Jilly Kevo

# Deriving a functional equation for Riemann's Zeta function by visually exploring Mellin transforms throughout phase plots

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Under the supervision of

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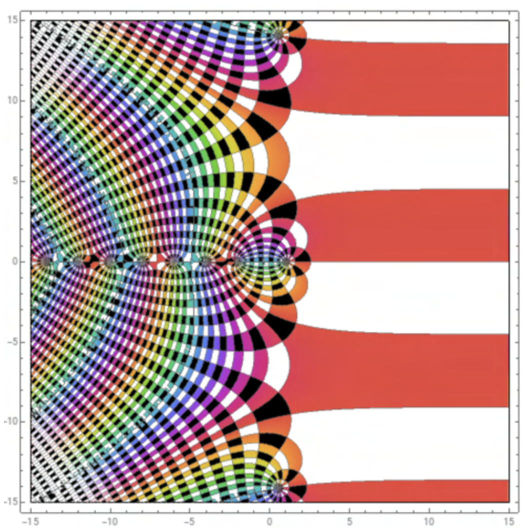
**Abstract**

The Mellin transform, just as any other analytic function of one complex variable, can be conveniently visualized by color-encoding its phase and mapping it onto a plane, creating a so-called *phase plot*, with all the important functional information perceivable at a glance. Since this kind of integral transform plays a crucial role in the theory of the Riemann Zeta function, it becomes possible to create a bridge between the functional equation of the latter and the properties of Mellin integrals - a bridge constructed upon illustrations. Towards the end, this functional equation morphs into the key for unraveling the mystery behind the value of $\zeta(0)$, while concomitantly performing an empirical residue calculation, again based on visual representations of $\zeta$.

I am very grateful for the continuous support I received from Dr. Thierry Meyrath during my journey through the realm of the complex.



The $\zeta$ function,
plotted in the square |Re(z)|< 15
and |Im(z)|< 15.

# Contents

*"Real mathematics must be justified as art, if it can be justified at all"* - G. H. Hardy

# 1 Introduction

## 1.1 What's this all about?

When associating a word (for example, one from the english language) to whatever specific idea that word might stand for, it is common for us to have some kind of thought-image in mind, a picture that helps us visualize and "make sense" of the (mostly rather abstract) concepts that are represented by the curvy little symbols we are all familiarized seeing on a daily basis. Imagine reading a book and not having any kind of cinema inside your head, wouldn't that be a fairly dull endeavor?! Texts in mathematics should make no exception to this, as long as a reasonable visualization *exists* (which is certainly more often the case than one would suspect at first). Luckily, complex functions (including $\zeta$ which will be of primary interest here) *can* be seen.

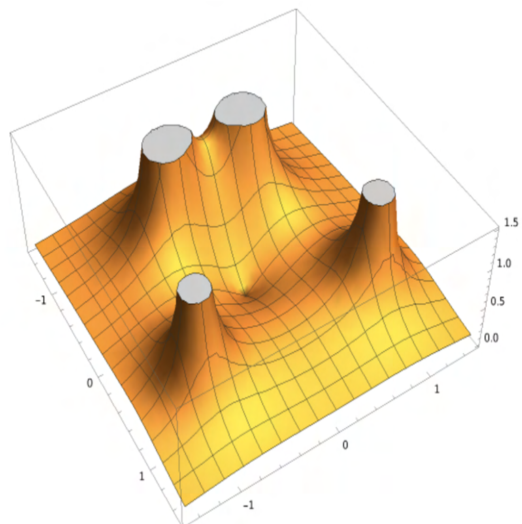
Now, in the case of an ordinary function of one real variable $g : E \subset \mathbb{R} \to \mathbb{R}$, it is not much of an ambiguity to find an appropriate visualization since the graph $G_g = \{(x, g(x)) \in \mathbb{R} \times \mathbb{R} : x \in E\}$ is a 1-dimensional curve that lives in 2-dimensional space. Matters are a little different for complex functions of a complex variable $f : D \subset \mathbb{C} \to \mathbb{C}$ with $z = x + iy$ and $(x, y) \in \mathbb{R}^2$ , the graph $G_f = \{(z, f(z)) \in \mathbb{C} \times \mathbb{C} : z \in D\}$ inhabiting *four*-dimensional space. Nonetheless, the situation is not quite as forlorn as it seems at first. A popular approach to study complex functions (and their global behavior) often involves so called *Riemann surfaces*, 2-dimensional surfaces (or 2-dimensional real analytic manifolds if you prefer) requiring only two real numbers ($x$ and $y$) to identify each point within them and therefore needing only 3-dimensional space to be visualized.

Especially in the case of multiple-valued functions (think for example of the complex square root), its different branches can live in one and the same Riemann surface, avoiding therefore the ambiguities of multiple-values by giving birth to a single-valued analytic function that is well-defined on this new domain. If you don't know what any of this means, don't worry about it since I won't be worrying about it either (meaning, none of this will show up again here). [1]

Another option that might be worth mentioning is the construction of the *modular surface* of $f(z)$, where the image point $f(z)$ of $z$ is represented by its distance from the origin $| f(z) |$, discarding the information of the angle $\arg(f(z))$ that is made with the real axis completely.

[1] If you are interested in knowing more about the fun topic of Riemann surfaces, you can consult Wegert [2012] for a nice introduction with pictures.

This structure is principally made by placing the complex plane horizontally in space and drawing, for every point $z$, another point at height $| f(z) |$ vertically above the first.

