiv p_k + |x_k| \mod 2 \\
&\Rightarrow p_{k+1} \equiv k + |x_k| \mod 2 \\
&\Rightarrow p_{k+1} \equiv k + 1 \mod 2
\end{aligned}
$$

■

We now have the tools to state and prove the following theorem :

Theorem 3.1. *The pattern P_n is similar to the pattern P_{n-3}, $\forall n > 3$*

Proof. The goal is to find a morphism ϑ that will transform the word ω into another word, that guarantees the odd-even drawing alternation required by the odd-even drawing rule. In order to show that, we have to construct ϑ such that :

1. ϑ preserves the parity of position (by proposition 3.1, we just have to show the parity of length)
2. The resulting angle of a pattern must be preserved or inverted by ϑ .

Let's first show point 1.
Let σ be the morphism that generates the infinite Fibonacci Word. ($\mathscr{S}$)
Then, we get [1] :

$$\begin{aligned}\sigma(00) &= 0101\\ \sigma(01) &= 010\\ \sigma(10) &= 001\end{aligned}$$

Hence, σ is not the morphism we are looking for, as $|\sigma(\omega)|$ can be odd for ω even.
Let's examine if σ^2 is the desired morphism.

$$\begin{aligned}\sigma^2(00) &= \sigma(0101) = 010010\\ \sigma^2(01) &= \sigma(010) = 01001\\ \sigma^2(10) &= \sigma(001) = 01010\end{aligned}$$

σ^2 doesn't meet our requirements, as $|\sigma^2(\omega)|$ can be odd for ω even.
Let us determine if σ^3 is the morphism we are looking for.

$$\begin{aligned}\sigma^3(00) &= \sigma(010010) = 0100101001\\ \sigma^3(01) &= \sigma(01001) = 01001010\\ \sigma^3(10) &= \sigma(01010) = 01001001\end{aligned}$$

Let's now verify that this statement also holds for odd numbers :

$$\begin{aligned}\sigma^3(0) &= \sigma(\sigma(\sigma(0))) = 01001\\ \sigma^3(1) &= \sigma(\sigma(\sigma(1))) = 010\end{aligned}$$

By induction and the fact that Fibonacci words are constructed by an iterative concatenation, it follows that for all words ω of odd (resp. even) length, $|\sigma^3(\omega)|$ is always odd (resp. even).

[1] We do not consider the case $\sigma(11)$ as the subword 11 does not exist by proposition 2.2

We successfully proved the first point : the morphism σ^3 preserves the parity of length, and hence the parity of position.

Let's now consider point 2 :
The goal is to prove that the resulting angle of a pattern must be preserved or inverted by σ^3.
Let $a(\omega)$ be the function that gives the resulting angle of a word ω through the odd-even drawing rule.
We have : [2]

$$\begin{aligned} a(0) &= \alpha \\ a(1) &= 0 \\ a(00) &= 0 \\ a(01) &= \alpha \\ a(10) &= -\alpha \end{aligned}$$

Following the OEDR introduced in the definition 3.1, one can also show that :

$$\begin{aligned} a(\sigma^3(0)) &= a(01001) = \alpha \\ a(\sigma^3(1)) &= a(010) = 0 \\ a(\sigma^3(00)) &= a(0100101001) = 0 \\ a(\sigma^3(01)) &= a(0100101001) = -\alpha \\ a(\sigma^3(10)) &= a(0100101001) = \alpha \end{aligned}$$

As before, the general case can be shown using induction.

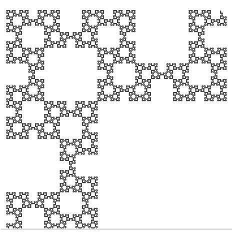
We successfully proved the second point : the resulting angle of a pattern is preserved or inverted by σ^3.

This concludes the proof that P_n is similar to P_{n-3}. ■

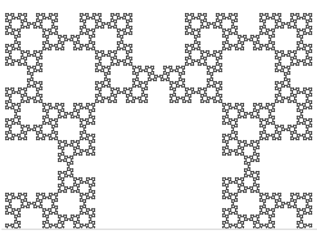
Remark 3.3. One can observe that one can extend this theorem to the $\alpha°$-variant of n-th Fibonacci Word Fractal.

3.3 The 3 different patterns for the Fibonacci Word Fractal

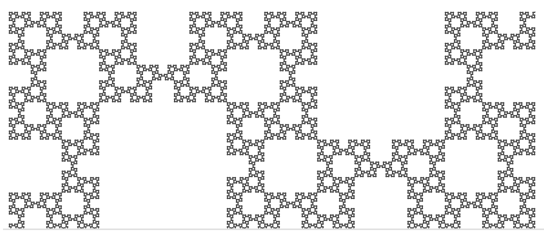
By proposition 3.1, we know that the pattern P_n is similar to the pattern P_{n-3}.
There exists therefore up to three patterns for the Fibonacci Word Fractal : P_{3k-2}, P_{3k-1} and P_{3k}.



P_n for $n = 22$



P_n for $n = 23$



P_n for $n = 24$

[2] One can check these easily with the definition 3.1

3.4 The symmetries of the patterns of the Fibonacci Word Fractal

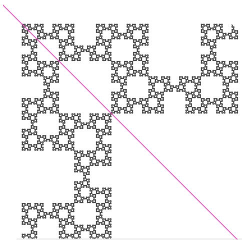
Proposition 3.2.

1. *The pattern P_{3k-2} has a diagonal axis symmetry.*
2. *The pattern P_{3k-1} has a orthogonal symmetry.*
3. *The pattern P_{3k} has a central symmetry.*

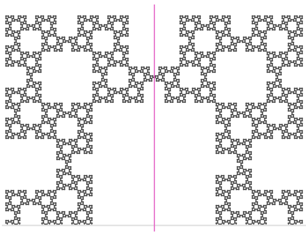
This proposition will not be proven here.

Remark 3.4.
An idea for the proof was to use complex numbers (see section 7 of the miscellaneous section).

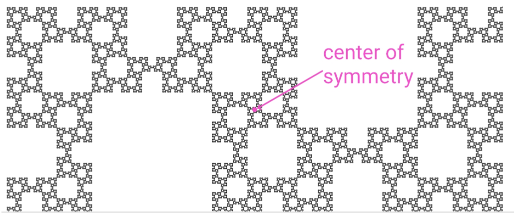
The following three figures illustrate the symmetries of the fractal.



P_n for $n = 22$



P_n for $n = 23$



P_n for $n = 24$

3.5 Some geometrical proportions of the pattern of the FWF

We here state and prove some properties of the Fibonacci Word Fractal that will be used afterwards.

Proposition 3.3. *The pattern P_n is built in the following way: it begins with a pattern P_{n-3}; after a right-angled turn, another P_{n-3} follows, then a single P_{n-6}, another P_{n-3}, and finally, after a right-angled turn in the opposite direction to the first, one last P_{n-3}.*

Proof. By corollary 2.1, we have

$$p_n = p_{n-3}\, ab\, p_{n-3}\, p_{n-6}\, p_{n-3}\, ba\, p_{n-3}.$$

As previously established in proposition 2.1, $(ab) \in \{(01), (10)\}$.
From this observation, it follows that the first two copies of P_{n-3} are orthogonal.
Consequently, the pattern P_n can be expressed as

$$P_n = P_{n-3}\ turn\ 90^\circ\ P_{n-3}\, P_{n-6}\, P_{n-3}\ turn - 90^\circ\, P_{n-3}.$$

■

Remark 3.5. Let's observe that we can extend this proposition to the α°-variant of n-th Fibonacci Word Fractal, for $\alpha \in [0, \frac{\pi}{2}]$.

Proposition 3.4. $L_n = 2L_{n-3} + L_{n-6}\ \forall n > 6$

Proof. We know that : $L_n = L_{n-3} \cdot \cos(90°) + L_{n-3} + L_{n-6} + L_{n-3} + \cos(90°) \cdot L_{n-3}$ by the previous proposition.
The result follows. ■

Remark 3.6. We can again generalize this proposition to the $\alpha°$-variant of n-th Fibonacci Word Fractal. The length would then be $L_n = 2L_{n-3} + 2L_{n-3}\cos(\alpha) + L_{n-6}$.

Theorem 3.2. *The contracting ratio r between P_n and P_{n-3} is $\frac{1}{1+\sqrt{2}}$.*

Proof. From proposition 3.4, we have $L_n = 2L_{n-3} + L_{n-6}$.
We also know that, by definition, the contracting ratio $r = \frac{L_{n-3}}{L_n} = \frac{L_{n-6}}{L_{n-3}}$.
By applying the relations established above and given that r is positive, it is possible to calculate:

$$\Rightarrow r = \frac{L_{n-3}}{L_n}$$
$$\Rightarrow r \cdot L_n = L_{n-3}$$
$$\Rightarrow r \cdot (2 \cdot L_{n-3} + L_{n-6}) = L_{n-3}$$
$$\Rightarrow r \cdot (2 \cdot L_{n-3} + r \cdot L_{n-3}) = L_{n-3}$$
$$\Rightarrow 2r + r^2 = 1$$
$$\Rightarrow r = \sqrt{2} - 1 = \frac{1}{1+\sqrt{2}}$$

■

Theorem 3.3. *The $\alpha°$-variant of the n-th Fibonacci Word Fractal is non-overlapping for $\alpha \in [0, \frac{\pi}{2}]$.*

This theorem will be assumed here.

Proposition 3.5. *The ratio $\frac{length}{height}$ of the P_{3k+2} rectangle is $\sqrt{2}$.*

Proof. Using proposition 3.3, we know that the height of the pattern P_n is equal to the length of P_{n-3} plus the height of P_{n-3}.
Hence, $H_n = L_{n-3} + H_{n-3}$.
Moreover, by theorem 3.2, we know that the contracting ratio between P_n and P_{n-3} is $\frac{1}{1+\sqrt{2}}$
Thus, $H_{n-3} = \frac{H_n}{1+\sqrt{2}}$ and $L_{n-3} = \frac{L_n}{1+\sqrt{2}}$.

$$\Rightarrow H_n = \frac{L_n}{1+\sqrt{2}} + \frac{H_n}{1+\sqrt{2}}$$
$$\Rightarrow (1+\sqrt{2})H_n = L_n + H_n$$
$$\Rightarrow \sqrt{2}H_n = L_n$$
$$\Rightarrow \frac{L_n}{H_n} = \sqrt{2}$$

■

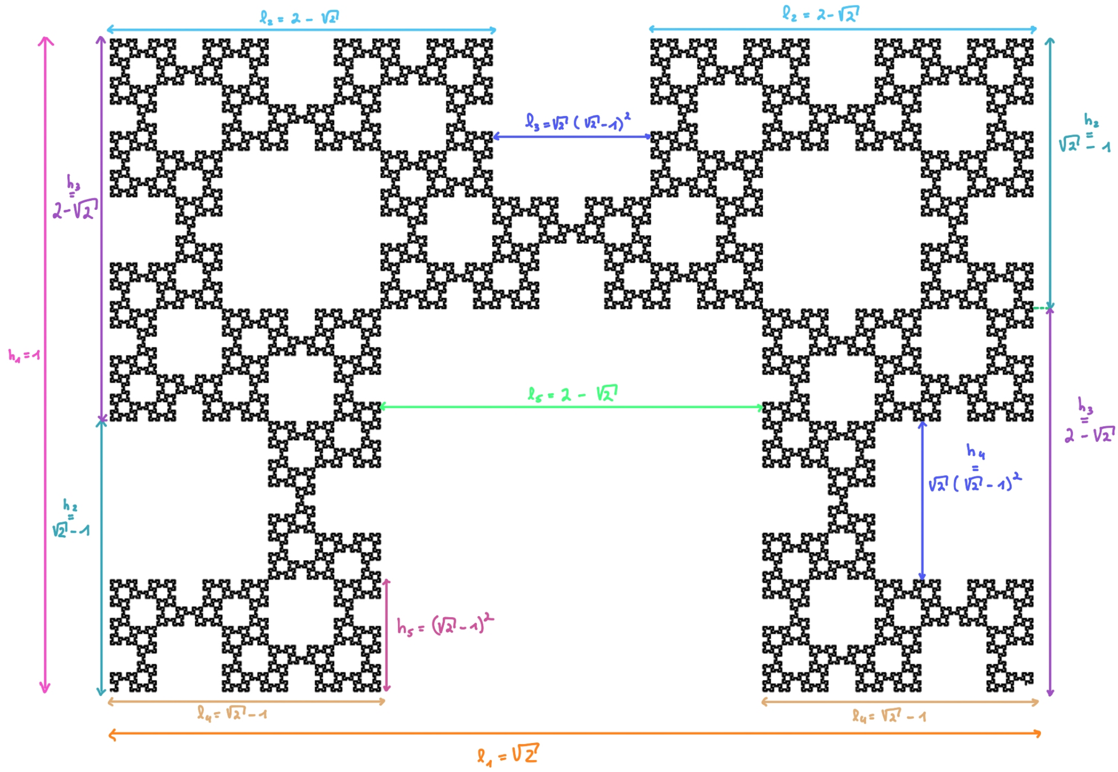


Figure 6: The geometrical proportions P_{3k+2}

On the pattern presented in figure 6, we have normalized the height of the fractal. Proposition 3.5 and theorem 3.2 are used to find the other proportions. The computations made are detailed in the following remark.

Remark 3.7.

- In orange, we have that the length l_1 of the fractal is $\sqrt{2}$, by proposition 3.5.
- The teal distance h_2 is $\sqrt{2}-1$ by the contraction ratio, as it's the height of P_{3k-3}.
- In violet, h_3 is a simple subtraction from the height of the pattern P_{3k} and $\sqrt{2}-1$. The distance is hence $1-(\sqrt{2}-1) = 2-\sqrt{2}$.
- The distance l_2 in cyan is equal to $2-\sqrt{2}$, as it is the length of the pattern P_{3k-3}, which is precisely the violet distance h_3 .
- The distance in blue l_3 is equal to $\sqrt{2}-(2-\sqrt{2})-(2-\sqrt{2}) = \sqrt{2}(\sqrt{2}-1)^2$.
- The distance in brown l_4 is the height of a pattern P_{3k-3}, which is precisely the teal distance h_2, it is hence equal to $\sqrt{2}-1$.
- The distance in green l_5 is the length of the fractal minus twice the brown distance l_4, it's hence equal to $\sqrt{2}-(\sqrt{2}-1)-(\sqrt{2}-1) = 2-\sqrt{2}$.
- The distance in purple h_5 is the height of a pattern P_{3k-6}, which is equal to : $\frac{\sqrt{2}-1}{1+\sqrt{2}} = (\sqrt{2}-1)^2$

4 The Fractal Dimension

In [20], Mandelbroot provides a mathematical answer to the Coastline Paradox, or the question "*How long is the coast of Britain?*". He shows, based on a data-driven work by Lewis Fry Richardson (1881-1983), that as the measuring scale becomes smaller and more precise, the estimated length of the coastline increases rather than converging to a fixed value. As this measure fails, the notion of dimension is introduced. In this section, we introduce and apply to the Fibonacci Word Fractal two dimensions used in Fractal geometry: the box-counting dimension, also known as the Minkowski-Bouligand dimension, and the Hausdorff dimension. We then compute the dimension of the boundary and of the curve[3] of the Fibonacci Word Fractal.

4.1 The box-counting dimension of the curve of the Fibonacci Word Fractal

This section aims to prove the box-counting dimension of the Fibonacci Word Fractal, which is $\frac{\log(\varphi^3)}{\log(1+\sqrt{2})}$.

Definition 4.1 (Box-counting dimension)**.** Let $F \subset \mathbb{R}^d$ a fractal.
For $r > 0$, denote by $N(r)$ the minimal number of closed boxes of side length r required to cover F.
The *box-counting dimension of F* is :

$$\dim_B(F) = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

The following theorem comes from a proof in document [21] (theorem of section 5.1).

Theorem 4.1. *Let $F \subset \mathbb{R}^2$ be a self-similar non empty non overlapping fractal. Then the box-counting dimension of F exists and equals*

$$\dim_B F = \frac{\log M}{\log(1/r')} = \frac{\log M}{-\log r'}.$$

We denote by M the ratio indicating how the number of segments increase after an iteration and by r' the contracting ratio after an iteration.

Next, we provide and prove a proposition required for the following theorem :

Proposition 4.1. $\frac{\log M}{\log(1/r')} = \frac{\log M^3}{\log(1/(r')^3)}$

Proof. Indeed, $\frac{\log M^3}{\log(1/(r')^3)} = \frac{3\log M}{3\log(1/r')} = \frac{\log M}{\log(1/r')}$. ∎

We know from theorem 3.1 that the pattern P_n is similar to the pattern P_{n-3}. Moreover, by theorem 3.2, we know the contracting ratio $r = (r')^3$ between P_{n-3} and P_n. We let $N := M^3$. These theorems together with proposition 4.1 enable us to state the following theorem :

[3] In the following sections and in literature, one calls the box-counting dimension or the Hausdorff dimension of the curve of a fractal the dimension of the fractal itself

Theorem 4.2. *Let F be the Fibonacci Word Fractal.*
$\dim_B(F) = \frac{\log N}{\log(1/r)}$ *for N the ratio indicating how the number of segments increase between P_{n-3} and P_n and for r the contracting ratio between P_{n-3} and P_n.*

Theorem 4.3. $N = \varphi^3$*, where N is defined in the previous theorem.*

Proof. By propositions 2.6 and 2.5, $\frac{|S_n|}{|S_{n-1}|} \to \varphi$:

$$N = \lim_{n\to\infty} \frac{|S_n|}{|S_{n-3}|} = \frac{|S_n|}{|S_{n-1}|} \cdot \frac{|S_{n-1}|}{|S_{n-2}|} \cdot \frac{|S_{n-2}|}{|S_{n-3}|} = \varphi^3$$

■

Theorem 4.4. *The box-counting dimension of the curve of the Fibonacci Word Fractal is* $\dim_B(F_C) = \frac{\log(\varphi^3)}{\log(1+\sqrt{2})}$

Proof. It follows from theorems 4.2, 3.2 and 4.3. ■

4.2 The box-counting dimension of the the $\alpha°$-variant of n-th Fibonacci Word Fractal

This section aims to extend the box-counting dimension of the curve to definition 3.2. The previous subsection stated and proved the box-counting dimension of the fractal with rotational angle 90°.
Here, we will state and prove its value for angles $\alpha°$ as long as the fractal does not overlap itself.

Theorem 4.5. *The box-counting dimension of the curve of the the Fibonacci Word Fractal for a rotational angle $\alpha \in [0, \frac{\pi}{2}]$, such that the fractal does not overlap itself, is :* $\frac{\log(\varphi^3)}{\log(1+\cos(\alpha)+\sqrt{(1+\cos(\alpha))^2+1})}$

Proof. We use theorem 4.2. We observe that the N from the theorem does not change with the change of rotational angle, it is still φ^3.
Let's calculate the r from the formula :
By generalisation of proposition 3.3, we have $L_n = 2L_{n-3} + 2L_{n-3}\cos(\alpha) + L_{n-6}$.
We also know that, by definition, the contracting ratio $r = \frac{L_{n-3}}{L_n} = \frac{L_{n-6}}{L_{n-3}}$. Hence,

$$\begin{aligned}
&\Rightarrow r = \frac{L_{n-3}}{L_n} = \frac{L_{n-6}}{L_{n-3}} \\
&\Rightarrow r \cdot L_n = L_{n-3} \\
&\Rightarrow r \cdot (2 \cdot L_{n-3} + 2L_{n-3}\cos(\alpha) + L_{n-6}) = L_{n-3} \\
&\Rightarrow r \cdot (2 \cdot L_{n-3} + 2L_{n-3}\cos(\alpha) + r \cdot L_{n-3}) = L_{n-3} \\
&\Rightarrow 2r + 2r\cos(\alpha) + r^2 = 1 \\
&\Rightarrow r^2 + 2(1+\cos\alpha)r - 1 = 0 \\
&\Rightarrow r = -1 - \cos\alpha \pm \sqrt{(1+\cos\alpha)^2 + 1} \\
&\Rightarrow r = -1 - \cos\alpha + \sqrt{(1+\cos\alpha)^2 + 1}
\end{aligned}$$

The last implication comes from the fact that r is positive.
This shows that the dimension is
$\dim_B F = \frac{\log N}{\log(1/r)} = \frac{\log(\varphi^3)}{-\log(-1-\cos(\alpha)+\sqrt{(1+\cos(\alpha))^2+1})} = \frac{\log(\varphi^3)}{\log(1+\cos(\alpha)+\sqrt{(1+\cos(\alpha))^2+1})}$,
where the last equality comes from properties of the logarithms. ■

4.3 The Hausdorff dimension

A common dimension used in fractal geometry is the Hausdorff dimension. This section is dedicated to show how thermodynamic formalism and potential theory can be used to compute the Hausdorff dimension of the Fibonacci Word Fractal. The goal is however wider, one with knowledge in dimension theory can note that the box-counting dimension and the Hausdorff dimension coincides, since the Fibonacci Word Fractal is a self-similarity satisfying the Open Set Condition (OSC), as later proved in Section 4.3.2. The following shows the Hausdorff dimension fractal under the previous hypothesises and strictly following thermodynamic formalism logic, as it was presented in the works of the mathematicians Bowen and Ruelle. A rigorous alternative is using potential theory, as in [14]. Nevertheless, although this method formally gives us a lower bound candidate for the Hausdorff dimension without the need of proving the OSC nor assuming that the Fibonacci Word Fractal never intersects itself, it requires complex computational techniques we won't present here. However, it relies on useful statements one can use to formally prove the equivalence between the box-counting dimension and the Hausdorff dimension through the construction of a Gibbs measure satisfying the Mass Distribution Principle (Proposition 4.2). Reminders of measure theory and the definition of the Hausdorff measure and dimension can be found in the Appendix, sections 8.1 and 8.2.

4.3.1 Computing the Hausdorff dimension of the Fibonacci Word Fractal

In the previous sections, we have visualized the Fibonacci Word Fractal using the OEDR rule, providing a deterministic mapping for each F_n we call pre-fractals. As $n \to \infty$, these pre-fractals converges to the Fibonacci Word Fractal one can see as a compact set called an attractor. A widely used technique in the computation of the Hausdorff dimension is introduced in [14] and requires the following affirmation:

Proposition 4.2 (Mass Distribution Principle)**.** *Let $E \subset \mathbb{R}^n$ be a Borel subset.If there is a Borelian probability[a] measure μ supported by E such that for all $x \in \mathbb{R}^n$, $r > 0$,*

$$\mu(B(x,r)) \leq Cr^s$$

for some constant $C > 0$, then one has that $\mathcal{H}^s(E) \geq \frac{1}{C}\mu(E)$. In particular, we have that

$$\dim_{\mathcal{H}}(E) \geq s$$

[a]The proof holds for any measure that satisfies $\mu(E) > 0$. In particular, a probability measure is a measure where $\mu(E) = 1$. An example of possible probability measure is the Gibbs measure.

Proof. Let $\{U_i\}_{i\in I}$ be a cover of E, where $I \subseteq \mathbb{N}$. Let $\{r_i\}_{i\in I}$ be such that for all $i \in I$, $r_i > |U_i|$ where $|\cdot|$ denotes the diameter of U_i. It follows that, for all $x_i \in U_i$:

$$\mu(U_i) \leq \mu(B(x_i, r_i)) \leq Cr_i^s$$

Let $r_i \to |U_i|$ for all $i \in I$. As $\{U_i\}_{i\in I}$ is a cover of E, we get the following inequality:

$$\frac{\mu(E)}{C} \leq \sum_{i\in I} \frac{\mu(U_i)}{C} \leq \sum_{i\in I} |U_i|^s$$

In particular, choosing $\{U_i\}_{i\in I}$ to be any $\delta-$cover of E and taking the infimum, we have that

$$\frac{\mu(E)}{C} \leq \mathcal{H}^s(E)$$

Moreover, by the definition of the Hausdorff dimension, since μ is a probability measure and C is a non-zero constant, we have that $0 < \mathcal{H}^s(E)$, meaning that $\dim_{\mathcal{H}}(E) \geq s$. ■

[3] provides consequences (Frostman's Lemma) and generalizations (Billingsley's Lemma) of the Mass Distribution Theorem, directly related to a formal computation of the Hausdorff dimension of such set E. However, we follow here a physic approach using potential theory as in [14]. Note that computing the Hausdorff dimension strictly using the Mass Distribution Principle requires checking every possible ball of the Fibonacci Word Fractal (attractor). Falconer proves that the local condition of this principle is satisfied if the following integral converges

$$I_s(\mu) = \iint \frac{d\mu(x)d\mu(y)}{|x-y|^s} = \int \Phi_s(x)d\mu(x)$$

where I_s is called the $s-$energy and Φ_s the $s-$potential at a point $x \in \mathbb{R}^n$ due to the mass distribution μ on E and convergence therefore implies that $\dim_{\mathcal{H}}(E) \geq s$. Although we don't delve in the theory of dynamical systems, we can consider the construction of the Fibonacci Word Fractal as an iterative process where the fractal is its limit. Constructing the pre-fractals is therefore a state in our system and the attractor is its final stable shape. Hence, as the fractal we get is a stable object, we associate it a *mass*, or a Borel probability measure for instance, as our set is well-defined on Borel subsets. The self-similarity property of the Fibonacci Word Fractal therefore ensures that the local mass $\mu(B(x,r))$ behaves like r^s. Intuitively, assigning all the points a "pressure", we can move from a local condition as described in Proposition 4.2 to a global one, where $\Phi(\mu)$ at a point x measures the *total* repulsive force that all $y \in \mathbb{R}^n \supset E$ exert on the fractal and I_s is the potential energy.

The following theorem will be very important for later computations:

Theorem 4.6 (Perron-Frobenius Theorem for primitive matrices [30])**.** *Let A be a non-negative primitive matrix, i.e. there exists $k > 0$ such that $A^k_{ij} > 0$ for all i, j. Then, there exists a positive eigenvalue $\rho(A)$, the spectral radius, such that for any other eigenvalue λ, $\rho(A) > |\lambda|$. Moreover, $\rho(A)$ has algebraic and geometric multiplicity 1, i.e. there exists a unique eigenvector $v > 0$ associated with $\rho(A)$.*

To the Fibonacci Word defined with the substitution rule $\mathscr{S}$ we can assign the following incidence matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\sigma(0)$ produces the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\sigma(1)$ the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The incidence matrix of a substitution σ is indeed a square matrix $M = m_{ij}$, where m_{ij} is the number of occurrence

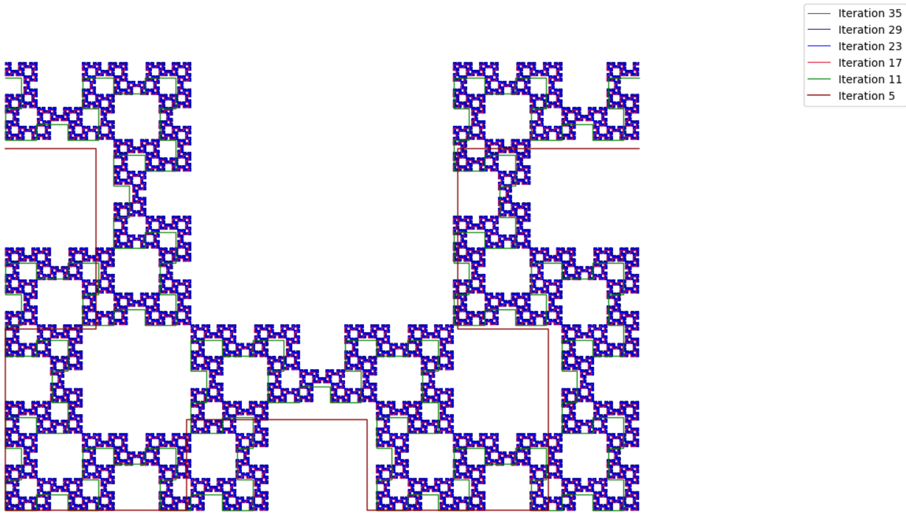


Figure 7: Convergence of the Fibonacci Word Fractal Attractor

of the letter i in $\sigma(j)$. As M is primitive (take $k = 2$), the theorem of Perron-Frobenius ensures the existence of a biggest eigenvalue, in our case the Golden ratio φ, corresponding to the spectral radius and telling how fast the number of segments of the Fibonacci Word Fractal is multiplying as $n \to \infty$. Moreover, inspired by the work of defining Fibonacci Snowflakes in the complex plane (see Section 8.3) let us represent the construction of the Fibonacci Word Fractal in $\mathbb{C}$. One can find the following relation

$$z_n = z_{n-1} + i^{|S_{n-1}|} z_{n-2}$$

where $z_n \in \mathbb{C}$ represents the total displacement vector of the $n-$th iteration of the Fibonacci Word Fractal F_n and i a 90° rotation. This construction emphasis that the geometric orientation of the pre-fractals of the Fibonacci Word resets every three iterations as $\pi(2) = 3$ [16], where $\pi(2)$ is called the second Pisano period, which is the period with which $(A_i)_{i\in\mathbb{N}}$ taken modulo 2 repeats, as shown in Proposition 3.1. This means that the average contracting ratio r per single substitution is indeed on average $\varrho = \left(\frac{1}{1+\sqrt{2}}\right)^{\frac{1}{3}}$, using Theorem 3.2.

Proposition 4.3 (Ruelle Operator/ Transfer Operator of the Fibonacci Word Fractal)**.** *Let* $(\{0,1\}^*, \sigma^{-1})$ *be a dynamical system, where* σ *is the substitution rule* $\mathscr{S}$*. Let* $\phi : \{0,1\}^* \to \mathbb{R} : y \mapsto -s\ln(\frac{1}{\varrho}) = s\ln(\varrho)$ *be the potential. The Ruelle operator of the Fibonacci Word Fractal is equivalent to*

$$\mathcal{L}_\phi = r^s \cdot M = M(s) = \begin{pmatrix} r^s & r^s \\ r^s & 0 \end{pmatrix} \tag{2}$$

where M *is this incidence matrix associated to the substitution rule* σ *and* $\varrho = \left(\frac{1}{1+\sqrt{2}}\right)^{\frac{1}{3}}$*.*

Remark 4.1. A complex proof using Gibbs state can show formally the link between the $s-$potential at x $\Phi_s(x) = \int |x-y|^{-s} d\mu(y)$ and $\phi = s\ln(\varrho)$. One can find further details in [26] Intuitively, one can see that the distance between two sequences $x, y \in \Sigma^* = \{0,1\}^*$

will be smaller if they share a greater common prefix of length k. Therefore we get $|x-y| \approx \varrho^k$. For instance, if x and y differ completely, then $k=0 \implies |x-y|=1$ which represents the whole fractal in a probabilistic view!

Proof. For any continuous function f in $\{0,1\}^*$ the Ruelle Operator is defined by $\mathcal{L}_\phi(f)(x) = \sum_{y\in\sigma(x)} e^{\phi(y)} f(y)$, see [32]. Since $\phi(y) = s\ln\varrho$, we have that:

$$\begin{aligned}\sum_{y\in\sigma(x)} e^{\phi(y)} f(y) &= \varrho^s \sum_{y\in\sigma(x)} f(y) \\ &= \varrho^s M f \end{aligned} \tag{3}$$

Note that (3) follows from the definition of the substitution σ. Indeed $\sum_{y\in\sigma(x)} f(y)$ counts how many times each symbol appears in $\sigma(x)$, when representing f as the vector

$$f = \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}$$

and therefore

$$\mathcal{L}_\phi(f)(x) = \varrho^s \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} \implies \mathcal{L}_\phi = \varrho^s M \tag{4}$$

■

Remark 4.2. It is direct that $M(s)$ is a primitive linear operator. Therefore, the Perron-Frobenius Theorem ensures that $\rho(M(s))$ exists and is well-defined and intuitively characterizes the $s-$mass of the Fibonacci Word Fractal.

Proposition 4.4 (Topological Pressure). *The topological pressure of the Fibonacci Word Fractal is given by*

$$P(s) = \log(\rho(M(s)))$$

Remark 4.3. As originally defined in [9] by Bowel, the topological pressure of such substitution systems can be derived from $P(s) = \lim_{n\to\infty} \frac{1}{n}\ln Z_n(s)$ where Z_n is a partition function. Moreover, assuming Z_n is here equivalent to $\rho(M(s)^n)$, we can easily link the fact that $\lim_{n\to\infty}\frac{1}{n}\log\rho(M(s)^n) = \log\rho(M(s))$ with the result shown in [10] and saying that, under the hypothesis of being in a IFSm, if there exists a positive number $\rho = \rho(\mathcal{L}_\phi)$, the spectral radius, and a positive function $h : \{0,1\}^* \to \mathbb{R}$ such that $\mathcal{L}_\phi(h) = \rho h$, then the limit $\lim_{n\to\infty}\frac{1}{n}\log\mathcal{L}_\phi(1)(x) = \log\rho$ exists and is therefore the topological pressure.

However, let us assume the following theorem, proved in [10], which applies directly to the case of the Fibonacci Word Fractal.

Theorem 4.7. *In a Iterated Function System with a measure (IFSm) where the potential ϕ is positive and continuous, if there exists a positive number $\rho = \rho(\mathcal{L}_\phi)$, the spectral radius, and a positive function $h : \{0,1\}^* \to \mathbb{R}$ such that $\mathcal{L}_\phi(h) = \rho h$, then the topological pressure is given by $P(s) = \log\rho(\mathcal{L}_\phi)$*

Proof of Proposition 4.4. By Proposition 4.3, we have that $\mathcal{L}_\phi = M(s)$ and is primitive, ensuring by Perron Frobenius theorem the existence of $\rho(M(s)) > 0$ and of a unique associated eigenvector $h > 0$. It follows that $P(s) = \log(\rho(\mathcal{L}_\phi)) = \log(\rho(M(s)))$. ■

The following theorem is the key to compute the Hausdorff dimension, proved by Bowen in [8] for quasi-circles and hyperbolic systems.