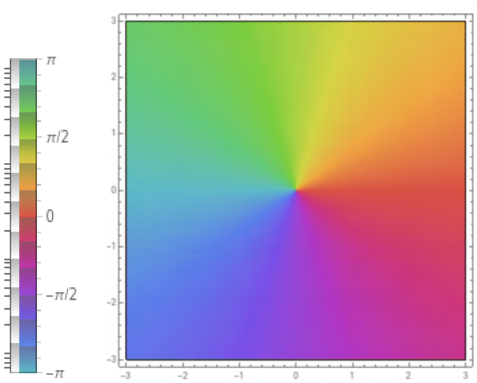



As an example, consider the modular surface of

$$f_1(z) = \frac{z}{1 + z + z^4}$$

, $f_1(z)$ having 4 poles and a zero at $z = 0$.

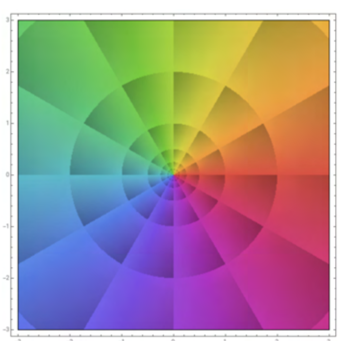
However, there are some ambiguities associated to this kind of representation. It would certainly be of great use to read as much vital functional information as possible off the plots, without having to look at it from different angles or reposition the whole construction just to be able to peak behind the towers of the poles to check if something is hiding behind. What we need is a way to display all the interesting elements (zeros, for example) in such a manner that all of them (or at least most of them) become perceivable at a glance. That's where *phase portraits* come into play. The idea behind is quite simple: instead of focusing on the modulus, one puts emphasis on the *phase*.



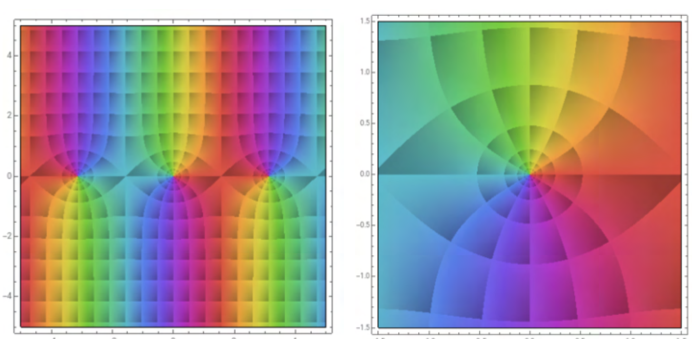

Since the phase of nonzero complex numbers resides entirely on the unit circle $\mathbb{T}$ and each point on a circle can be associated to a color (think of the color wheel used by artists), one obtains a 2-dimensional image of the function in question by translating its values into colors. At the left, you can see the phase portrait of the identity function $f(z) = z$.

Lets remark that contrary to the argument $\arg f$, which is defined up to an additive multiple of $2\pi$, the phase of $f$, outlined by $f/ | f |$, is single valued and therefore eligible for the 1-dimensional color-coded representation previously discussed. Although this enables us to immediately see where entire regions of the complex plane are going after application

of $f$ (giving us an insight about the general behavior of the function, all perceivable at an instant) this is not the end of the story. Besides the phase $\Phi_f : D \to \mathbb{T}, z \to \Phi(f(z))$ and the associated graph $P_f = \{(z, \Phi_f(z)) : z \in D\}$, it is still possible to integrate some information of the modulus into the phase plot by drawing contour lines of constant $\mid f \mid$. Usually, one adds a new (grey) function $g$, such that $g = \lceil \log \mid f \mid \rceil - \log \mid f \mid$, where $\lceil \log \mid f \mid \rceil$ denotes the *ceiling function* of $\mid f \mid$. For example, $\lceil \pi \rceil = 4$ and $\lceil 7 \rceil = 7$. Why using $\log \mid f \mid$ instead of $\mid f \mid$ alone you may ask? Well, first of all $\mid f \mid$ could have a really wide range, so "contracting" the information gives certainly a more compact representation. Secondarily, $\log \mid f \mid$ and $\arg f$ are *conjugate harmonic* functions. [2] To see the latter, start by assuming that $f$ is nonzero and analytic on some domain D. Therefore, both $f'(z)$ and $1/f(z)$ exist and must also be analytic, implying that $f'(z)/f(z)$ is analytic, too. Hence, the integral of $f'(z)/f(z)$, which just happens to be $\log \mid f(z) \mid + i \arg(f(z))$, is analytic as well and that at least in some simply connected subset of D. Since $\log \mid f(z) \mid + i \arg(f(z)) = \log(f(z))$, the conclusion follows.
Now, one can go even further and add contour lines of constant phase, creating an *enhanced* phase portrait. Let's remark that in the case of the functions depicted below, the contour lines of constant phase and modulus are *perpendicular* to each other, a property that can be traced back to their analyticity (in short, just as ordinary real and differentiable functions of one variable are approximately *straight lines* when one zooms very closely onto them, *complex* differentiable functions preserve infinitesimal *squares* ).



Enhanced phase plot of $f(z) = z$

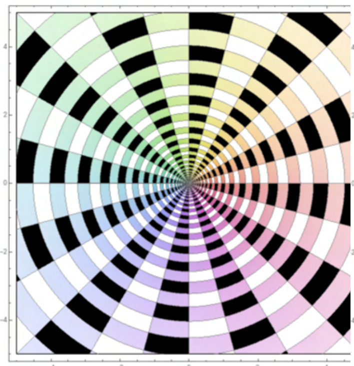
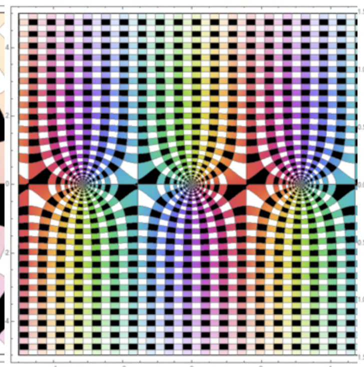
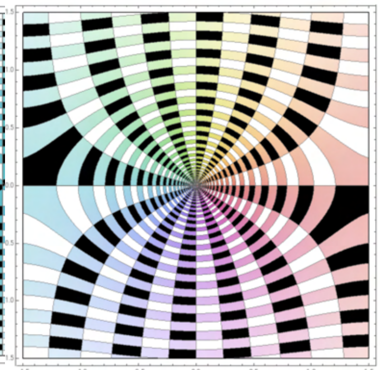


Complex sine (observe how the "eye" in the middle becomes very similar to $f(z) = z$ when zooming very closely onto it, giving already a hint at its Taylor series...)