Theorem 4.8 (Bowen's Equation)**.** *The Hausdorff dimension* $\dim_{\mathcal{H}}(E)$ *is the unique solution to* $P(s) = 0$. *That is, in the case of the Fibonacci Word Fractal, the Hausdorff dimension can be found computing*

$$\rho(M(s)) = 1$$

Therefore, the Hausdorff dimension of the Fibonacci Word Fractal is equal to $\frac{3\ln(\varphi)}{1+\sqrt{2}}$

Proof.

$$\begin{aligned}\rho(M(s)) = 1 &\iff \rho(\varrho^s M) = 1\\ &\iff \varrho^s \rho(M) = 1\\ &\iff \left(\frac{1}{1+\sqrt{2}}\right)^{\frac{s}{3}} \varphi = 1\\ &\iff \ln(\varphi) - \frac{s}{3}\ln(1+\sqrt{2}) = 0 \iff \ln(\varphi) = \frac{s}{3}\ln(1+\sqrt{2})\end{aligned}$$

Therefore, we have that

$$s = \dim_{\mathcal{H}}(F_C) = \frac{3\ln(\varphi)}{\ln(1+\sqrt{2})}$$

Which precisely corresponds to the result found in Theorem 4.4. ■

4.3.2 On the equivalence of the box-counting dimension and the Hausdorff dimension

Definition 4.2 (Open Set Condition [14])**.** Let $S_i(F)$ be the components (*the pre-fractals*) of an attractor F (*the Fibonacci Word Fractal*)[a]. We say that the S_i satisfy the open set condition if there exists $V \neq \emptyset$, open and bounded, such that:

$$\bigsqcup_{i=1}^{m} S_i(V) \subset V$$

where $\bigsqcup$ is the disjoint union operator.

[a] This notation comes directly from [14] and may not be confused with the notation S_n for the $n-$th Fibonacci Word.

Kenneth proves that, under the Open Set Condition, **the box-counting dimension and the Hausdorff dimension are equivalent**.

Theorem 4.9. *The Fibonacci Word Fractal* F *verifies the Open Set Condition for all* F_n.

Proof. In the article of the American Mathematical Society *Separation Properties for Self-Similar Sets*, Andreas Schief [27] shows the following theorem:

Theorem 4.10. *Let K be a self-similar set such that $K = \bigcup_{i=1}^{m} S_i(K)$. The following are equivalent:*

1. *The components of K verify the Open Set Condition*
2. *$H^\alpha(K) > 0$*

where α is the unique solution satisfying $\sum_{i=1}^{m} r_i^\alpha = 1$ and r_i is the ratio of the similarity S_i for each $i \in \{1, \dots, m\}$.

We have already shown the the Fibonacci Word Fractal is a self-similar set (Proposition 3.1). One can see that all the possible patterns the Fibonacci Word Fractal can be written as a (disjoint) union of 5 self-similarities. As in Figure 6, taking 1 as the height of the Fibonacci Word Fractal, one can compute that four self-similarities have ratio $r_{\{1,2,3,4\}} = \sqrt{2} - 1$ and one $r_5 = (\sqrt{2} - 1)^2$. We therefore have that:

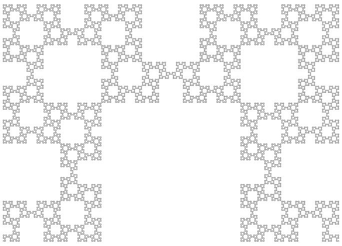
$$\sum_{i=1}^{5} r_i^\alpha = 4(\sqrt{2} - 1)^\alpha + (\sqrt{2} - 1)^{2\alpha} = 1$$
$$\iff 4(\sqrt{2} - 1)^\alpha + (\sqrt{2} - 1)^{2\alpha} - 1 = 0$$

Let $X = (\sqrt{2} - 1)^\alpha$. We solve $X^2 + 4X - 1 = 0$. The positive solution is $X = -2 + \sqrt{5}$. Substitution back, we get that $(\sqrt{2} - 1)^\alpha = -2 + \sqrt{5} \iff \alpha = \log_{(\sqrt{2}-1)}(-2 + \sqrt{5}) = \frac{\log(-2+\sqrt{5})}{\log(\sqrt{2}-1)} \approx 1,63\dots$, which is precisely the Hausdorff dimension of the Fibonacci Word Fractal computed in Theorem 4.8. We thereofore have that $\dim_{\mathcal{H}}(F_C) = \frac{\log(-2+\sqrt{5})}{\log(\sqrt{2}-1)} = \alpha \implies H^\alpha > 0$. In conclusion, all the similarities of the Fibonacci Word Fractal satisfy the Open Set Condition. This implies and checks that $\dim_{\mathcal{H}}(F_C) = \dim_B(F_C)$.

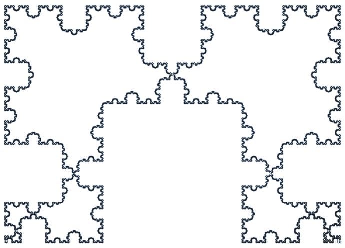
■

4.4 The fractal dimension of the boundary of the Fibonacci Word Fractal

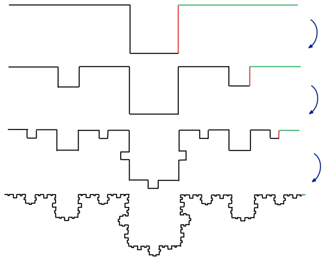
The dimension of the curve of the Fibonacci Word Fractal is not the same as that of its boundary. One can define the boundary of the fractal as a self-similar set itself. Assuming the Open Set Condition, we compute in this section the dimension of the boundary of the Fibonacci Word Fractal using only the box-counting dimension.



Curve of the 23^{rd} FWF



Boundary of the 23^{rd} FWF



Iterative steps to construct the boundary of the FWF

At each iteration, there are segments of two different lengths. This leads us to the following definition :

Definition 4.3 (long segment)**.** We define *long segments* as the longest segments at an iteration.
We denote by $l(n)$ the number of long segments at the n-th iteration.

Definition 4.4 (small segment)**.** We define *small segments* as the smallest segments at an iteration.
We denote by $s(n)$ the number of small segments at the n-th iteration.

The boundary of the Fibonacci Word Fractal is constructed iteratively from an initial segment of length k, by the following method :

1. Transform the long segment of length k into 2 segments of length $\frac{k}{1+\sqrt{2}}$ on the sides, and 3 segments placed in the center, with right angles, of length $\frac{k}{(1+\sqrt{2})^2}$
2. Iterate this rule for each long segment.

Remark 4.4. By construction, at each iteration, the long segments will generate 2 long segments and 3 short segments, while the short segments will become 3 long segments.

Theorem 4.11. *The box-counting dimension of the boundary of the Fibonacci Word Fractal is* $\dim_B(F_B) = \frac{\log(3)}{\log(1+\sqrt{2})}$

Proof. Let's use theorem 4.1 as the boundary of the fractal is not intersecting itself.
We will more precisely use the formula $\dim_B F = \frac{\log M}{-\log r'}$, where r' denotes the contracting ratio after 1 iteration and M the ratio indicating how the number of segments increase after an iteration.

We already know that the contracting ratio $r' = 1 + \sqrt{2}$, by the construction described above.

Let's now find M :
By remark 4.4, we get that :

- $l(n+1) = 2l(n) + s(n)$
- $s(n+1) = 3l(n)$

Hence, $l(n) = 2l(n-1) + s(n-1) = 2l(n-1) + 3l(n-2)$
The initial conditions are l(0)=0 and l(1)=1.
This is a linear induction of order 2.
Its characteristic equation is $\lambda^2 = 2\lambda + 3$.
Its roots are $\lambda_1 = 3$ and $\lambda_2 = -1$.
We therefore have that $l(n) = \alpha \cdot 3^n + \beta \cdot (-1)^n, \text{ for } \alpha, \beta \in \mathbb{R}$.
By the initial conditions, we get that :

$$l(n) = \frac{1}{4}(3^n - (-1)^n)$$

If n is even, we have $l(n) = \frac{3^n-1}{4} = \lfloor \frac{3^n}{4} + \frac{1}{2} \rfloor$.
If n is odd, we have $l(n) = \frac{3^n+1}{4} = \lfloor \frac{3^n}{4} + \frac{1}{2} \rfloor$.

Hence,

$$l(n) = \left\lfloor \frac{3^n}{4} + \frac{1}{2} \right\rfloor$$

$$s(n) = 3 \cdot \left\lfloor \frac{3^{n-1}}{4} + \frac{1}{2} \right\rfloor$$

Remark 4.5. The long and small segments don't have the same length.
It is easily seen that the length of the long segments are $1 + \sqrt{2}$ longer then the small segments.
We then define t(n) the total number of segments, of the same length as the small segments, at the n-th iteration.
We get that $t(n) = l(n) \cdot (1 + \sqrt{2}) + s(n)$.
Now,

$$\begin{aligned}
M &= \lim_{n \to +\infty} \frac{t(n+1)}{t(n)} \\
&= \lim_{n \to +\infty} \frac{\left\lfloor \frac{3^{n+1}}{4} + \frac{1}{2} \right\rfloor \cdot (1 + \sqrt{2}) + 3 \cdot \lfloor \frac{3^n}{4} + \frac{1}{2} \rfloor}{\lfloor \frac{3^n}{4} + \frac{1}{2} \rfloor \cdot (1 + \sqrt{2}) + 3 \cdot \lfloor \frac{3^{n-1}}{4} + \frac{1}{2} \rfloor} \\
&= \lim_{n \to +\infty} \frac{\left\lfloor \frac{3^{n+1}}{4} \right\rfloor \cdot (1 + \sqrt{2}) + 3 \cdot \lfloor \frac{3^n}{4} \rfloor}{\lfloor \frac{3^n}{4} \rfloor \cdot (1 + \sqrt{2}) + 3 \cdot \lfloor \frac{3^{n-1}}{4} \rfloor} \\
&= \frac{3 \cdot (1 + \sqrt{2}) + 3}{1 + \sqrt{2} + 3 \cdot \frac{1}{3}} \\
&= \frac{6 + 3\sqrt{2}}{2 + \sqrt{2}} \\
&= 3
\end{aligned}$$

Hence, $\dim_B(F_B) = \frac{\log(3)}{\log(1+\sqrt{2})}$ ■

5 The Fibonacci Snowflake

In order to understand the definitions and theorems stated below, please refer to the section 8.3, in the Appendix, where the terminology is explained and some facts used in the proofs of this section are proven.

5.1 Constructions towards the definition of the Fibonacci snowflake

Definition 5.1. We define a sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathcal{T}^*$ by

$$q_0 = \varepsilon, \qquad q_1 = R,$$

and for every $n \geq 2$,

$$q_n = \begin{cases} q_{n-1}\, q_{n-2}, & \text{if } n \equiv 2 \pmod 3, \\ q_{n-1}\, \overline{q_{n-2}}, & \text{if } n \equiv 0, 1 \pmod 3). \end{cases}$$

Remark 5.1. The first terms of the sequence $(q_n)_{n\in\mathbb{N}}$ are :

- $q_0 = \varepsilon$,
- $q_1 = R$,
- $q_2 = R$,
- $q_3 = RL$,
- $q_4 = RLL$,
- $q_5 = RLLRL$,
- $q_6 = RLLRLLRR$,
- $q_7 = RLLRLLRRLRRLR$,
- $q_8 = RLLRLLRRLRRLRRLLRLLRR$.

Proposition 5.1. *The length of the n-th term $|q_n|$ is the Fibonacci number A_n.*

Proof. The proof is analogous to the one of proposition 2.5.
We also prove this by induction. The base cases for $n = 0$ and $n = 1$ are done, as shown above. Let's now assume that $|q_{n-1}| = A_{n-1}$ and $|q_{n-2}| = A_{n-2}$. We have $|q_n| = A_n$ as $|q_{n-2}| = |\overline{q_{n-2}}|$. ■

Proposition 5.2. *Let $n \in \mathbb{N}$. There exists an antipalindrome t, two palindroms p, r and a letter $a \in \{L, R\}$ such that:*

- $q_{3n+1} = ta$
- $q_{3n+2} = pa$
- $q_{3n+3} = r\overline{a}$

Proof. Let's prove the three points by induction.

- <u>Base case : $n = 0$</u>
  $q_1 = \varepsilon R$
  <u>Inductive step :</u>

Assume that $q_{3n+1} = ta$.
Then :

$$\begin{aligned} q_{3n+4} &= q_{3n+3}\overline{q_{3n+2}} \\ &= q_{3n+2}\overline{q_{3n+1}}q_{3n+2} \\ &= pa\overline{ta}\overline{pa} \\ &= \underbrace{pa\bar{t}\overline{ap}}_{\text{antipalindrome}} \underbrace{\bar{a}}_{\text{letter}} \end{aligned}$$

- Base case : $n = 0$
  $q_2 = \varepsilon R$
  Inductive step :
  Assume that $q_{3n+2} = pa$.
  Then :

$$\begin{aligned} q_{3n+5} &= q_{3n+4}q_{3n+3} \\ &= q_{3n+3}\overline{q_{3n+2}}q_{3n+3} \\ &= r\overline{apa}r\bar{a} \\ &= \underbrace{r\overline{apa}r}_{\text{palindrome}} \underbrace{\bar{a}}_{\text{letter}} \end{aligned}$$

- Base case : $n = 0$
  $q_3 = RL$
  Inductive step :
  Assume that $q_{3n+3} = r\bar{a}$.
  Then :

$$\begin{aligned} q_{3n+6} &= q_{3n+5}\overline{q_{3n+4}} \\ &= q_{3n+4}q_{3n+3}\overline{q_{3n+4}} \\ &= pa\overline{tapa}r\overline{apat}ap\ a \\ &= \underbrace{pa\overline{tapa}r\overline{apat}ap}_{\text{palindrome}} \underbrace{a}_{\text{letter}} \end{aligned}$$

■

Proposition 5.2 tells us that the winding number can only take three values.
These values are specified in the next proposition.

Proposition 5.3. *Let $n \in \mathbb{N}$. Then, we have :*

- $\Delta(q_{3n}) = 0$
- $\Delta(q_{3n+1}) = \Delta(q_{3n+2}) = (-1)^{n+1}$

Proof.

- We have $q_{3n+3} = q_{3n+2}\overline{q_{3n+1}} = q_{3n+1}q_{3n}\overline{q_{3n+1}}$.

  By additivity of the winding number, we get that :

$\Delta(q_{3n+3}) = \Delta(q_{3n+1}) + \Delta(q_{3n}) + \Delta(\overline{q_{3n+1}})$.

By the fourth property of proposition 8.2, we get that $\Delta(q_{3n+3}) = \Delta(q_{3n})$.
Moreover, $\Delta(q_3) = 0$ as $q_3 = RL$.

Hence, by induction, $\Delta(q_{3n}) = 0 \ \forall n \in \mathbb{N}$.

- We have $q_{3n+2} = q_{3n+1}q_{3n}$.

  By additivity of the winding number, we get that :
  $\Delta(q_{3n+2}) = \Delta(q_{3n+1}) + \Delta(q_{3n})$.
  Hence, by the previous point, $\Delta(q_{3n+2}) = \Delta(q_{3n+1})$.

  Moreover, by proposition 5.2, we know that $q_{3n+1} = ta$,
  for t a antipalindrome and a a letter in $\{R, L\}$.

  By propositions 2.7 and 5.1, we know that $|q_{3n+1}|$ is odd.
  This implies that $|t|$ is even.

  Furthermore, we know that $\Delta(t) = 0$, as t is an antipalindrome of even length.
  Hence, $\Delta(q_{3n+1}) = \Delta(t) + \Delta(a) = \Delta(a) = \pm 1$.

  Let's now show by induction that $\Delta(q_{3n+1}) = (-1)^{n+1}$.
  Base case : $n = 0$
  $\Delta(q_1) = -1$

  Inductive step :
  Assume that $\Delta(q_{3n+1}) = (-1)^{n+1}$.
  Recall that $q_{3n+4} = q_{3n+2}\overline{q_{3n+1}}\ \overline{q_{3n+2}}$ Then :

$$\begin{aligned}\Delta(q_{3n+4}) &= \Delta(q_{3n+2}) + \Delta(\overline{q_{3n+1}}) \ + \Delta(\overline{q_{3n+2}}) \\ &= \Delta(\overline{q_{3n+1}}) \\ &= -\Delta(q_{3n+1}) \\ &= (-1)^{n+2}\end{aligned}$$

  This shows that $\Delta(q_{3n+1}) = \Delta(q_{3n+2}) = (-1)^{n+1}$.

■

Definition 5.2. We define Q_n to be the smallest rectangle box containing q_n^-.
The sides of Q_n are determined by the origin and the endpoint of q_n^-.

Remark 5.2. Each Q_n is contained in Q_{n+1} as q_n^- is a prefix of q_{n+1}^-

Theorem 5.1. *For all $n \in \mathbb{N}$, the paths q_n satisfy the following properties:*

1. *The path $(q_{3n+1})^3\, q_{3n+1}^-$ is closed.*
2. *The path $(q_{3n+2})^3\, q_{3n+2}^-$ is closed.*
3. *The path q_n is non-intersecting.*
4. *The path $(q_{3n+1})^3\, q_{3n+1}^-$ is non-intersecting.*

Proof.

1. The goal is to show that $\alpha q_{3n+1} = \pm i\alpha$.
This will imply that $(q_{3n+1})^3 q_{3n+1}^-$ is closed by theorem 8.3.
By proposition 5.2, we know that $q_{3n+1} = ta$, for t an even antipalindrome and $a \in \{R, L\}$.
By theorem 8.2, we know that $\alpha w = i^{\Delta(w)}\alpha$.
As already stated in the proof of proposition 5.3, we have $\Delta(ta) = \Delta(a) = \pm 1$ as t is an even antipalindrome.

   Using the recalled properties above, we have : $\alpha q^{3n+1} = i^{\pm 1}\alpha = \pm i\alpha$

2. It follows from the first point together with the fact that $\Delta(q_{3n+1}) = \Delta(q_{3n+2})$ by proposition 5.3.

3. By construction, each q_n is a prefix of q_{n+1}.
Hence, if q_n has an intersection, then q_{n+1} also has an intersection.
It therefore suffices to show that q_{3n} is non-intersecting $\forall n \in \mathbb{N}$.

   As q_{3n} is intersecting itself iff q_{3n}^- is, we will work on q_{3n}^-, to be able to afterwards work with Q_n.

   Let's first of all prove that $q_{3n}^- = q_{3n-2} q_{3n-1}^-$.
<u>Base case : n=1</u>
We have $q_3^- = R,\ q_1 = R,\ q_2^- = \varepsilon$.
Hence, $q_3^- = q_1 q_2^-$

   <u>Inductive step:</u> Assume that $q_{3n}^- = q_{3n-2} q_{3n-1}^-$.
Then :

$$\begin{aligned}
q_{3n+3}^- &= q_{3n+2} \cdot \overline{q_{3n+1}^-} \\
&= q_{3n+1} q_{3n} \cdot \overline{q_{3n}} q_{3n-1}^- \\
&= q_{3n+1} q_{3n} \cdot \overline{q_{3n-1}} q_{3n-2} \cdot q_{3n-1}^- \\
&= q_{3n+1} \cdot q_{3n} \overline{q_{3n-1}} \cdot q_{3n-2} q_{3n-1}^- \\
&= q_{3n+1} \cdot q_{3n+1} \cdot q_{3n}^- \\
&= q_{3n+1} \cdot q_{3n+2}^-
\end{aligned}$$

   This ends the inductive proof that $q_{3n}^- = q_{3n-2} q_{3n-1}^-$.
The statement $q_{3n}^- = q_{3n-2} q_{3n-1}^-$ implies that $Q_{3n} = Q_{3n-2} Q_{3n-1}$.
Let's now assume that q_{3n-1}^- is non-intersecting.
Then, q_{3n-k}^- is non-intersecting $\forall k \in \mathbb{N}$ such that $1 \leq k \leq 3n - 1$.

$\Rightarrow$ there are no intersections in the rectangle Q_{3n-1} nor in the rectangle Q_{3n-2}.
$\Rightarrow$ there is no intersection in Q_{3n}.
$\Rightarrow$ q^{-}_{3n} is non-intersecting.
$\Rightarrow$ q_{3n} is non-intersecting.
$\Rightarrow$ q_n is non-intersecting.

4. By theorem 8.3, we know that the word $w^3 w^-$ defines a closed path if $\alpha w = \pm i\, \alpha$. It suffices then to show that $(q_{3n+1})^2$ is non-intersecting.

$$\begin{aligned}(q_{3n+1})^2 &= q_{3n}\overline{q_{3n-1}} \cdot q_{3n}\overline{q_{3n-1}} \\ &= q_{3n} \cdot \overline{q_{3n-1}} q_{3n-1} \cdot \overline{q_{3n-2}}\ \overline{q_{3n-1}}\end{aligned}$$

We know by the third point that none of these factors are intersecting.

This implies that the path $(q_{3n+1})^3\, q^{-}_{3n+1}$ is non-intersecting.

■

The first and fourth points of theorem 5.1 make sense of the following definition :

Definition 5.3. A Fibonacci Snowflake of order n is a polynomino $T_F(n)$ represented by the word $(q_{3n+1})^3\, q^{-}_{3n+1}$, where $n \in \mathbb{N}$.

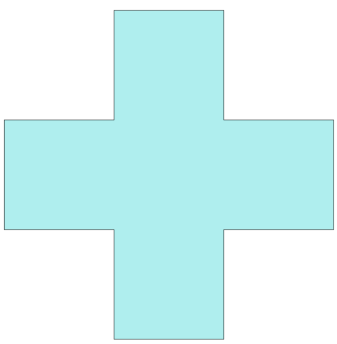
5.2 Generating the Fibonacci Snowflake

Given Definitions 5.1 and 5.3, we generate the Fibonacci Snowflakes using `Turtle` Library in Python.

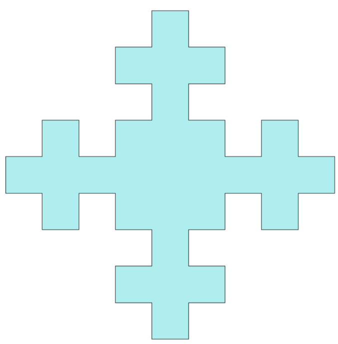
```
import turtle

def q_sequence(n):

    q = ["", "R"]  # q0 = \epsilon, q1 = R

    for k in range(2, n+1):
        if k % 3 == 2:
            q.append(q[k-1] + q[k-2])
        else:
            q.append(q[k-1] + invert(q[k-2]))

    return q[n]

def draw_fibonacci_flake(word, step=5, angle=90):

    turtle.tracer(0, 0) #To directly output the Snowflake.
    t = turtle.Turtle()
    t.speed(0)
    t.penup()
    t.goto(-200, 0)
    t.pendown()
```

```
    #The program recognizes that the drawn path is indeed closed as the turtle gets back to its initial position. This allows filling the Snowflake with a colour:
    t.color("black", "paleturquoise")
    t.begin_fill()

    for c in word:
        if c == 'R':
            t.right(angle)
        elif c == 'L':
            t.left(angle)
        t.forward(step)


    t.end_fill()
    t.hideturtle()
    turtle.update()
    turtle.done()


n = int(input("Enter a natural number less or equal than 7: "))
#From Theorems 4.2 and 4.3 and as Proposition 5.1 links the sequence q_n to A_n, we have that this program has a complexity of \Theta(\phi^{3n}), explaining why we limit the input at n = 7.
k = 3*n + 1 #By Definition 5.3
q = q_sequence(k)
flake_word = q + q + q + q
draw_fibonacci_flake(flake_word, step=4, angle=90)
```

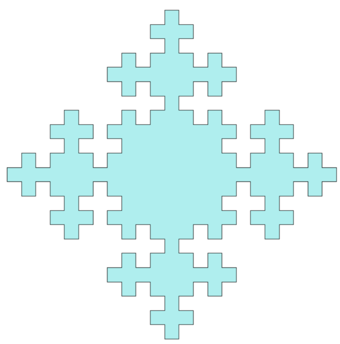
Remark 5.3. The following figures illustrate the Fibonacci Snowflakes corresponding to the five first orders.



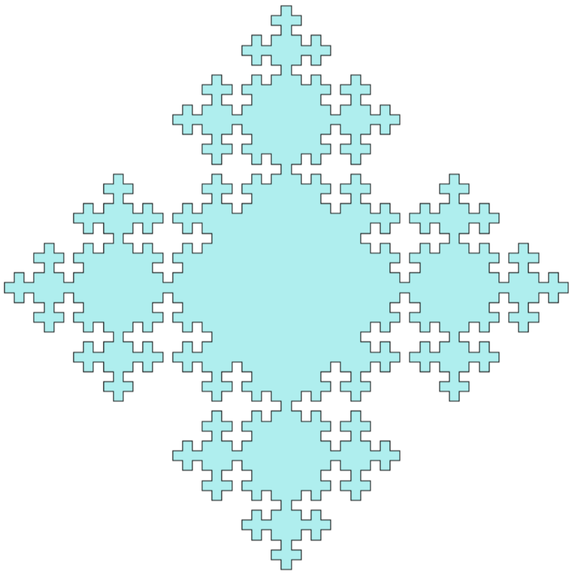
$T_F(1)$



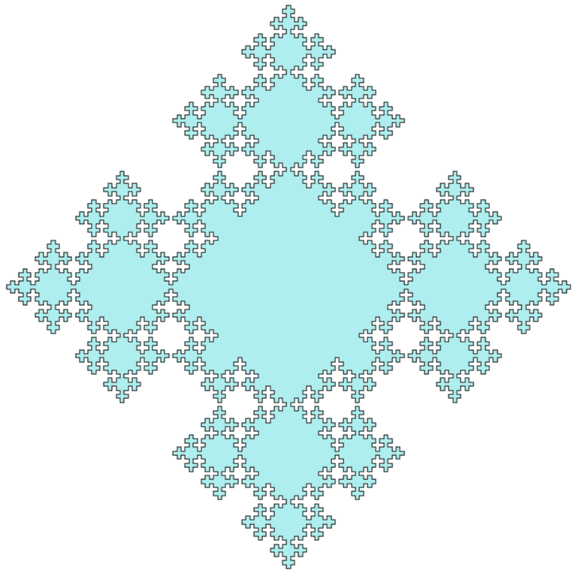
$T_F(2)$



$T_F(3)$



$T_F(4)$



$T_F(5)$

Remark 5.4. As for the n-th Fibonacci Fractal, see Remark 3.1, there exists a more elaborate version of the code in `The Fibonacci Word Fractal - Construction and Geometric Properties - Final > Codes`. There, we provide a solution to optimize the size of a step in a given window, using the same technique as described in Remark 3.1.

5.3 Perimeter of the Fibonacci Snowflake

Theorem 5.2. *The perimeter of a Fibonacci Snowflake of order n is*
$L(n) = 4A_{3n+1} = \frac{4}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{3n+1} - \frac{4}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{3n+1}$.

Proof. By definition, the Fibonacci Snowflake of order n is represented by the word : $(q_{3n+1})^3\, q^{-}_{3n+1}$.
The perimeter is then four times the length of q_{3n+1}.
Moreover, $|q_{3n+1}| = A_{3n+1}$, by proposition 5.1.
This shows that : $L(n) = 4A_{3n+1}$.
The second equality comes from the Binet formula, which will not be proven in this document. One can find the proof in [15]. ■

5.4 Area of the Fibonacci Snowflake

5.4.1 Some useful propositions for the next section

This subsection contains three propositions that will be used in the proof of theorem 5.3. We will assume them to be true.

Proposition 5.4. $P(n+1)^2 + P(n)^2 = P(2n+1)$ *for $P(n)$ the n-th Pell number.*

Proposition 5.5. *Let $w \in T^*$ and Γ its corresponding polygonal line. Let $\alpha \in E$ and M be the midpoint of the vector $z = \overrightarrow{\alpha w}$.*

1. *w is a palindrome if and only if the perpendicular to $\overrightarrow{\alpha w}$ at M is a symmetry axis for Γ.*

2. *w is an antipalindrome if and only if Γ is symmetric with respect to M.*

Proposition 5.6. *Let $\alpha \in \mathcal{E}$. Then for all $n \in \mathbb{N}$, we have :*

$$\begin{cases} \overrightarrow{\alpha q_{3n}^{-}} = \alpha \cdot (P(n), (-1)^n P(n)) \\ \overrightarrow{\alpha q_{3n+1}^{-}} = \alpha \cdot (P(n+1), (-1)^n P(n)) \\ \overrightarrow{\alpha q_{3n+2}^{-}} = \alpha \cdot (P(n) + P(n+1), 0) \end{cases}$$

5.4.2 The area of the Fibonacci Snowflake

Theorem 5.3. *The area of a Fibonacci snowflake of order n is $S(n) = P(2n+1)$, where $P(n)$ is the n-th Pell number.*

Proof. We know by proposition 5.2 that q_{3n+1}^{-} is an antipalindrome.
By the first point of theorem 8.3 and proposition 5.5, we know that the area determined by the word $(q_{3n+1})^3 q_{3n+1}^{-}$ is the area of the square of side $\overrightarrow{\alpha q_{3n+1}^{-}}$.
By proposition 5.6, we know that $\overrightarrow{\alpha q_{3n+1}^{-}} = \alpha \cdot (P(n+1), (-1)^n P(n))$.
Hence,

$$\begin{aligned} S(n) &= \alpha^2 \cdot (P(n+1)^2 + (-1)^{2n} P(n)^2) && (5)\\ &= P(n+1)^2 + (-1)^{2n} P(n)^2 && (6)\\ &= P(n+1)^2 + P(n)^2 && (7)\\ &= P(2n+1) && (8) \end{aligned}$$

The first equality comes from the fact that the area of a square is the square of the length of its side.
The second equality holds as $\alpha^2 \in \{-1, 1\}$ and $\alpha^2 \neq -1$, as the surface would otherwise be negative.
The fourth equality comes from proposition 5.4.

We successfully showed that the area of a Fibonacci snowflake of order n is $S(n) = P(2n+1)$. ■

6 A Generalization of the Fibonacci Word Fractal in Higher Dimensions

This section introduces a stochastic generalization of the Fibonacci Word Fractal (FWF) in $\mathbb{R}^n$. Unlike the traditional deterministic construction, this approach incorporates probabilistic plane switching to govern the fractal's trajectory. To the best of our knowledge, this n-dimensional stochastic extension represents a novel contribution to the literature.

Definition 6.1 (The Generalized Fibonacci Word Fractal)**.** Let $n \in \mathbb{N}_{\geq 2}$ be the ambient dimension, $\varphi \in \mathbb{R}$ be the rotation angle (previously denoted as α), and $p \in [0, 1]$ be a stochastic switching probability. Let $\mathcal{W} = (w_i)_{i\in\mathbb{N}}$ be the infinite Fibonacci Word.

Fractal Construction
The fractal is defined as the discrete trajectory of points $(X_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$. Let the state at step i be given by $(X_i, \mathcal{B}_i)$, where $X_i \in \mathbb{R}^n$ is the position and $\mathcal{B}_i = (u_1^{(i)}, u_2^{(i)}, \ldots, u_n^{(i)})$ is an orthonormal basis representing the local orientation.
Initialization: $X_0 = \mathbf{0}$ and $\mathcal{B}_0 = (e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$.
Iteration: For each step $i \geq 0$, let symbol $L = w_i$.

1. **Orientation Update:**
   - If $L = 1$: The orientation remains unchanged, $\mathcal{B}_i^* = \mathcal{B}_i$.
   - If $L = 0$:
     (a) *Stochastic Plane Switch (if $n > 2$):* With probability p, swap the vector $u_2^{(i)}$ with a randomly selected vector $u_k^{(i)}$ where $k \in \{3, \ldots, n\}$.
     (b) *Rotation:* Rotate the first two vectors $(u_1^{(i)}, u_2^{(i)})$ in their spanned plane by an angle θ_i, determined by the OEDR rule:

     $$\theta_i = (-1)^i \varphi$$

     The new basis $\mathcal{B}_i^*$ is obtained by updating u_1 and u_2 via the standard 2D rotation matrix $R(\theta_i)$.

2. **Position Update:** Regardless of the symbol s, the position advances in the direction of the current heading:

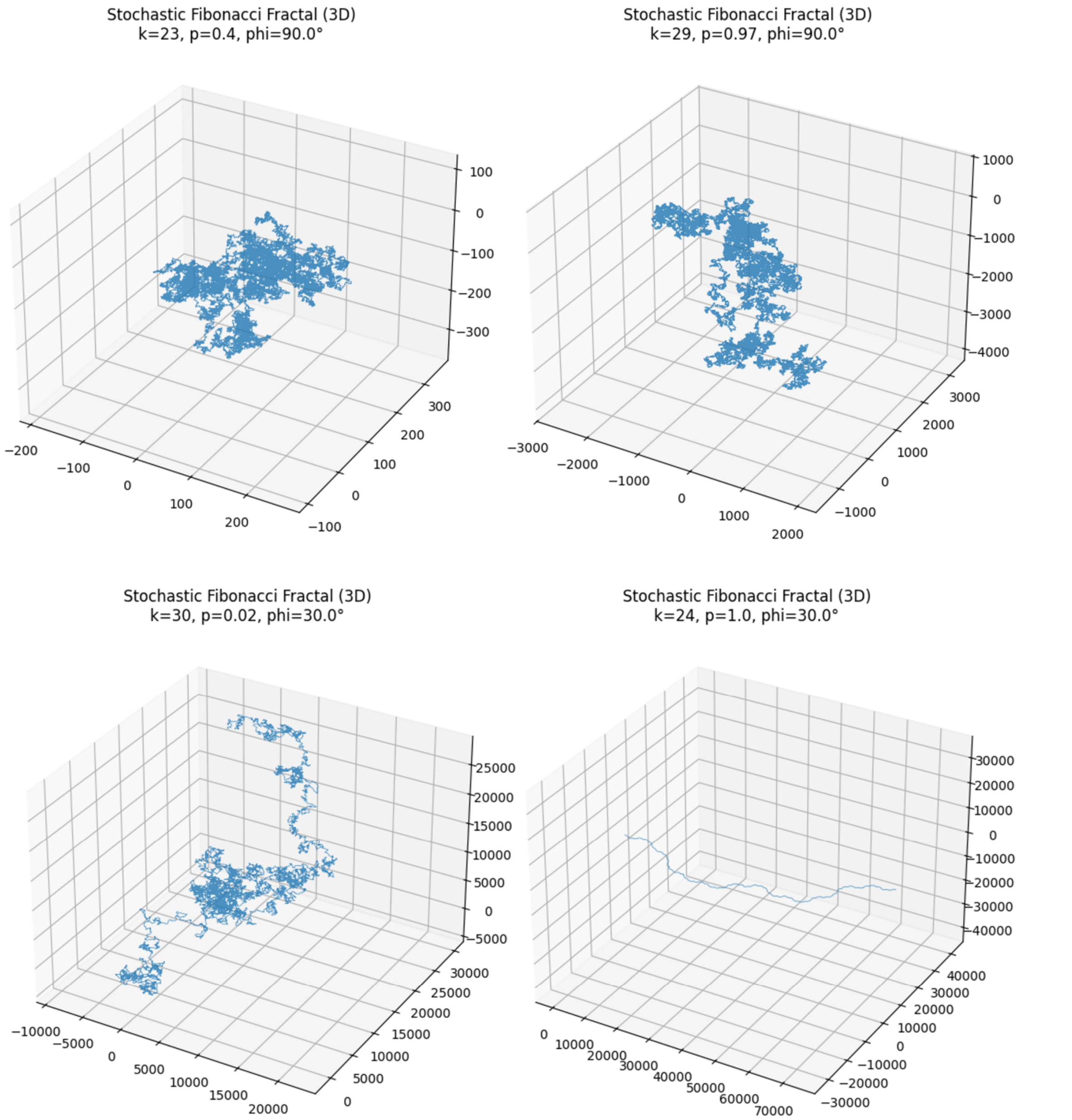
$$X_{i+1} = X_i + u_1^{*(i)} \quad \text{and} \quad \mathcal{B}_{i+1} = \mathcal{B}_i^*$$

6.1 Exploration of the Newly Defined Fractal

We created three Python scripts with the aim to explore both the properties and the shape of the Generalized Fibonacci Word Fractal (GFWF). Code generation assistance was provided by Gemini 3 Deep Think[4]. Our first objective has been to visualise the 3-dimensional fractal at different orders, angles, and values of p.