My personal favorite when it comes to complex function visualization is however only obtained when adding, on top of the phase plot, a colored *mesh function* (in this case, "colors" alternating between black and white) to the plot which takes over the role of the

[2] A function is, roughly speaking, said to be harmonic in an open set if it satisfies the Laplace (differential) equation. If f = u + iv is analytic, then the two functions u and v are conjugate harmonic.

grey contour lines we previously encountered. The properties of the contour lines are all conserved and the coloring scheme is the same, however it does (sometimes) constitute a clearer representation of functions for our purposes, so I will freely alternate between these two kinds of representations. Notice moreover how the length of the modulus is translated by the intensity of the colors, in the sense that lighter colors stand for a larger absolute value and vice versa. Below are depicted the identity function and complex sine, for comparison.

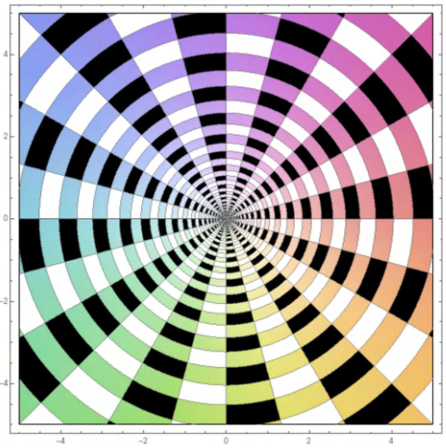
### 1.2 The experimental approach

Some last words before embarking on our exploration of phase plots. It is certainly clear that experiments cannot replace a proper mathematical argument, in the sense that they do not *prove* anything. Of course, empirically collected data can be quite suggestive and it would be unfair to declare that it did not lead to some impressive discoveries over the centuries, but it still remains a *tool* (as far as mathematics is concerned at least) and not an instance that we can rely on unconditionally , as it is often done in the natural sciences. This will be our way of "poking it" (the function) "with a stick" (the phase plots) in our laboratory (the computer).

## 2 Getting acquainted with phase plots

### 2.1 Zeros, poles and other singularities

Consider the identity function $f(z) = z$ again and pay special attention to the order of the colors when going around counterclockwise on a simple closed curve (say, a circle) that encloses the origin. Now, examine the inverse $g(z) = 1/z$ (depicted below). Clearly, the order of the colors has changed! This is because $f(z)$ has a *zero* at $z = 0$ whereby $g(z)$ has a *pole* at the origin.



Let's reformulate this as our first experimental observation (EO):

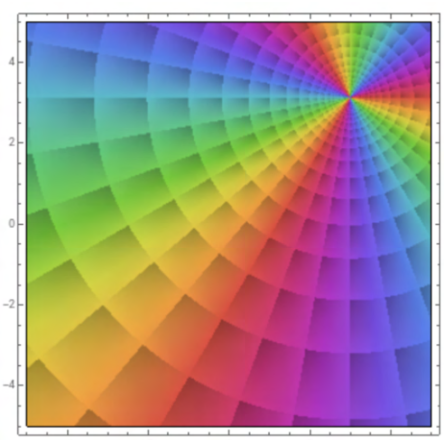
**EO1**: If a function has a zero at some point $z_0$, the order of colors, when going counterclockwise on a simple closed curve that encloses $z_0$ is, red, yellow, green, blue. If $z_0$ is a pole, this order is inverted.

(Note also how the colors become gradually stronger when moving away from the origin, which is quite natural since $g$ becomes very small for large $z$.)

To understand why this is the case, let $z_0$ be an isolated singularity ($z_0$ is therefore fixed in $\mathbb{C}$) of a function $f(z)$ and $(a_n)$ a sequence of complex numbers. Furthermore, assume that $f(z)$ is analytic on the punctured disk $0 <\mid z - z_0 \mid< r$ with $r \in \mathbb{R}$ such to have a *Laurent series* expansion around $z_0$:

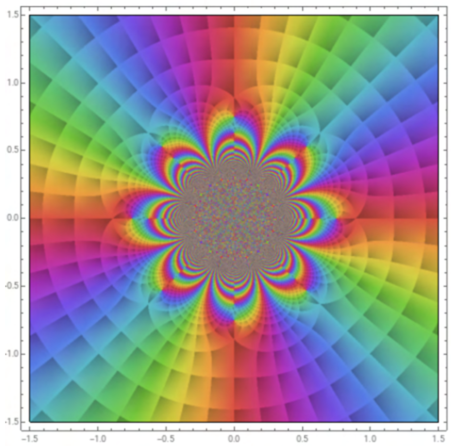
$$f(z) = \sum_{-\infty}^{\infty} a^k (z - z_0)^k$$

First of all, $z$ is a *zero* of $f$ if and only if $f(z) = 0$. By definition, a *pole* of $f$ is a zero of $1/f$. Therefore, for a meromorphic function (meromorphic at least in the neighborhood of $z_0$), there must be an integer $m$ such that $(z - z_0)^m f(z)$ is holomorphic, but $(z - z_0)^{m-1} f(z)$ is not. We call $z_0$ a pole of order $m$, provided that $m > 0$.