[4] All used codes are available in the document `The Fibonacci Word Fractal - Construction and Geometric Properties - Final > Codes` and have been almost thoroughly apprehended. Only Shapiro-Wilk and D'Agostino's K^2 tests have not been entirely understood as the mathematics behind them were too complex for us to grasp.

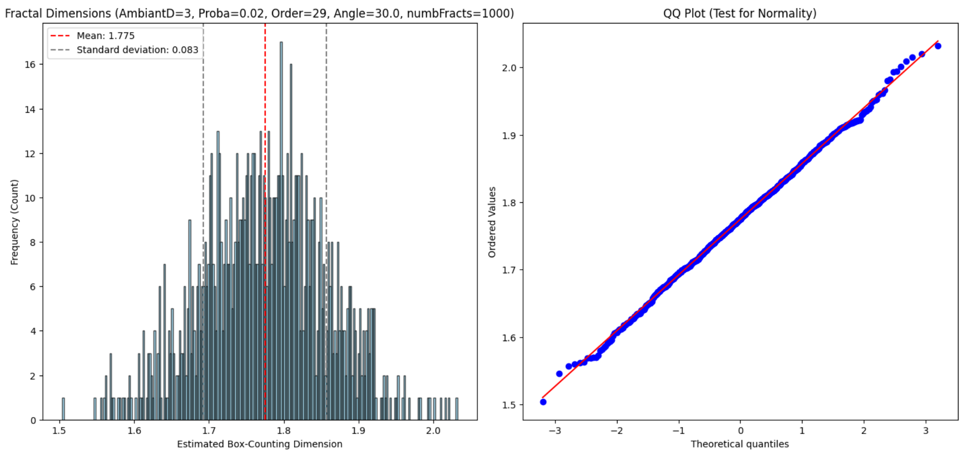
Remark 6.1. The definition used in the codes to generate the Fibonacci words of a user-set order differs from the one used elsewhere in this report. Indeed, the former begins the recursion with "0" and "01" whereas the latter starts it with "1" and "0". This does not change the limit as the order approaches infinity. However, if you wish to retrieve our normal Fibonacci Word Fractal, you must shift the indexes by 2.

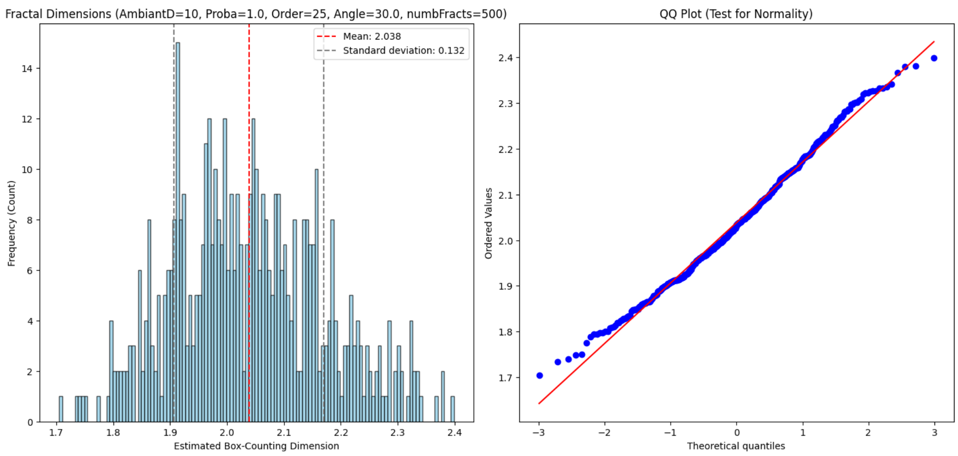


Looking at the graphs, we were amazed to discover how different they appeared. When p is low (as seen in the third plot, $p = 0.02$), the fractal tends to remain "trapped" within a primary plane for longer sequences. This results in a structure that appears locally planar but undergoes occasional "bursts" into the third dimension. Conversely, when p is high (as seen in the first and second plots, $p = 0.4$ and $p = 0.97$), the frequent switching between the vectors leads to a structure resembling a pseudo-random walk in a finite box. The fourth plot ($k = 24, p = 1.0, \varphi = 30°$) reveals a phenomenon that first surprised us but ended up being absolutely logical. When the probability of switching the rotation plane is maximised, the trajectory loses its dense "clumping" and begins to resemble the 2D corresponding Fibonacci Word Fractal. Because there are only 3 directions in 3D, when $p = 1$,the system becomes deterministic, and you lose any randomness. These visual variations suggest that the GFWF does not possess a single Box-Counting dimension, but rather a spectrum of dimensions depending on the choice of p and φ.

6.2 A Seemingly Normal Distribution of the Dimension

When we first decided to numerically approximate the Box-Counting dimension, our code had some errors in it that had gone unnoticed. Nonetheless, it was correct enough to make us realize that given fixed parameters (n, k, p, φ), the approximated dimension could greatly vary between trials. Thus, for our second code, we opted to run multiple simulations with the fixed parameters and then display the approximated dimensions in a histogram. Once more, we were surprised to observe that the results seemed to be normally distributed for the values tested (except for the case $p = 0$ and the case $p = 1, n = 3$).[5]





[5]We tested only values with angles in the range $]0, \frac{\pi}{2}]$ and did not write them down at the beginning. Nonetheless, here are some values that have been tested with the format (dimension, order, probability, angle, number of simulations) : (2, 37, 0.0, 30, 1),(3, 22, 0.02, 30, 2000),(3, 24, 0.0, 30, 100),(3, 24, 1.0, 30, 2000),(3, 29, 0.02, 30, 1000),(3, 29, 0.5, 30, 1000),(3, 30, 0.0, 30, 10),(3, 34, 0.00001, 30, 100),(3, 34, 0.02, 30, 50),(10, 25, 1.0, 30, 500)

Following an external suggestion, we employed a Q-Q plot to verify normality. We were drawn to its simplicity, though we recognize that it relies primarily on visual assessment. Thus, we continued our search for more rigorous tests and stumbled upon D'Agostino's K^2 and Shapiro-Wilk tests. We implemented them into our code and the results are mixed. We do not really understand why they sometimes reject the null hypothesis while the Q-Q plot appears to be in favour of it. The opposite is also true. It has happened that the Q-Q plot appeared to deviate from normality and still they did not reject it. However, what we can be sure of is that the analysis of the Q-Q plots from those two pictures, the distribution seems to be right-skewed. This is coherent with the fact that the dimension is bounded below by the topological dimension of the curve which is 1 and above by the ambient dimensions which are respectively 3 and 10, both further away from the mean than 1. Unfortunately, every one of our attempts to explain the reason behind this apparent normality has come short of doing so.

```
--- Multi-Fractal Simulator (Parallelized) ---
Number of Simulations m (e.g., 20): 1000
Order k (e.g., 14): 29
Probability p (0.0 - 1.0): 0.02
Rotation Angle phi (DEGREES): 30
Dimension n (e.g., 3): 3

Starting 1000 simulations on 3-dimensional fractal...

--- Results ---
Time Taken: 2393.76s
Mean Estimated Dimension: 1.7747
Standard Deviation: 0.0825

--- Statistical Normality Tests ---
Null Hypothesis (H0): The data IS normally distributed.
If p-value < 0.05, we REJECT H0 (Data is NOT normal).
If p-value > 0.05, we CANNOT reject H0 (Data looks normal).

1. Shapiro-Wilk Test:
   Statistic=0.9990, p-value=8.7660e-01
   -> Result: Distribution looks Normal.

2. D'Agostino's K^2 Test:
   Statistic=0.1608, p-value=9.2276e-01
   -> Result: Distribution looks Normal.
```

```
--- Multi-Fractal Simulator (Parallelized) ---
Number of Simulations m (e.g., 20): 500
Order k (e.g., 14): 25
Probability p (0.0 - 1.0): 1
Rotation Angle phi (DEGREES): 30
Dimension n (e.g., 3): 10

Starting 500 simulations on 10-dimensional fractal...

--- Results ---
Time Taken: 1136.50s
Mean Estimated Dimension: 2.0384
Standard Deviation: 0.1320

--- Statistical Normality Tests ---
Null Hypothesis (H0): The data IS normally distributed.
If p-value < 0.05, we REJECT H0 (Data is NOT normal).
If p-value > 0.05, we CANNOT reject H0 (Data looks normal).

1. Shapiro-Wilk Test:
   Statistic=0.9915, p-value=5.6481e-03
   -> Result: Distribution is NOT Normal.

2. D'Agostino's K^2 Test:
   Statistic=8.6534, p-value=1.3211e-02
   -> Result: Distribution is NOT Normal.
```

Figure 8: These texts have been generated alongside the two distribution. One can observe more clearly the time taken, the mean estimated dimensions and the standard deviations

6.3 Influence of the Parameter p on the Estimated Box-Counting Dimension

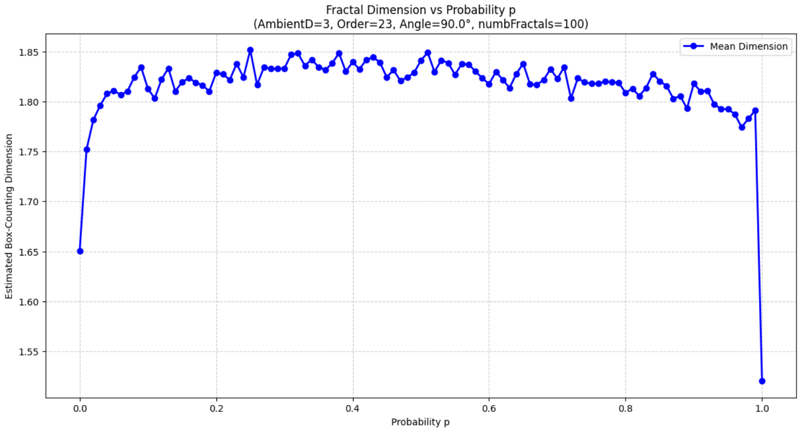
Having observed that the fractal dimension seems to follow a right-skewed normal distribution for some fixed parameters[6], we now turn our attention to the functional relationship between the switching probability p and the resulting mean dimension. By performing a parameter sweep across the interval $p \in [0, 1]$ with fixed ambient dimension, angles and orders, we observed a non-linear and again surprising evolution of the fractal's space-filling properties. We mainly focused on the 3 dimensional variant but believe to have possibly found an inherent pattern. Indeed, as illustrated in the various "Sweep" plots below, the dimension exhibits two distinct regimes of behaviour:

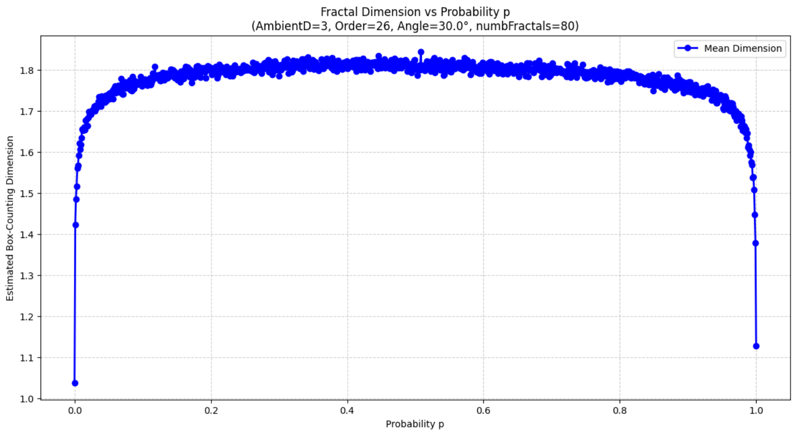
- The Jump ($p \to 0^+$): The most striking feature is the extreme sensitivity of the dimension to the introduction of randomness. For $p = 0$, the trajectory is strictly planar and its dimension is minimized. However, even an infinitesimal probability

[6] The theoretical derivation of this distribution remains an open problem for future research.

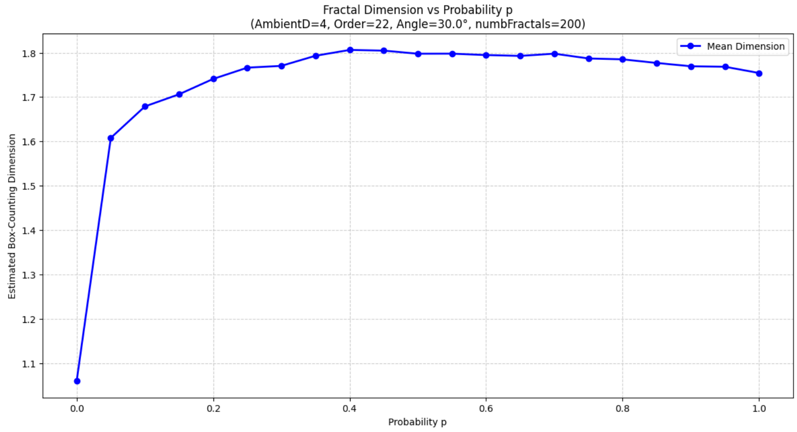
(e.g., $p = 10^{-4}$) is sufficient to "break" the planar confinement, causing the dimension to leap toward a significantly higher value. Indeed, we believe that the sharp increase occurring around $p = 0$ is due to the exponential length of the fibonacci sequence and subsequently the exponential number of 0's in the word.

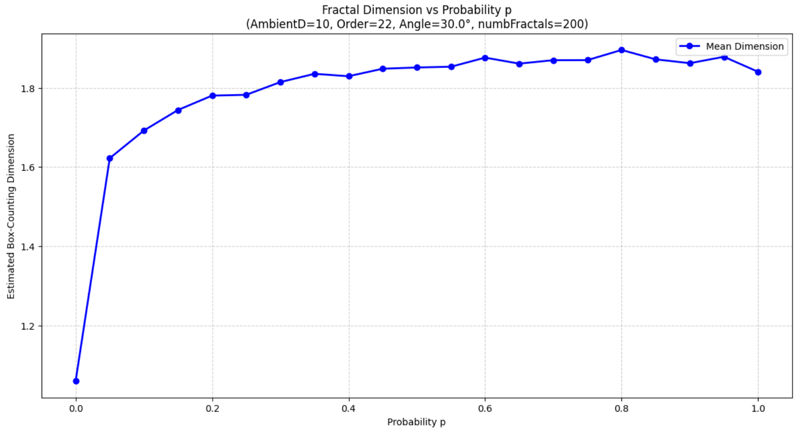
- The Plateau: After the sharp increase and over a wide range of intermediate values, the estimated dimension remains remarkably stable, fluctuating slightly around a mean plateau. This suggests that once the trajectory has enough freedom to explore the ambient space, increasing the frequency of the plane-switching does not necessarily increase the "density" of the resulting fractal; and if it does, it does so very slowly.





Remark 6.2. The Plateau behaviour is significantly disturbed when the ambient dimension is chosen to be 3. Indeed, near $p = 1$, the graph plunges rapidly. We believe that this is due to the form of deterministic "order" in the construction, preventing the chaotic exploration found at lower probabilities and causing the trajectory to simplify its path, thereby reducing its box-counting dimension. In the 3-dimensional case, $p = 1$ represents a state where the rotation plane is switched at every possible iteration. with only one other possibility.





Remark 6.3. Preliminary observations from our 3D, 4D, and 10D data suggest that the "plateau" of the fractal dimension scales with n, but the precise nature of this growth—whether linear, logarithmic, or asymptotic—remains unknown.

7 The Fibonacci Word Fractal in the complex plane

This section will present the Fibonacci Word Fractal as a fractal living in the complex plane.
By convention, the Fibonacci Word Fractal starts at the origin of the complex plane.