As an example, take $f(z) = 1/(z - 3 - \pi i)^3$. Since multiplying $f(z)$ by $(z - 3 - \pi i)^3$ would result in a holomorphic function (whereby multiplication by $(z - 3 - \pi i)^2$ would not), $z = 3 + \pi i$ is a pole of order 3. This can easily be read off the phase plot of $f(z)$ alone, only by considering the order of the colors (when going along a simple closed curve containing $z = 3 + \pi i$ of course) one concludes that $z = 3 + \pi i$ must be a pole and by counting the number of times one particular isochromatic line appears (say for instance blue) when going around that curve that it must, in fact, be of order 3.

To see that matters are a little different for *essential singularities*, let's examine the function $f(z) = \sin(1/z^4)$ and it's Laurent series:



$$\sin(1/z^4) = 1/z^4 - 1/3!z^{12} + 1/5!z^{20} - 1/7!z^{28} + ...$$

Clearly, the series expansion of $f(z)$ contains an infinite number of negative powers of $z$. When taking a look at the phase plot, one immediately sees that the functions exhibits a rather wild behavior in the neighborhood of such points, a behavior that does not bear much resemblance to the one of the other isolated singularities encountered before. In fact, what one witnesses is the following:

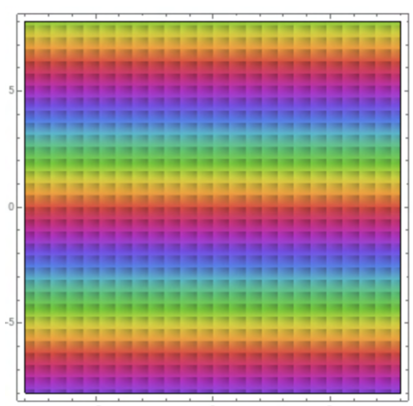
**EO2**: Any neighborhood of an essential singularity of an analytic function intersects infinitely many isochromatic lines of one and the same color infinitely often.

This arises from the following classic result:

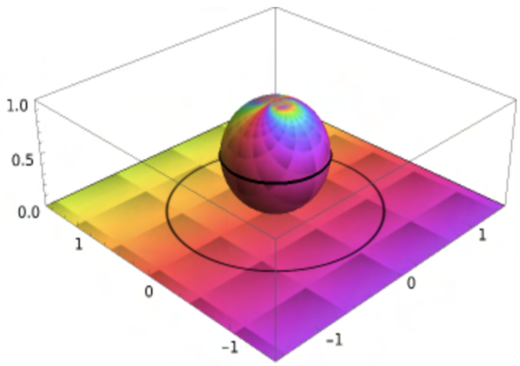
**Casorati-Weierstrass theorem**: Let $f$ be holomorphic in $\mathbb{C} \setminus \{z_0\}$ and $z_0$ an essential singularity. If V is any neighborhood of $z_0$, then $f(V \setminus \{z_0\})$ is dense in $\mathbb{C}$.

Or, the considerably stronger statement:

**Great Picard's theorem**: Any analytic function with an essential singularity at $z_0$ takes on all possible complex values, with at most a single exception, infinitely often in any punctured neighborhood of $z_0$.

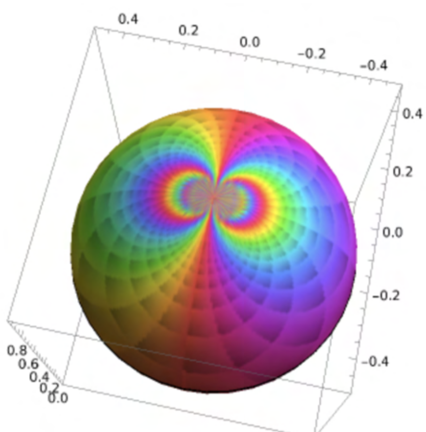


This is all perfectly fine, however there is still another issue to settle. Contrary as to what one would expect, it is not the case that such singularities are always perceivable at a glance. Even innocent looking functions like $f_2(z) = e^z$ (depicted on the left) or $f_3(z) = \sin(z)$ posses essential singularities, despite their phase plots being of seemingly much different nature than the one from our previous examples. How can this be?




To unravel the mystery, let's jump a dimension higher and take a look at the familiar exponential function, plotted on the *Riemann sphere*. The key idea behind is that the *extended complex plane* $\mathbb{C} \cup \{\infty\}$ is projected onto a sphere, visualizing not only any ordinary point of the complex plane but also the *point at infinity*. This, it turns out, is *precisely* where this essential singularity kept hiding itself.

(Note how the unit circle from the plane (represented in black) is mapped onto the equator of the sphere. ) Now, to verify that this all is indeed just like it seems to be, it suffices to confirm that $\lim_{z\to\infty} e^z$ does not exist.




If the limit would exist, then the function would have to converge to one and the same limit point along any path. However, by fixing $x$ (from $z = x + iy$) and letting $y$ go off to infinity, it is clear , with Euler's fabulous formula, that

$$e^z = e^x(\cos(y) + i\sin(y))$$

can't converge to anything.

Interestingly, the exponential function does not form an exception with its essential singularity at infinity because, it turns out, that this is the case for every entire function that is not rational.

Alright then, but what happens with the zeros of $f(z) = \sin(1/z^4)$? A classic statement tells us that if a nonzero function is analytic in a domain D and K is a compact subset of D, then the number of zeros that function has in K must be finite. However, this does not mean that the function can't possess infinitely many zeros in D itself. In this case, the zeros must accumulate at the "boundary" of D, or, as it is the case for our example $f(z) = \sin(1/z^4)$ (the function being analytic on $\mathbb{C} \setminus \{0\}$ and having infinitely many zeros) all zeros accumulate at a single point, namely the origin.

Lets remark that *removable singularities* can not be seen at all in phase plots.

### 2.2 Uniqueness principle for phase plots

To be certain that the results we will be deriving throughout phase plots in subsequent sections do really apply to the function *of interest* and not to other entities, we are in need of an uniqueness principle that guarantees us that if two analytic functions share the same phase plots, they are substantially identical. This is easily seen to be true by considering two analytic (at least in some open subset) functions $f$ and $g$ verifying this condition. Naturally, their phases must be of equal nature, meaning that $g/f$ is real and positive (provided $f$ has no zeros in the subset in question). Or, in other words:

**Theorem: Uniqueness principle for phase plots**

If $f$ and $g$ are analytic in a domain D and have identical phase plots on some open subset U of D, $U \subset D$, then

$$g(z) = cf(z)$$

for all $z \in D$ and $c$ a positive, real constant.

## 3 Deriving the functional equation for Riemann's Zeta function

### 3.1 The Zeta function as a Mellin transform

What sparked Riemann's initial interest in studying the (complex valued) Zeta function that now bears his name, is it's surprising connection to the prime numbers. A result due to Euler states that:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p\ prime} \frac{1}{1 - p^{-z}}$$

However, since $\mid n^z \mid = n^{Re(z)}$, this expression only converges (absolutely) for Re(z)> 1. In order to obtain an expression (a *functional* equation) for $\zeta(z)$ that lives on a larger domain than the half-plane it currently resides, one can make use of *analytic continuation.* Zeta being analytic on its current domain of definition, the uniqueness principle guarantees that it has *at most* one such extension. Furthermore, since analytic functions are completely determined by their values on an arbitrarily small open set, finding a function that *coincides* with Zeta for Re(z)> 1 but is defined on a larger domain would fulfill the task (whereby Weierstrass's disk chain method would certainly not be so straightforward). To this end, start [3] with the very definition of the gamma function (for $t \in \mathbb{R}_+$ and the principal branch of $t^z = \exp[z \log(t)]$):

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \qquad \mathrm{Re}(z) > 0$$

By examining the more general family of integrals given by

$$I_n = \int_0^{+\infty} t^{z-1} e^{-nt} dt$$

for $n$ a natural number, and changing variables to $u = nt$ (thus obtaining $dt = du/n$), we are left with:

$$I_n = \int_0^{+\infty} \frac{u}{n}^{z-1} e^{-u} \frac{du}{n} = \frac{n^{1-z}}{n} \int_0^{+\infty} u^{z-1} e^{-u} du = \frac{\Gamma(z)}{n^z}$$

Since $u$ is just a dummy-variable of integration, we might as well change it back to $t$. Alright. Now, to conjure the Zeta function it suffices to sum over the positive integers

$$\sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{I_n}{\Gamma(z)}, \qquad \mathrm{Re}(z) > 1$$

and by interchanging [4] sums and integrals, one arrives at the following:

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{+\infty} t^{z-1} \sum_{n=1}^{\infty} e^{-nt} dt = \frac{1}{\Gamma(z)} \int_0^{+\infty} t^{z-1} \frac{1}{e^t - 1} dt, \qquad \mathrm{Re}(z) > 1$$

The last step justifies itself because $\sum_{n=1}^{\infty} e^{-nt} = 1/(e^t - 1)$ is a *geometric series* (remember, $t \in \mathbb{R}_+$ ). This new entity with which we are confronted now is the so called *Mellin transform* of $1/(e^t - 1)$:

**Definition (Mellin transform):** Given a function $f(t)$ defined on the positive real axis $0 < t < \infty$, the Mellin transform $\mathcal{M}$ of $f(t)$ maps the function $f(t)$ into the complex-valued function $F(z)$ defined by the relation

$$\mathcal{M}[f; z] \equiv F(z) = \int_0^{+\infty} t^{z-1} f(t) dt$$

In our case, $F(z) = \zeta(z)\Gamma(z)$:

$$\zeta(z)\Gamma(z) = \mathcal{M}\left[\frac{1}{e^t - 1}; z\right]$$

[3] Parts of this derivation can be found in [9]

[4] That this is a valid step can be seen by applying the dominated convergence theorem

Now, one has certainly many possibilities about how to proceed. Instead of trying to find a convenient combination of functions that extend $\zeta(z)$ directly, we will keep focusing on $F(z)$. By rewriting it a little as (still for Re(z)> 1)

$$\zeta(z)\Gamma(z) = \int_0^1 \frac{t^{z-1}}{e^t - 1}dt + \int_1^\infty \frac{t^{z-1}}{e^t-1}dt + \frac{1}{z-1} - \frac{1}{z-1} + \frac{1}{2z} - \frac{1}{2z} \tag{1}$$