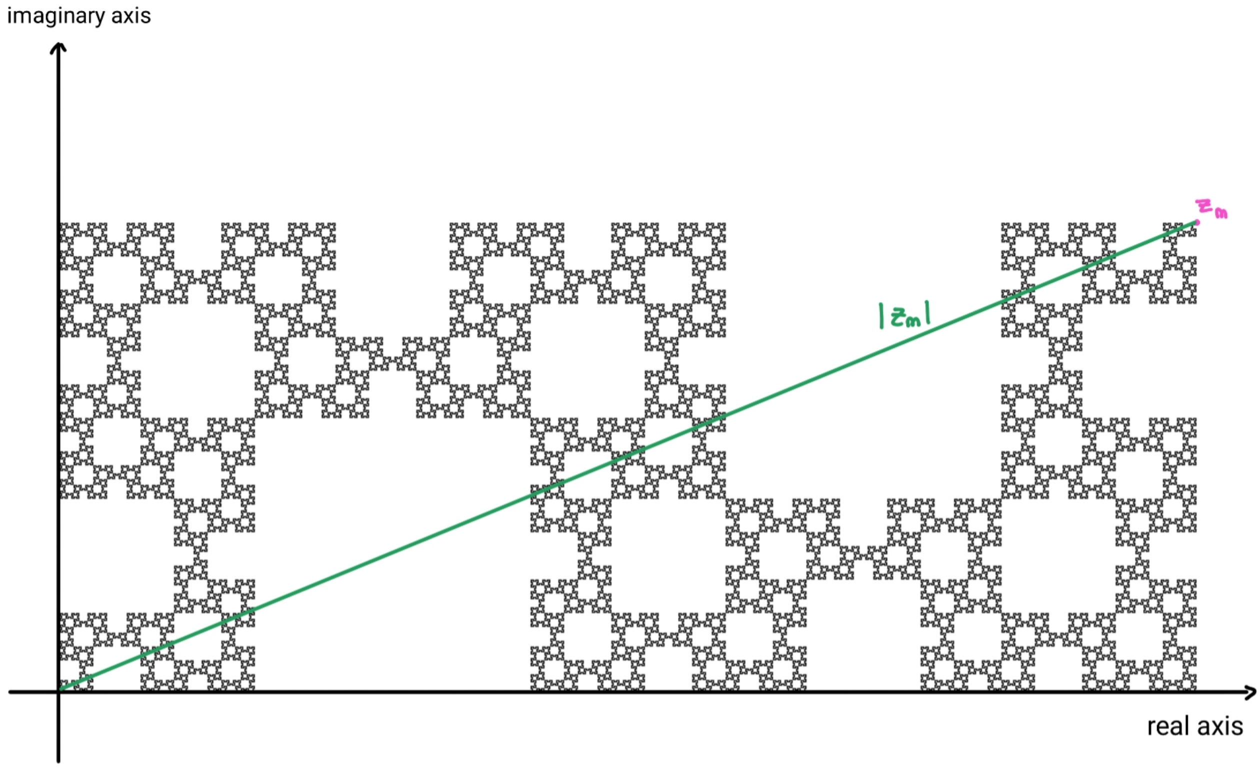


Figure 9: F_{24} in the complex plane

This approach of using the complex plane is motivated by the fact that we can approach complex numbers algebraically, and represent them geometrically.

Definition 7.1. The *complex number associated to the n-th Fibonacci Word* is the complex number z_n corresponding to the coordinate of the end point of n-th Fibonacci Word Fractal.

The following table contains the first 30 Fibonacci Words (FW), their associated complex number $z_n = a_n + ib_n$, their modules to the square and the orientation of the final line segment of their Fibonacci Word Fractal.

n	FW	z_n	$\lvert z_n\rvert^2$	**Orientation**
1	1	$1i$	1	N
2	0	$1i$	1	E
3	01	$1+i$	2	E
4	010	$2+i$	5	S
5	0100	3	9	E
6	...	$4+2i$	20	O
7		$4+5i$	41	N
8		$7i$	49	E
9		$5+11i$	146	E
10		$12+11i$	265	S
11		17	289	E
12		$28+12i$	928	O
13		$28+29i$	1625	N
14		$41i$	1681	E
15		$29+69i$	5602	E
16		$70+69i$	9661	S
17		99	9801	E
18		$168+70i$	33 124	O
19		$168+169i$	56 785	N
20		$239i$	57 121	E
21		$169+407i$	194 210	E
22		$408+407i$	332 113	S
23		577	332 929	E
24		$984+408i$	1 134 720	O
25		$984+985i$	1 938 481	N
26		$1393i$	1 940 449	E
27		$985+2377i$	6 620 354	E
28		$2378+2377i$	11 305 013	S
29		3363	11 309 769	E
30		$5740+2378i$	38 602 484	O

A lot of interesting properties can be found. Here is a list of a few of them :

- z_n for $n \equiv 5 \mod 6$ is real.
- z_n for $n \equiv 2 \mod 6$ is pure imaginary. (see in orange in the table)
- The Pell number $P_n = b_{3n-2}$ for n odd and $P_n = a_{3n-2}$ for n even.
- The module of a the complex numbers z_n for $n = 2 \mod 6$ and $n = 5 \mod 6$ are integers. (this comes fom the fact that either a_n or b_n is zero)
- For $n = 2 \mod 6$ and $n = 5 \mod 6$, $z_n = 6z_{n-1} - z_{n-2}$.
- $n = 0 \mod 3 \Rightarrow \frac{a_n}{b_n} \to \sqrt{2} + 1$ as $n \to +\infty$
- $n = 1 \mod 3 \Rightarrow \frac{a_n}{b_n} \to 1$ as $n \to +\infty$
- $n = 2 \mod 6 \Rightarrow \frac{a_n}{b_n} = 0$
- $n = 5 \mod 6 \Rightarrow \frac{b_n}{a_n} = 0$
- The sequence of the modules is strictly increasing for $n > 1$.

Remark 7.1. The non-overlappingness of the Fibonacci Word Fractal is assumed in the document. A heuristic approach involved using the modules of the complex number associated to the n-th Fibonacci Word, along with the properties of concatenation. However, owing to time limitations, this investigation remains an open avenue for future research.

The following code generates the n-th Fibonacci Word, the orientation of the final line of their Fibonacci Word Fractal and its associated complex number.

```
def fibonacci_word(n):
    if n == 1:
        return "1"
    elif n == 2:
        return "0"
    else: return fibonacci_word(n-1)+fibonacci_word(n-2)

def turn(orientation, direction):
#This function updates the direction towards which the pen will continue drawing the fractal.
    compass = ["N", "E", "S", "W"] #Plays the role of a compass.
    i = directions.index(orientation)

    if direction == 'right':
        return compass[(i + 1) % 4]
    elif direction == 'left':
        return compass[(i - 1) % 4]
    else:
        return orientation

def forward(orientation):
#Given the orientation (N, E, S, W) of the pen, this function adds to the current position of the pen in the complex space its new displacement vector (a complex number).
    step_size = 1

    match orientation:
        case "N":
            return complex(0,1)*step_size
        case "E":
            return complex(1,0)*step_size
        case "S":
            return complex(0,-1)*step_size
        case "W":
            return complex(-1,0)*step_size
        case _:
            return "A problem occured executing forward()"

def complex_fibonacci_fractal(n):

#This function gives the final position of the pen once it has completely drawn the n-th Fibonacci Word Fractal in the complex plane.
```

```
    word = fibonacci_word(n)
    compass = "N" # Initial orientation of the pen
    complex_position = complex(0,0) # Initial position of the pen in the complex plane

    for index, character in enumerate(word):

        complex_position += forward(compass)

        if character == "0":
            if (index + 1) % 2 == 0:
                compass = turn(compass, "left")
            else:
                compass = turn(compass, "right")

    return [complex_position, compass]

def sequence_of_complex_fibonacci(last_term):
#This function gives all final positions of the pen for the Fibonacci Words S_1, ..., S_last_term.

return [(f"S_{a+1}", complex_fibonacci_fractal(a+1)[0]) for a in range(last_term)]
```

8 Appendix

8.1 Measure theory [12], [7], [28], [25]

Definition 8.1 ($\sigma-$algebra)**.** Let X be a set. A non-empty collection of subsets of X $\mathcal{A}$ is called a $\sigma-$algebra if the following properties are satisfied:

1. $X \in \mathcal{A}$.
2. If $A \in \mathcal{A}$, then $\overline{A} \in \mathcal{A}$, where $\overline{A}$ is the complementary of A in X.
3. For any family $(A_i)_{i \in I}$ of elements of A, with I a countable set ($|I| = |\mathbb{N}|$), then $\bigcup_{i \in I} A_i \in A$.

Remark 8.1. We have $\emptyset \in A$ and A closed under countable intersection, as it is closed under complementation and countable union. The pair (X, A) is called measurable space. Moreover, every topological space (X, T), where T defines a topology on X, with for elements open subsets of X, is naturally associated to a $\sigma-$algebra, the **Borel $\sigma-$algebra** $\mathcal{B}_X$. It is the $\sigma-$algebra generated by T and its elements are called Borel sets. A proof showing that a Borel $\sigma-$algebra is indeed a $\sigma-$algebra can be found in [12]. In further definitions, we work on $\mathcal{A} = \mathcal{B}_{\mathbb{R}^n}$ to ensure ensure consistency.

Definition 8.2 (Measure)**.** Let $(X, \mathcal{A})$ be a measurable set. A measure μ on $\mathcal{A}$ is an application

$$\begin{aligned} \mu : \; & \mathcal{A} \to [0, +\infty] \\ & A \mapsto \mu(A) \end{aligned}$$

that verifies the two following properties:

1. $\mu(\emptyset) = 0$,
2. μ is countably additive. That is, for any sequence $(A_i)_{i \in \mathbb{N}}$ of $\mathcal{A}$ such that $\forall i \neq j \in \mathbb{N}$, $A_i \cap A_j$,
$$\mu(\bigcup_{i \geq 1} A_i) = \sum_{i \geq 1} \mu(A_i) \tag{9}$$

The triplet $(X, \mathcal{A}, \mu)$ is called a measure space.

Simple but important consequences follow from this definition. Let $(X, \mathcal{A}, \mu)$ be a measure space and $A, B \in \mathcal{A}$ such that $B \subseteq A$. If $\mu(B) < +\infty$, we have that $\mu(A \backslash B) = \mu(A - B) = \mu(A) - \mu(B)$. In particular, if $B \subseteq A$, then $\mu(B) \leq \mu(A)$. The proof follows considering $\mu(A) = \mu(B \cup A \backslash B)$ since $B \subseteq A$ and using (9) as $B \cap (A \backslash B) = \emptyset$.
Moreover, for any arbitrary sequences of set that belongs to $\mathcal{A}$ $(A_i)_{i \in \mathbb{N}}$,

$$\mu(\bigcup_{i \geq 1} A_i) \leq \sum_{i \geq 1} \mu(A_i)$$

Indeed, let $(B_i)_{i \in \mathbb{N}}$ be defined as $B_1 = A_1$, $B_k = A_k \backslash \bigcup_{i=1}^{k-1} A_i$. Note that $\forall i \neq j \in \mathbb{N}$,$B_i \in \mathcal{A}$ and $B_i \cap B_j = \emptyset$. We therefore have $\bigcup_{i \geq 1} A_i = \bigcup_{i \geq 1} B_i$ by construction and $\mu(\bigcup_{i \geq 1} B_i) = \sum_{i \geq 1} \mu(B_i)$. Furthermore, by the previously statement consequence of the definition of a measure, as $\forall k \in \mathbb{N}, B_k \subseteq A_k$, $\mu(B_k) \leq \mu(A_k)$. It then follows:

$$\mu(\bigcup_{i\geq 1} A_i) = \mu(\bigcup_{i\geq 1} B_i) = \sum_{i\geq 1} \mu(B_i) \leq \sum_{i\geq 1} \mu(A_i)$$

Countable additivity cannot be guaranteed for arbitrary subsets, since some sets are non-measurable and have no consistent generalization of "area." To overcome this, one first introduces the less restrictive notion of outer measure, from which a measure can later be constructed.

Definition 8.3. Outer measure Let X be a set and $\mathscr{P}(X)$ denote its powerset, the collection of all the subsets of X. An outer measure $\mu^* : \mathscr{P}(X) \to [0, +\infty]$ is an application satisfying the following properties:

1. $\mu^*(\emptyset) = 0$,
2. For $A, B \in \mathscr{P}(X)$, if $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$ (monotonicity),
3. μ^* is countably subadditive. That is, for any sequence $(A_i)_{i\in\mathbb{N}}$ of $\mathcal{A}$ such that $\forall i \neq j \in \mathbb{N}$, $A_i \cap A_j$:
$$\mu^*(\bigcup_{i\geq 1} A_i) \leq \sum_{i\geq 1} \mu^*(A_i) \tag{10}$$

Moreover, we say that $A \in \mathscr{P}(X)$ is μ^*-measurable if $\forall B \in \mathscr{P}(X)$,

$$\mu^*(B) = \mu^*(A \cap B) + \mu^*(\overline{A} \cap B)$$

In other words, A is μ^*-measurable if it divides X in a way that the measured "sizes" of all subsets of X by μ^* add properly. Such subsets satisfy $\mu^*(A) = 0 \ \vee \ \mu^*(\overline{A}) = 0$. Note that $\mu^*(B) \leq \mu^*(A \cap B) + \mu^*(\overline{A} \cap B)$ is provided by the countably subadditivity of the outer measure. Assuming for instance $\mu^*(A) = 0$, we have $\mu^*(A \cap B) \leq \mu^*(A) = 0 \implies \mu^*(A \cap B) = 0$ and $\mu^*(\overline{A} \cap B) \leq \mu^*(B)$ by monotonicity of μ^*, implying $\mu^*(B) \geq \mu^*(A \cap B) + \mu^*(\overline{A} \cap B)$ and thus, A is μ^*-measurable.[7]

The next theorem is fundamental in the construction of a measure from outer measures. For a proof see Theorem 1.3.6 of [12].

Theorem 8.1. *Let X, be a set and μ^* an outer measure on X. Let $\mathcal{M}_{\mu^*} := \{A \in \mathscr{P}(X) \mid A$ is μ^*-measurable $\}$. Then:*

1. *$\mathcal{M}_{\mu^*}$ is a $\sigma-$algebra,*
2. *The restriction of μ^* on $\mathcal{M}_{\mu^*}$ is a measure on $\mathcal{M}_{\mu^*}$*

Example 8.1. *The Lebesgue outer measure on $\mathbb{R}$ is an application $\mathcal{L}_1^* : \mathscr{P}(X) \to [0, +\infty]$, such that:*

$$\mathcal{L}_1^*(A) = \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) \middle| A \subset \bigcup_{i=1}^{\infty}[a_i, b_i]\right\}$$

Note that for any open or closed intervals of $\mathbb{R}$, $\mathcal{L}_1^*((a-b)) = \mathcal{L}_1^*([a-b]) = a - b$. The Lebesgue outer measure indeed extends the notion of "length" to any arbitrary subsets of $\mathbb{R}$.

[7]The proof for $\mu^*(\overline{A}) = 0 \implies A$ is μ^*-measurable is analogous.

The Lebesgue outer measure is an outer measure, as it satisfies $\mathcal{L}_1^*(\emptyset) = 0$ *and monotonicity. Moreover, considering the countable subadditivity, let* $(A_i)_{i\in\mathbb{N}}$ *be an arbitrary sequence of subsets of* $\mathbb{R}$*. Let us assume* $\sum_i \mathcal{L}_1^*(A_i) < +\infty$ *(otherwise, the inequality holds) and let* $\epsilon > 0$ *be arbitrary. For each* i*, choose* $(a_{i,k}, b_{i,k})$ *such that* $A_i \subset \bigcup_{k=1}^{\infty}[a_{i,k}, b_{i,k}]$ *and*

$$\sum_{k=1}^{\infty}(b_{i,k} - a_{i,k}) < \mathcal{L}_1^*(A_i) + \frac{\epsilon}{2^i} \tag{11}$$

Using for instance Cantor's pairing function, we combine $(a_{i,k}, b_{i,k})$ $\forall i$ *in* (a_j, b_j)*. We have that:*

$$\sum_{j=1}^{\infty}(b_j - a_j) \overset{(11)}{<} \sum_{i=1}^{\infty}\left(\mathcal{L}_1^*(A_i) + \frac{\epsilon}{2^i}\right) = \sum_{i=1}^{\infty}\left(\mathcal{L}_1^*(A_i)\right) + \epsilon$$

Notice that $\bigcup_i A_i \subseteq \bigcup_j (a_j, b_j)$*. Letting* $\epsilon \to 0$ *and using Lebesgue outer measure definition, we then have:*

$$\mathcal{L}_1^*\left(\bigcup_{i\geq 1} A_i\right) \leq \mathcal{L}_1^*\left(\bigcup_{j\geq 1}(a_i, b_j)\right) = \sum_{j=1}^{\infty}(b_j - a_j) \leq \sum_{i=1}^{\infty}\left(\mathcal{L}_1^*(A_i)\right)$$

Showing (10)*. Therefore, the Lebesgue outer measure is an outer measure. Moreover, applying Theorem 8.1, one can show that* $\mathcal{M}_{\mathcal{L}_1^*}$*, the collection of Lebesgue measurable subsets of* $\mathbb{R}$*, corresponds to the Borel* $\sigma-$*algebra* $\mathcal{B}_{\mathbb{R}}$ *(Proposition 1.3.7,[12])*[8]*. Analogously, if* $A = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid \forall i \in \{1, ..., n\}, \exists a_i, b_i \in \mathbb{R} : a_i \leq x_i \leq b_i\}$ *with* $\mathrm{vol}^n(A) = \prod_{i=1}^{n}(b_i - a_i)$*,* A *is called a coordinate parallepiped in* $\mathbb{R}^n$*. Now, we define the* n*-dimensional Lebesgue outer measure* $\mathcal{L}_n^*$ *on any arbitrary set* $A \subseteq \mathbb{R}^n$*:*

$$\mathcal{L}_n^*(A) = \inf\left\{\sum_{i=1}^{\infty}\mathrm{vol}^n(A_i)\middle| A \subset \bigcup_{i=1}^{\infty} A_i\right\}$$

Note that $\mathcal{L}_n^*(A) = \mathrm{vol}^n(A)$ *if* A *is a coordinate parallelepiped in* $\mathbb{R}^n$*. The* n*-dimensional Lebesgue outer measure* $\mathcal{L}_n^*$ *is an outer measure (the proof is analogous to the one of* $\mathcal{L}_1^*$*, see Proposition 1.3.4 in [12]), that generalizes the notion of "length" (if* $n = 1$*,* $\mathcal{L}_n^* = \mathcal{L}_1^*$*), of "area" (*$n = 2$*), of "volume" (*$n = 3$*) and* n*-dimensional volume. Moreover,* $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, \mathcal{L}_n)$ *is a measure space, where* $\mathcal{L}_n := \mathcal{L}_n^* : \mathcal{B}_{\mathbb{R}^n} \to [0, +\infty]$ *[12].*

However, mathematicians like Hausdorff showed the existence of objects which dimension n is not a natural number but either a rational or an irrational number.[9] To measure subsets of lower dimensions than n, like fractal sets, we define the Hausdorff measure, a generalization of the Lebesgue measure.