and realizing that $-1/(z-1)$ and $1/z$ are just

$$-\frac{1}{z-1} = -\int_0^1 t^{z-2}dt, \qquad \text{Re}(z) > 1$$

and

$$\frac{1}{z} = \int_0^1 t^{z-1}dt, \qquad \text{Re}(z) > 0$$

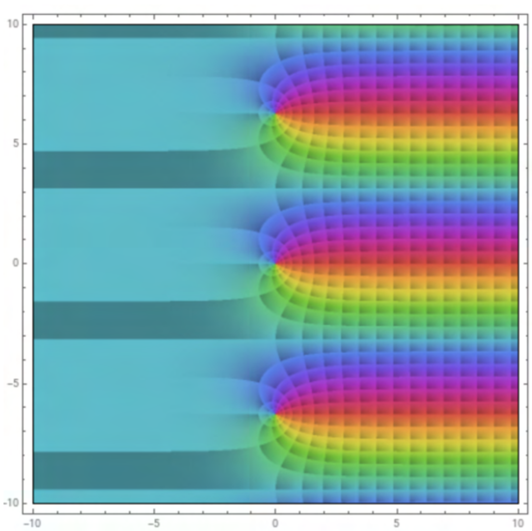
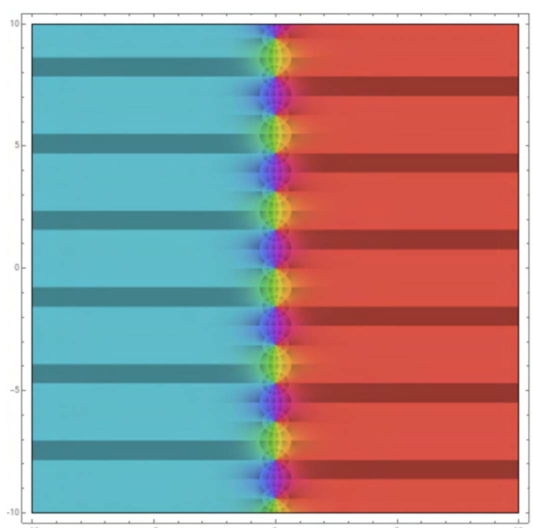
we can immediately absorb these two terms into the left most integral of (1) without having to worry about convergence, provided Re(z)> 1:

$$\zeta(z)\Gamma(z) = \int_0^1 (\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2})t^{z-1}dt + \int_1^\infty \frac{t^{z-1}}{e^t-1}dt + \frac{1}{z-1} - \frac{1}{2z} \tag{2}$$

Alright. Now, to get around the ambiguity of convergence, it will prove itself crucial to take a closer look at the function $f(z) = 1/(e^t - 1)$ on its own before continuing with our investigation.

## 3.2 A useful series from hyperbolic geometry

By comparing the phase plots of the two functions $f(z) = 1/(e^z - 1)$ (left) and $g(z) = \coth(z)$ (right), where $g(z)$ is the hyperbolic cotangent function, it seems promising to search for some kind of relationship:

Remembering that coth$(t)$ is defined as coth$(t) = (e^{2t}+1)/(e^{2t}-1)$ we obtain $1/(e^t-1) = \frac{1}{2}(\coth(\frac{t}{2})-1)$. Now, it is known that coth$(t)$ can be expanded in the following series:

$$\coth(t) = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{\pi^2 n^2 + t^2}$$

And a little algebra gives us:

$$1/(e^t-1) = \frac{1}{2}(\coth(\frac{t}{2})-1) = \frac{1}{2}(\frac{2}{t} + \sum_{n=1}^{\infty} \frac{t}{\pi^2 n^2 + \frac{t^2}{4}} - 1) = \frac{1}{t} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2t}{4\pi^2 n^2 + t^2}$$

Inserting this into the leftmost integral of (2) yields:

$$\zeta(z)\Gamma(z) = \int_0^1 \left( \sum_{n=1}^{\infty} \frac{2t}{4\pi^2 n^2 + t^2} \right) t^{z-1} dt + \int_1^{\infty} \frac{t^{z-1}}{e^t - 1} dt + \frac{1}{z-1} - \frac{1}{2z} \qquad (3)$$

At a first glance, it might seem as if we have complicated matters even further. However, it turns out, we are now in a *much* more favorable situation. First of all, $1/(z-1)$ and $1/2z$ are meromorphic functions on $\mathbb{C}$ (they are analytic everywhere except at $z = 1$ and $z = 0$, correspondingly). Furthermore, the right most integral (where $t$ ranges from 1 to $\infty$) is *entire* and as far as the other one is concerned, we can interchange sums and integrals to obtain

$$\int_0^1 \left( \sum_{n=1}^{\infty} \frac{2t}{4\pi^2 n^2 + t^2} \right) t^{z-1} dt = 2 \sum_{n=1}^{\infty} \int_0^1 \frac{t^z}{4\pi^2 n^2 + t^2} dt$$

which convergences [5] for Re(z) $> -1$. Hence, (3) also lives in the part of the complex plane for which Re(z) $> -1$. This is amazing because it allows us to rewrite $1/(z-1)$ and $1/2z$ as

$$\frac{1}{z-1} = -\int_1^{+\infty} t^{z-2} dt, \qquad \mathrm{Re}(z) < 1$$

and

$$-\frac{1}{2z} = \frac{1}{2} \int_1^{+\infty} t^{z-1} dt, \qquad \mathrm{Re}(z) < 0$$

Therefore, as long as $-1 <$Re(z) $< 0$, all these terms can reside in one and the same equation:

$$\zeta(z)\Gamma(z) = \int_0^1 \left( \sum_{n=1}^{\infty} \frac{2t}{4\pi^2 n^2 + t^2} \right) t^{z-1} dt + \int_1^{\infty} \left( \frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt$$

[5] It is easily seen that for Re(z) $\geq 0$ the integrals must converge and for Re(z) $= -1$ diverge, forcing the ones for Re(z) $< -1$ to diverge as well (by comparison). For $-1 <$Re(z) $< 0$, rewrite the exponent as $t^{-1/n}$ for $n \in ]1; +\infty)$ and notice that $\int_0^1 t^{-1/n} dt$ is an upper bound

Inserting the series expression from before into the right most integral, taking the two terms together and interchanging sums and integrals, we arrive at:

$$\zeta(z)\Gamma(z) = \sum_{n=1}^{\infty} \int_0^{+\infty} \left( \frac{2t}{4\pi^2 n^2 + t^2} \right) t^{z-1} dt = 2 \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{t^z}{4\pi^2 n^2 + t^2} dt$$

Or, by using Mellin transform notation:

$$\zeta(z)\Gamma(z) = 2 \sum_{n=1}^{\infty} \mathcal{M}[g_n(t); z]$$

where $g_n(t) = t/((2n\pi)^2 + t^2)$.