8.2 Defining the Hausdorff dimension

To define de Hausdorff dimension, we first need to define the Hausdorff measure. Some definitions are required.

[8]Theorem 8.1 is an essential key to a more general theorem we don't state in this project, the Carathéodory's extension theorem, which guarantees that restricting an outer measure requiring a natural *pre-measure* (i.e., the usual length, vol^n, etc.) to the Borel σ-algebra is a measure.

[9]Some literature refers to this as a *fractional dimension*, although the correct term is *fractal dimension*.

Definition 8.4 (The diameter of U)**.** Let $U \subset \mathbb{R}^n$ be non-empty. We define the diameter of $|U|$ as

$$|U| = \sup\{|x-y| \big| x, y \in U\}$$

where $|x-y|$ denotes the Euclidean norm.

Definition 8.5 (δ-cover)**.** A $\delta-$cover of a non-empty subset E of $\mathbb{R}^n$ is a finite or countable collection of sets $\{U_i\}_{i\geq 1}$ that satisfies:

1. $E \subset \bigcup_{i\geq 1} U_i$
2. $\forall i \in \mathbb{N}, 0 < |U_i| < \delta$

Definition 8.6 ($\delta-$neighborhood)**.** Let E be a non-empty subset of $\mathbb{R}^n$. We call the $\delta-$neighborhood of E the set $E_\delta := \{x \in \mathbb{R}^n \big| |x-y| \leq \delta \text{ for } y \in E\}$

We now define the Hausdorff measure, for any positive real parameter s, that will determine the scaling property of the measure. For any subsets A of $\mathbb{R}^n$ and any $\lambda \in \mathbb{R}$, the Hausdorff measure with parameter s, denoted $\mathcal{H}^s$, must satisfy $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$, where $\lambda A = \{\lambda x \mid x \in A\}$. s denotes the dimension of A when the Hausdorff measure is positive and finite.

Example 8.2. *Let us consider the unit square $C = [0,1]^2 \subset \mathbb{R}^2$, a Borel set. It has a dimension $s = 2$. Since $2 \in \mathbb{N}$, we consider the Hausdorff measure of parameter 2 as the 2-dimensional Lebesgue measure. We then have $\mathcal{H}^2(C) = \mathcal{L}_2(C) = \text{vol}^2(C) = 1$. Scaling by $\lambda \in \mathbb{R}$, we indeed have that $\mathcal{H}^2(\lambda C) = \mathcal{H}^2([0,\lambda]^2) = \lambda^2 = \lambda^2 \cdot \mathcal{H}^2(C)$.*

Definition 8.7. Let $E \subset \mathbb{R}^n$, $s \geq 0$. Then, for any positive δ, we define:

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s \Big| \{U_i\}_{i\geq 1} \text{ is a } \delta\text{-cover of } E\right\} \tag{12}$$

The s-dimensional Hausdorff measure of E is an outer measure [25][a], and is given by:

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta$$

[a]The restriction of $\mathcal{H}^s_\delta$ on the Borel σ- algebra is a measure. (See footnote 8)

Remark 8.2. Taking $\delta' < \delta$, we have that $\mathcal{H}^s_{\delta'}(E) \leq \mathcal{H}^s_\delta(E)$. Hence, as $\delta \to 0$, both of the number of δ-covers and $\sum_{i=1}^{\infty} |U_i|$, where $\{U_i\}_{i\geq 1}$ is a δ-cover of E decrease (or remain constant), guaranteeing the existence of a limit of $\mathcal{H}^s_\delta$, as it must be positive by construction.

The Hausdorff measure satisfies the scaling property mentioned above.

Proposition 8.1. *Let $E \subset \mathbb{R}^n$, $\lambda > 0$. The Hausdorff measure $\mathcal{H}^s$ satisfies the scaling property, meaning that*

$$\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$$

Proof.

- For any δ-cover and $\lambda > 0$, if $\{U_i\}_{i\geq 1}$ is a δ-cover of E, then $\{\lambda U_i\}_{i\geq 1}$ is a $\lambda\delta$-cover of λE and $\sum_{i=1}^{\infty} |U_i|^s \leq \sum_{i=1}^{\infty} |\lambda U_i|^s = \lambda^s \sum_{i=1}^{\infty} |U_i|^s$, as $\lambda > 0$. This means that $\mathcal{H}^s_{\lambda\delta}(\lambda E) \leq \lambda^s \mathcal{H}^s_\delta(E)$. But $\lambda\delta \overset{\delta\to 0}{\to} 0$, hence $\mathcal{H}^s(\lambda E) \leq \lambda^s \mathcal{H}^s(E)$.
- Let $\mathcal{H}^s(\lambda E)$. By the previous inequality and multiplying by $\frac{\lambda^s}{\lambda^s}$, we get:

$$\lambda^s \mathcal{H}^s(E) \leq \frac{\lambda^s}{\lambda^s} \mathcal{H}^s(\lambda E) = \mathcal{H}^s(\lambda E)$$

  as $\frac{1}{\lambda^s} > 0$.

In conclusion, we have shown that $\lambda^s \mathcal{H}^s(E) \leq \mathcal{H}^s(\lambda E) \wedge \mathcal{H}^s(\lambda E) \leq \lambda^s \mathcal{H}^s(E) \iff \mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$. ■

We need to study the parameter s. It is a critical parameter satisfying that if $t < s$, $\mathcal{H}^t(E) = \infty$ and if $t > s$, $\mathcal{H}^s(E) = 0$. Intuitively, it respectively corresponds to compute the Hausdorff measure of a cube using line segments $(1 = t \leq s = 3)$ or 4-hypercubes $(4 = t \geq s = 3)$. This drop from ∞ to 0 can be shown using the definition of $\mathcal{H}^s_\delta$ (8.7). If $t > s$, we have that:

$$\sum_{i=1}^{\infty} |U_i|^t = \sum_{i=1}^{\infty} |U_i|^{t-s} |U_i|^s \leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s$$
$$\implies 0 \leq \mathcal{H}^t_\delta(E) \leq \delta^{t-s} \mathcal{H}^s_\delta(E)$$

Supposing that $\mathcal{H}^s_\delta(E) < \infty$ and letting $\delta \to 0$, we have that $\mathcal{H}^t(E) = 0$. The same logic applies to show that $\mathcal{H}^t(E) = \infty$ for $t < s$. We define the Hausdorff measure as this critical parameter s.

Definition 8.8 (Hausdorff dimension)**.** The Hausdorff dimension of $E \subset \mathbb{R}^n$ is defined as follows:

$$\dim_{\mathcal{H}}(E) = \sup\{s \geq 0 \mid \mathcal{H}^s(E) = \infty\} = \inf\{s \geq 0 \mid \mathcal{H}^s(E) = 0\} \tag{13}$$

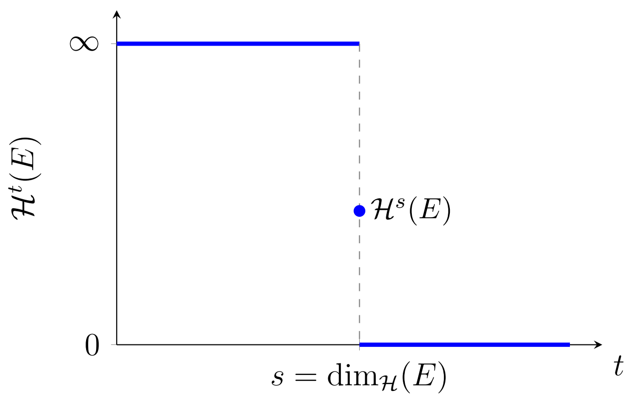


Figure 10: Representation of the Hausdorff measure, with the drop from ∞ to 0 around the critical parameter s

8.3 Prerequisites for the Fibonacci Snowflake [6]

These prerequisites are based on the document [6].
Let's start with some definitions.
We first define what a path is:

Definition 8.9. A *path* in $\mathbb{Z} \times i\mathbb{Z}$ is a polygonal path made of the elementary unit translations :

$$i = (0, 1), i^2 = (-1, 0), i^3 = (0, -1), i^4 = (1, 0)$$

Let's now state some definitions related to some particular types of paths.

Definition 8.10.
Let $|w|_\alpha$ denote the number of occurrences of the letter α in the word w, where $\alpha \in \{i, -i, 1, -1\}$.
A path is said to be closed if it satisfies $|w|_i = |w|_{-i}$ and $|w|_1 = |w|_{-1}$, ie if the two extremities of the polygonal line coincide.
A simple path is a word such that none of its proper factors is a closed path.
A boundary word is a non-empty closed path.
A polynomino is the inner region bounded by a boundary path.

Remark 8.3. A path is fully determined by the starting step $\alpha \in \{i, -i, 1, -1\} := \mathcal{E}$ and the sequence of direction indications, which are left (L), right(R), forward (F) and backwards (B).
We will only consider the two directions L and R, as by the construction of the Fibonacci Word Fractal.

In the following we define some functions which will later be used.

Definition 8.11.

- $\bar{\cdot}$ exchanges the letters R and L.
- $\tilde{\cdot}$ reverses the letters of the word, ie, for a word $w = w_1 w_2 \dots w_n$, then $\tilde{w} = w_n w_{n-1} \dots w_1$
- Let $w = w_1 w_2 \dots w_{n-1} w_n$. We define w^- as $w^- := w_1 w_2 \dots w_{n-1}$

Definition 8.12. The words satisfying $w = \tilde{w}$ are called palindromes.
The words satisfying $\widehat{w} := \overline{\tilde{w}} = w$ are called antipalindromes.

Definition 8.13. Let $\mathcal{T}^*$ be the set of all finite words one can build from the alphabet $\mathcal{T} = \{L, R\}$.
The winding number $\Delta : \mathcal{T}^* \to \mathbb{Z}$ is defined by $\Delta(w) = |w|_L - |w|_R$

Here are some properties of the winding number

Proposition 8.2.
Let u and v be words, ϵ be the empty word.

- $\Delta(\epsilon) = 0$
- $\Delta(uv) = \Delta(u) + \Delta(v)$
- $\Delta(w) = \Delta(\widetilde{w})$
- $\Delta(\overline{w}) = -\Delta(w) = \Delta(\widehat{w})$
- $\Delta(wL) = \Delta(w) + 1$
- $\Delta(wR) = \Delta(w) - 1$

Remark 8.4. The winding number of an antipalindrome w is $\Delta(w) = 0$.

Definition 8.14. The right action $\mathcal{E} \times T^* \to \mathcal{E}$ is defined as follows. For each $\alpha \in \mathcal{E}$, we set :

1. $\alpha L = i \cdot \alpha$
2. $\alpha R = -i \cdot \alpha$,
3. $\alpha\varepsilon = \alpha$,

Remark 8.5. By associativity, we have $\alpha(uv) = (\alpha u)v$.

Theorem 8.2. *We have $\alpha w = i^{\Delta(w)} \alpha$.*

Proof. Let's prove this by induction :
The base case is fulfilled as $\alpha\varepsilon = \alpha$ and $\Delta(\epsilon) = 0$.
Let's now assume that this claim is true for w.
Then : $\alpha wL = (\alpha w)L = i^{\Delta(w)}\alpha L = i^{\Delta(w)+1}\alpha = i^{\Delta(wL)}\alpha$.
Similarly, we have : $\alpha wR = (\alpha w)R = i^{\Delta(w)}\alpha R = i^{\Delta(w)-1}\alpha = i^{\Delta(wR)}\alpha$. ■

The next proposition is describes the action of $\overline{w}$ on α

Proposition 8.3. *Let $w \in T^*$ and $\alpha \in \mathcal{E}$. Then*

$$\alpha\overline{w} = \begin{cases} \alpha w, & \text{if } \Delta(w) \equiv 0 \pmod 2 \\ -\alpha w & \text{otherwise.} \end{cases}$$

Proof.

- If $\Delta(w) \equiv 0 \pmod 2$

$$\begin{aligned} &\Rightarrow \Delta(w) = 2m \text{ for } m \in \mathbb{Z} \\ &\Rightarrow \alpha\overline{w} = i^{-\Delta(w)}\alpha = i^{-2m}\alpha \\ &\Rightarrow \alpha\overline{w} = (-1)^{-m}\alpha \\ &\Rightarrow \alpha\overline{w} = (-1)^{m}\alpha \\ &\Rightarrow \alpha\overline{w} = i^{2m}\alpha \\ &\Rightarrow \alpha\overline{w} = i^{\Delta(w)}\alpha \\ &\Rightarrow \alpha\overline{w} = \alpha w \end{aligned}$$

- If $\Delta(w) \equiv 1 \pmod 2$

$$\begin{aligned}
&\Rightarrow \Delta(w) = 2m + 1 \text{ for } m \in \mathbb{Z} \\
&\Rightarrow \alpha\overline{w} = i^{-\Delta(w)}\alpha = i^{-2m-1}\alpha \\
&\Rightarrow \alpha\overline{w} = ((-1)^{-m} i^{-1})\alpha \\
&\Rightarrow \alpha\overline{w} = ((-1)^{m} \cdot -i)\alpha \\
&\Rightarrow \alpha\overline{w} = i^{2m} \cdot (-i)\alpha \\
&\Rightarrow \alpha\overline{w} = -i^{2m+1}\alpha \\
&\Rightarrow \alpha\overline{w} = -i^{\Delta(w)}\alpha \\
&\Rightarrow \alpha\overline{w} = -\alpha w
\end{aligned}$$

■

To each pair $(\alpha, w) \in \mathcal{E} \times T^*$ we associate a polygonal path γ defined as follows. The first side of γ is the vector α. The subsequent sides are obtained by reading successively the instructions L and R given by the word w.

Let $z_0 = 0, \quad z_1 = \alpha, \quad \dots, \quad z_{|w|+1}$ be the sequence of vertices of γ. For $0 \leq \ell \leq |w|$, the vertices are given by

$$z_{\ell+1} = \alpha \sum_{k=1}^{\ell} i^{\Delta(w[1,k])},$$

where $w[1,k] = w_1 w_2 \cdots w_k$ denotes the prefix of w of length k, with the convention $w[1,0] = \varepsilon$.

For simplicity, we denote by $\overrightarrow{\alpha w} := \overrightarrow{z_0 z_{|w|+1}}$ the vector joining the initial and final vertices of the path γ.

Remark 8.6. By abuse of terminology, we often identify a word w with the path γ it defines.

Remark 8.7. If $w \in \mathcal{T}$ is a path, then $\widehat{w}$ is also a path, which is precisely the reverse path, *ie* the path obtained by moving along the original path backwards.

Proposition 8.4. *If a path $w \in \mathcal{T}^*$ is closed, then $\Delta(w) \equiv 0 \mod 3$*

This won't be proved and will be accepted in this document.

Theorem 8.3. *Let $w \in \mathcal{T}^*$ and $\alpha \in \mathcal{E}$.*
We have the following properties :

1. *If $\alpha w = \pm i\, \alpha$, then the word $w^3 w^-$ defines a closed path.*
2. *If $\alpha w = -\, \alpha$, then the word $w w^-$ defines a closed path.*
3. *If $\alpha w = \alpha$, then either w^- defines a closed path, in which case the sequence $(w^n)_{n \geq 1}$ is bounded as $n \to \infty$, or w^- is open and*

$$\left|\overrightarrow{\alpha w^n}\right| = c\, n \quad \textit{for some constant } c > 0.$$

Proof. 1. We assume that $\alpha w = \pm i\alpha$.
We want to show that $\overrightarrow{\alpha(w^3 w^-)} = 0$.
Let's first rewrite it : $\overrightarrow{\alpha(w^3 w^-)} = \overrightarrow{\alpha(wwww^-)}$.
By additivity, we get : $\overrightarrow{\alpha(wwww^-)} = \overrightarrow{\alpha w} + \overrightarrow{(\alpha w)w} + \overrightarrow{(\alpha w^2)w} + \overrightarrow{(\alpha w^3)w^-}$.
Moreover, $\alpha w = \pm i\alpha \Rightarrow \alpha w^k = (\pm i)^k \alpha$.
Therefore,

$$\begin{aligned}\overrightarrow{\alpha(w^3 w^-)} &= \pm i\alpha + (\pm i)\overrightarrow{\alpha w} \; + (\pm i)^2\overrightarrow{\alpha w} \; + (\pm i)^3\overrightarrow{\alpha w^-} \\ &= \pm i\alpha + (\pm i)(\pm i\alpha) \; + (\mp i\alpha) \; + (\pm i)^3(\pm i\alpha) \\ &= \pm i\alpha - \alpha \mp i\alpha + \alpha \\ &= 0\end{aligned}$$

2. We assume that $\alpha w = -\alpha$.
We want to show that $\overrightarrow{\alpha(ww^-)} = 0$.
By additivity, we get :

$$\begin{aligned}\overrightarrow{\alpha ww^-} &= \overrightarrow{\alpha w} \; + \overrightarrow{(\alpha w)w^-} \\ &= \overrightarrow{\alpha w} \; + \overrightarrow{(-\alpha)w^-} \\ &= \overrightarrow{\alpha w} \; - \overrightarrow{\alpha w} \\ &= 0\end{aligned}$$

3. We assume that $\alpha w = \alpha$.
Hence, $\alpha w^k = \alpha \quad \forall k \in \mathbb{N}$.
Let $v := \overrightarrow{\alpha w}$.
By additivity, we get : $\overrightarrow{\alpha w^n} = \overrightarrow{\alpha w} \; + \overrightarrow{(\alpha w)w} + \ldots + \overrightarrow{(\alpha w^{n-1})w}$
$\Rightarrow \overrightarrow{\alpha w^n} = n \cdot \overrightarrow{\alpha w} = n \cdot v$

- <u>if w^- is closed</u>:

$$\begin{aligned}&\Rightarrow \overrightarrow{\alpha w^-} = 0 \\ &\Rightarrow v = \overrightarrow{\alpha w} = 0 \\ &\Rightarrow \overrightarrow{\alpha w^n} = n \cdot 0 = 0 \; \forall n \\ &\Rightarrow w^n \text{ is bounded as n} \to +\infty\end{aligned}$$

- <u>if w^- is open</u>:

$v = \overrightarrow{\alpha w} \neq 0$
$\Rightarrow \overrightarrow{\alpha w^n} = n \cdot v$
Taking the norm :$|\overrightarrow{\alpha w^n}| = n|v|$
Let $c := |v|$
$\Rightarrow |\overrightarrow{\alpha w^n}| = cn$, for c > 0

■

8.4 Observing the overlapping property of the $\alpha = 100^\circ$ and $\alpha = 120^\circ$ variants of F_{23} using **Manim**

```python
import manim, math
import numpy as np

def fibonacci_word(n) #This function outputs the n-th Fibonacci Word and can be found in previously given codes.
def generate_points(word, step=0.1, alpha=90) #This functions copies draw_fibonacci_word_fractal for a generalized alpha variant of the Fibonacci Word Fractal but instead outputs the set of points "points" that belongs to the curve.

def build_fractal(n, angle, step):
    word = fibonacci_word(n)
    pts = generate_points(word, step, angle)
    fractal = VMobject().set_points_as_corners(pts).set_stroke(color=BLUE_A, width=1.5)

    #Note: One can here implement the centering of the fractal.

class FibonacciWordComparison(Scene):
    def construct(self):
        n = 23
        step = 0.02
        fractal = build_fractal(n, 90, step) #Constructing the 90deg 23-FWF
        fractal.set_height(5)
        self.play(Create(fractal, run_time=6, rate_func=linear))
        self.wait(1)

        def transition(old_fractal, new_angle):

            new_fractal = build_fractal(n, new_angle, step)
            new_fractal.set_height(5)

            self.play(
                ReplacementTransform(old_fractal, new_fractal),
                run_time=3
            )

            return new_title, new_fractal

        fractal = transition(fractal, 100)
        fractal = transition(fractal, 120)
        self.wait(2)
```

9 Conclusion

In this study, we have characterized the Fibonacci Word Fractal as an object at the intersection of combinatorics on words, geometry and measure theory. Starting from the recursive structure of the Fibonacci word and the foundational work of Monnerot-Dumaine, we demonstrated that the resulting fractal can be as well studied algebraically. Our geometric analysis confirmed the self-similarity of the curve and allowed precise calculations of its contracting ratio, $(1+\sqrt{2})^{-1}$. A significant portion of this work was dedicated to the rigorous determination of the fractal dimension. Using thermodynamic formalism, we established the Hausdorff dimension of the curve as $s = 3\ln(\phi)(\ln(1+\sqrt{2}))^{-1}$. Furthermore, by verifying the Open Set Condition (OSC), we formally confirmed the equivalence of the Hausdorff and Box-counting dimensions and concluded the section computing the Box-Counting dimension of the fractal boundary. The study of the Fibonacci Snowflake complementarily illustrated how these curves can be closed to form polyominoes with areas and perimeters defined by Pell and Fibonacci numbers, respectively. Beyond these deterministic models, this project explored a possible extension into higher-dimensional spaces ($n \geq 2$) through the introduction of a switching probability p. This Generalized Fibonacci Word Fractal (GFWF) introduces stochastic elements into the fractal's trajectory. Preliminary experimental data suggest that even a minimal jump parameter may be sufficient to break planar confinement. Our observations also point toward a "plateau" effect, where the estimated box-counting dimension appears to remain relatively stable across a broad range of p values. However, these behaviors—including the seemingly right-skewed normal distribution of the dimensions—remain empirical observations at this stage, as their formal theoretical derivation constitutes a significant open problem.

9.1 Avenues for further research

The scope of this project was vast, and time constraints meant that many intriguing avenues were left unexplored. For future students undertaking this lab, we propose the following paths for further investigation:

- **Generalization of the Fibonacci Word to a finite alphabet with more letters:** Extending the Fibonacci word construction to an alphabet with more than two letters opens the door to a rich variety of combinatorial sequences. One could investigate the preservation of recurrence relations and self-similarity within the resulting fractals. This could lead to new fractal curves in the plane with different symmetries, offering a broader understanding of how the Fibonacci structure generalizes beyond binary sequences.
- **Generalization to the $\alpha°$-variant of n-th Fibonacci Word Fractal:** Many of the propositions necessary to compute the box-counting dimension of the curve of the Fibonacci Word Fractal for a rotational angle $\alpha \in [0, \frac{\pi}{2}]$ were stated as remarks following the proves for the 90° n-th Fibonacci Word Fractal. Providing rigorous proves for these propositions would formalize the understanding of the $\alpha°$-variant of n-th Fibonacci Word Fractal.
- **Compare the Fibonacci Word Fractal with other sequences :** This can reveal how combinatorial rules influence geometric properties and dimension. Such a comparison may uncover universal patterns or highlight the unique structure of the Fibonacci-based fractals.

- **The Fibonacci Snowflake :** In this document, we have stated and shown many properties, to finally define the Fibonacci snowflake, as well as computed its perimeter and area. Future work could focus on exploring other properties such that its box-counting dimension or the measure of its complexity.
- **The Generalized Fibonacci Word Fractal :** Much remains to be understood regarding the properties of this mathematical object. Specifically, establishing a formal proof—or disproof—of the observed right-skewed normality would be a significant step forward. Such a breakthrough, alongside the determination of the exact box-counting dimensions of the GFWF, would be instrumental in deriving a general closed-form function that maps the ambient dimension, the rotation angle, and the switching probability to the resulting fractal dimension. Furthermore, while our current analysis has primarily focused on sweeps of the probability parameter p, a natural and promising extension of this work would involve implementing a systematic sweep across the ambient dimension n. Investigating how the fractal dimension behaves as $n \to \infty$ for a fixed p could reveal a "saturation point" where the trajectory's complexity is limited by the Fibonacci sequence itself rather than the available spatial degrees of freedom. Such a sweep would be essential to fully characterize the space-filling efficiency of the GFWF in higher dimensions.
- **Relation to the complex plane :** Some possible objectives for future research include proving the symmetries of the three patterns of the Fibonacci Word Fractal and rigorously confirming its non-overlapping property, by representing the FWF in the complex plane and using the algebraical properties of the complex numbers.
- **Explore real-life applications of the Fibonacci Word / Fibonacci Word Fractal** The Fibonnaci Word and the Fibonacci Word Fractal are linked to the Golden ratio, a number one can find in various natural structures. It has for instance been established that φ plays an important role in phyllotaxis, the arrangement of leaves on a plant stem. Moreover, current studies link the Fibonacci Word properties to quasicrystals, ordered but non-periodic structures.

9.2 Methodological Recommendations for Future Research

Based on the computational and statistical challenges encountered during the exploration of the Generalized Fibonacci Word Fractal (GFWF), we formulate the following recommendations for future studies on this topic. These guidelines aim to optimize computational efficiency and ensure statistical rigor.

9.2.1 Computational Optimization and Resource Management

The exponential growth of the Fibonacci sequence length ($|w_k| \approx \phi^k$) imposes severe memory and processing constraints.[10]

- **Compilation and Typing:** The use of interpreted Python code is insufficient for orders $k \geq 25$. It is much better to use Just-In-Time (JIT) compilation (e.g., `Numba`) or static languages (C++/Rust) to handle the coordinate generation and basis updates. Boolean arrays should be strictly preferred over string manipulation for the Fibonacci word generation to minimize memory footprint.

[10] Be prepared to let your computer run for an entire night without being able to access it.

- **Parallelization:** The stochastic nature of the GFWF requires large sample sizes (m) to obtain reliable dimension estimates. As demonstrated in our implementation, parallel processing (e.g., `ProcessPoolExecutor`) is essential. Future implementations should ensure that random seeds are distinct across concurrent processes to avoid pseudo-replication.
- **Memory-Efficient Box-Counting:** The standard Box-Counting algorithm is memory-intensive. For high orders ($k \geq 30$), storing all points in memory becomes unfeasible. We recommend implementing sparse matrix representations or hashing coordinate bins (as done via `np.unique` on discrete bins) rather than instantiating full dense 3D grids.

9.2.2 Statistical Analysis of the Fractal Dimension

Our results indicate that the distribution of estimated dimensions is not always perfectly Gaussian.

- **Normality Testing:** Visual inspection via Q-Q plots is necessary but insufficient. We recommend systematically pairing them with quantitative tests such as the Shapiro-Wilk test (for $m < 5000$) or D'Agostino's K^2 test. However, researchers should note that with large sample sizes, these tests may reject the null hypothesis even for negligible deviations from normality. (We recommend to look for better tests)
- **Skewness Interpretation:** The observed right-skew in the dimension distribution is tied to the fact that the fractal is bounded by the topological dimension (1) and the ambient dimension (n). Future statistical models should consider using skew-normal distributions or other similar distributions rather than assuming a pure Gaussian model, especially when the mean dimension is close to the ambient dimension or the topological dimension.

9.2.3 Parameter Space Exploration

The functional relationship between the switching probability p and the fractal dimension D exhibits non-linear sensitivity.

Sampling near $p = 0$: The "Jump" phenomenon observed likely implies that the fractal dimension changes drastically with infinitesimal randomness. Linear sampling of p is inadequate in the range $[0, 0.05]$. We strongly recommend using a logarithmic scale or a finer mesh near $p = 0$ to accurately capture the phase transition from planar to spatial behavior.

References

[1] Luís Barreira. *Thermodynamic formalism and applications to dimension theory /*. eng. Progress in mathematics ; 294. Basel: Birkhäuser, 2011. ISBN: 9783034802062. URL: https://proxy.bnl.lu/login?url=https://link.springer.com/10.1007/978-3-0348-0206-2.

[2] Jean Berstel. "Fibonacci words: a survey". In: *L.I.T.P., Université Paris VI* ().

[3] Christopher J. Bishop and Y. Peres. *Fractals in probability and analysis /*. eng. Cambridge studies in advanced mathematics ; 162. Cambridge: Cambridge University Press, 2017. ISBN: 9781316460238. URL: https://proxy.bnl.lu/login?url=https://doi.org/10.1017/9781316460238.

[4] Alexandre Blondin Massé et al. "Christoffel and Fibonacci Tiles". In: *Discrete Geometry for Computer Imagery* 6022 (2009), pp. 67–78.

[5] Alexandre Blondin Massé et al. "Complexity of the Fibonacci snowflake". In: *arXiv* (2011).

[6] Alexandre Blondin Massé et al. "Fibonacci Snowflakes". In: *Annales des Sciences Mathématiques du Québec* 35.2 (2011), pp. 141–152.

[7] V. I. Bogachev. *Measure theory /*. eng. Berlin ; Springer, 2007. ISBN: 9783540345145. URL: https://proxy.bnl.lu/login?url=https://link.springer.com/10.1007/978-3-540-34514-5.

[8] Rufus Bowen. "Hausdorff dimension of quasi-circles". en. In: *Publications Mathématiques de l'IHÉS* 50 (1979), pp. 11–25. URL: https://www.numdam.org/item/PMIHES_1979__50__11_0/.

[9] Rufus Bowen and J.-R. Chazottes. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms /*. eng. 2nd rev. ed. / Jean-René Chazottes, editor ; preface by David Ruelle. Lecture notes in mathematics, 470. Berlin: Springer, 2008. ISBN: 3540776958. URL: https://proxy.bnl.lu/login?url=https://link.springer.com/10.1007/978-3-540-77695-6.

[10] Jader E. Brasil, Elismar R. Oliveira, and Rafael Rigão Souza. "Thermodynamic Formalism for General Iterated Function Systems with Measures". eng. In: *Qualitative theory of dynamical systems* 22.1 (2023), pp. 19–. ISSN: 1575-5460.

[11] Catherine Bruce. *The Hausdorff dimension of quasi-circles: a result of Ruelle and Bowen*. 2015. arXiv: 1509.07558 [math.DS]. URL: https://arxiv.org/abs/1509.07558.

[12] Donald L. Cohn. *Measure theory /*. eng. 2nd ed. Birkhäuser advanced texts. New York, NY: Birkhäuser, 2013. ISBN: 9781461469568. URL: https://proxy.bnl.lu/login?url=https://link.springer.com/10.1007/978-1-4614-6956-8.

[13] Kim Da Cruz. "Lët'z box-counting!" Bachelor Thesis. University of Luxembourg, 2021.

[14] K. Falconer. "Fractal Geometry: Mathematical Foundations and Applications". eng ; jpn. In: *Biometrics* 46.3 (1990), pp. 886–887. ISSN: 0006-341X.

[15] *Formule de Binet*. Accessed: November 20, 2025. URL: https://major-prepa.com/mathematiques/formule-binet/.

[16] Martin Hansen. *Substitution drawing rules on the Fibonacci word*. 2023. arXiv: 2304.00458 [math.CO]. URL: https://arxiv.org/abs/2304.00458.

[17] Tyler Hoffman and Benjamin Steinhurst. *Hausdorff Dimension of Generalized Fibonacci Word Fractals*. 2016. arXiv: 1601.04786 [math.MG]. URL: https://arxiv.org/abs/1601.04786.

[18] Julian Ivaldi. *An Introduction To Fractal Analysis*. Last accessed: January 6, 2026. 2021. URL: https://math.uconn.edu/wp-content/uploads/sites/3655/2021/05/Ivaldi_DRP_Presentation_Spring_2021-1.pdf.

[19] Christian Kuehn. *Introduction to Potential Theory via Applications*. 2008. arXiv: 0804.4689 [math.CA]. URL: https://arxiv.org/abs/0804.4689.

[20] Benoît Mandelbrot. *Les objets fractals : forme, hasard et dimension*. fre. Nouvelle bibliothèque scientifique 1482. Paris: Flammarion, 1975. ISBN: 2-08-210647-0.

[21] Alexis Monnerot-Dumaine. "The Fibonacci Word fractal". Feb. 2009. URL: https://hal.science/hal-00367972.

[22] José L. Ramírez and Gustavo N. Rubiano. "On the k-Fibonacci words". In: *Acta Universitatis Sapientiae, Informatica* 5.2 (2013), pp. 212–226.

[23] José L. Ramírez and Gustavo N. Rubiano. "Properties and Generalizations of the Fibonacci Word Fractal". In: *The Mathematica Journal* 16 (2014).

[24] José L. Ramírez, Gustavo N. Rubiano, and Rodrigo de Castro. "A Generalization of the Fibonacci Word Fractal and the Fibonacci Snowflake". In: *arXiv* (2018).

[25] Tomi Rossini. *An Introduction to Hausdorff and Box Counting Dimension*. University of Chicago. 2023. URL: https://math.uchicago.edu/~may/REU2023/REUPapers/Rossini.pdf.

[26] David Ruelle. *Thermodynamic formalism : the mathematical structures of equilibrium statistical mechanics /*. eng. Second edition. Cambridge mathematical library. Cambridge: Cambridge University Press, 2004. ISBN: 9780511617546. URL: https://proxy.bnl.lu/login?url=https://doi.org/10.1017/CBO9780511617546.

[27] Andreas Schief. "Separation properties for self-similar sets". In: *Proceedings of the American Mathematical Society* 122.1 (1994), pp. 111–115. ISSN: 0002-9939.

[28] Franck Sueur. *Integration and measure theory*. Lecture notes of the course on Integration and Measure Theory, Bachelor in Mathematics, University of Luxembourg, Summer semester 2024-25. 2025.

[29] Tangente. *Les nombres complexes*. Éditions Pôle, 2018.

[30] *The Perron–Frobenius Theorem*. Cours MPRI 2-33-1; Last accessed: January 6, 2026. 2024. URL: https://wikimpri.dptinfo.ens-paris-saclay.fr/doku.php?id=cours:c-2-33-1.

[31] Jiemeng Zhang, Zhixiong Wen, and Wen Wu. "Some properties of the Fibonacci sequence on an infinite alphabet". In: *The Electronic Journal of Combinatorics* 24.2 (2017), P2.52.

[32] M. Zinsmeister. *Formalisme thermodynamique et systèmes dynamiques holomorphes*. Panoramas et syntheses : no.4, 1996. Société mathématique de France, 1996. ISBN: 9782856290514. URL: https://books.google.lu/books?id=-OaoMQEACAAJ.