### 3.3 Visually exploring Mellin transforms

Now, before continuing, it will prove useful to first investigate the Mellin transform of $h_0(t) = 1/(1+t)$. Changing variables to $t + 1 = 1/(1-u)$, we obtain $t = u/(1-u)$ and $\mathrm{d}t = 1/(1-u)^2 \mathrm{d}u$ which yields

$$\mathcal{M}[1/(1+t); z] = \int_0^1 (\frac{u}{1-u})^{z-1} (1-u) \frac{1}{(1-u)^2} du$$

or equivalently

$$\mathcal{M}[1/(1+t); z] = \int_0^1 u^{z-1} (1-u)^{-z} du$$

But wait a second, this is just the Beta function! Having recognized this, we can immediately deduce that

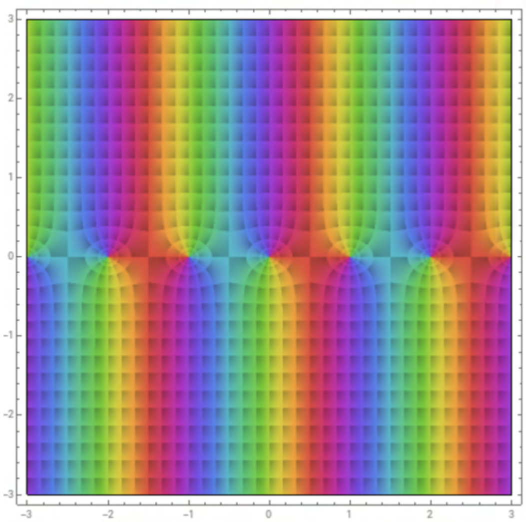
$$\mathcal{M}[1/(1+t); z] = \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1)} = \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

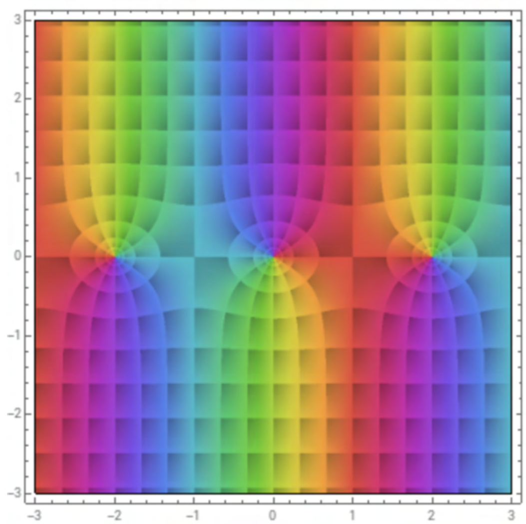
The last equality follows from Euler's fabulous reflection formula. So, what is it precisely that distinguishes our initial function $g_n(t) = t/((2n\pi)^2 + t^2)$ from $h_0(t)$? Well, first of all, the variable $t$ in $h_0(t)$ has been squared, than $h_1(t) = 1/(1+t^2)$ has been multiplied by $t$ and, lastly, $t$ in $h_2(t) = t/(1+t^2)$ has been scaled by $1/(2n\pi)$ (and $h_{3,n}(t) = \frac{t/(2n\pi)}{1+t^2/(2n\pi)^2}$ multiplied by $1/(2n\pi)$ such that $g_n(t) = \frac{1}{2n\pi} h_{3,n}(t)$). Now, instead of integrating any of the $h_k(t)$'s ($k = 1, 2, 3$) directly, we will determine $\mathcal{M}[g_n(t); z]$ by means of an *experimental* approach based on phase plots. But in order to deduce any kind of useful information from the plots *without* explicitly knowing the Mellin transform of any of the $h_k(t)$'s, the following few lines of *Mathematica* code will be of great use:

```
ComplexPlot[ MellinTransform[1/(1+t), t, z],
```

{z, -3 -3 I, 3+3I}, Sequence[
PlotLegends → Automatic, ColorFunction → ”CyclicLogAbsArg”,
Method → {”RasterSize” → 800}, ImageSize → 200]]

Now, while plotting $\mathcal{M}[h_0(t); z]$ (left) and $\mathcal{M}[h_1(t); z]$ (right) side by side, one can already deduce several things just by inspection:





Clearly, $\mathcal{M}[h_0(t); z]$ has simple poles all along the integers, whereby the poles of $\mathcal{M}[h_1(t); z]$ are scattered more sparsely, residing only at the even integers (and zero). This already suggests that the argument of $\mathcal{M}[h_1(t); z]$ must be divided by 2, as compared to $\mathcal{M}[h_0(t); z] = \frac{\pi}{\sin(\pi z)}$. In other words:

$$\mathcal{M}[h_1(t); z] = k_1 \frac{\pi}{\sin(\frac{\pi}{2} z)}$$

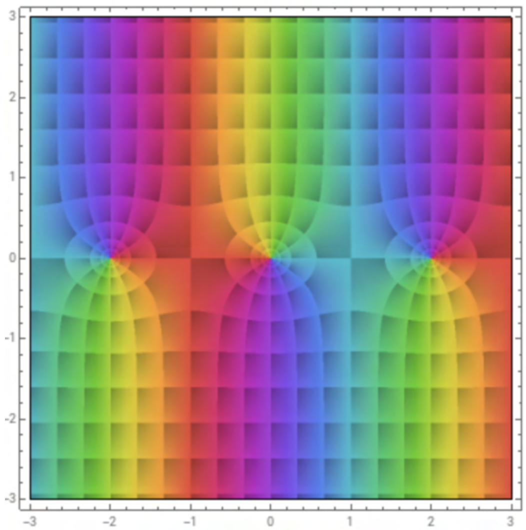
where $k_1$ is a constant that needs to be determined. However, we can already put some restrictions on $k_1$:

*$k_1$ must be real and positive since the placements and orders of the colors around the poles in the phase plots of $\mathcal{M}[h_0(t); z]$ and $\mathcal{M}[h_1(t); z]$ are identical.*

This is because multiplication by some $k_1$ that fails to fulfill these criteria would result in modifying the *phase* of $\mathcal{M}[h_1(t); z]$(and therefore the colors in its phase plot), which we do not want. To launch into the search, start by defining the following quantity:

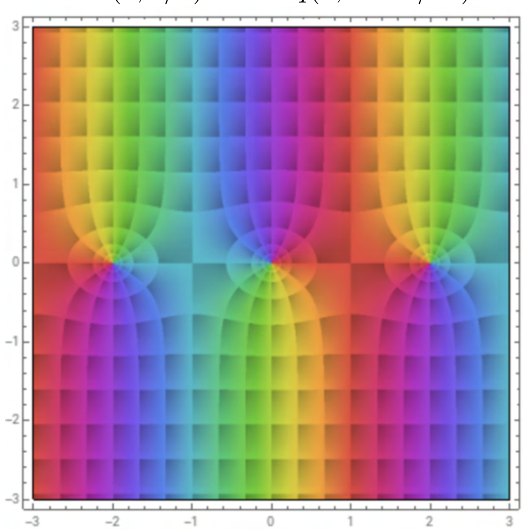
$$q(z, k) = \mathcal{M}[h_1(t); z] - k_1 \frac{\pi}{\sin(\frac{\pi}{2} z)}$$

Clearly, if the right $k_1$ is found than $q(z,k) = 0$. However, the interval $(0,\infty)$ is nevertheless *very* large and we cant possibly test every number in it, especially since nothing guarantees us that $k_1$ is not *irrational*, leaving us with an *uncountable* infinity of potential candidates. What we need is therefore a systematic way of extracting $k_1$ out of $q(z,k)$. To this end, start by picking any number in $(0,\infty)$, say 5, and plotting $q(z,k=5)$:



In contrast to $\mathcal{M}[h_1(t);z]$, the colors here are undoubtedly arranged differently. It seems just as the whole plot of $\mathcal{M}[h_1(t);z]$ has been rotated by an angle of $\pi$ in the clockwise sense. As far as $q(z,k=5)$ is concerned, we can immediately deduce that the latter is now a *negative* multiple of $\mathcal{M}[h_1(t);z]$ since this seemingly new coloration reveals that to the phase of every $z$ has been added some odd multiple of $\pi$. In other words, our choice of $k_1$ is an *overestimate*. This allows us already to discard every number in $[5,\infty)$.

Next, in the interval that is left, that is in $(0;5)$, take the *midpoint*, which is $5/2$ here, and plot $q(z,t=5/2)$. Since $q(z,t=5/2)$ looks exactly the same as $q(z,k=5)$, I'll omit it. However, this means that $k=5/2$ still overshoots $k_1$ and the interval we need to comb shrinks to $(0;5/2)$. After applying this technique two more times, one ends up with the interval $(0;5/8)$ and a $q(z,k=5/16)$ that wants to be tested:



Even though $q$ is still nonzero for $k=5/16$, this situation is a little different. Here, $q$ is a *positive* multiple of $\mathcal{M}[h_1(t);z]$, meaning that $k=5/16$ *underestimates* $k_1$. Hence, we'll discard $(0;5/16]$ and focus on $(5/16;5/8)$. After two more iterations, we find that $k_1$ must hide somewhere in $(15/32;35/64)$. Since $15/32 = 0,46875$ and $35/64 = 0,546875$, a reasonable guess for $k_1$ would be $1/2$. Alright then, let's plot $q(z,t=1/2)$ (not depicted).

And see there, we obtain a completely black screen, meaning that $q$ is indeed 0 for that particular value of $k_1$ (remember that z = 0 is the only complex number without a defined angle). Hence:

$$\mathcal{M}[h_1(t); z] = \frac{1}{2}\frac{\pi}{\sin(\frac{\pi}{2}z)}$$

More generally, the algorithm we have used so far can be reformulated as follows:

1. Let $\mathcal{M}[H(t); z] = k$ G(z) and $k \in (a; b) \subseteq \mathbb{R}$, $a < b$. Define $q$ such that $q(z, k) = \mathcal{M}[H(t); z]-$ kG(z) and pick some number n in $(a; b)$.

2. Draw the phase plot of $q(z, n)$ and compare it with the phase plot of $\mathcal{M}[H(t); z]$.

3. Three cases can present themselves:

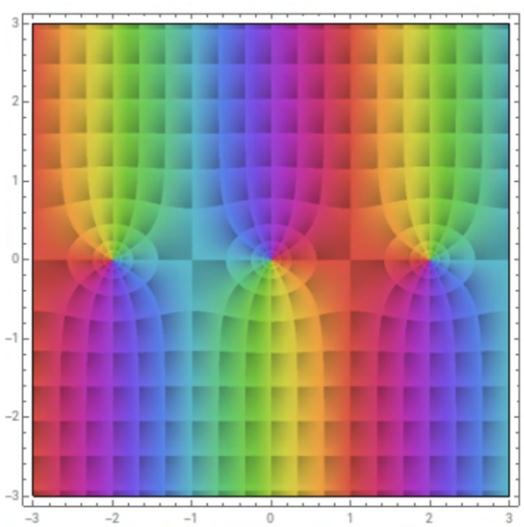
   (a) The chosen n is an overestimate. Discard the interval $[n; b)$ and continue with $(a; n)$. Define a new $n_2$ such that $n_2 = (a + n)/2$ is the midpoint of $(a; n)$.

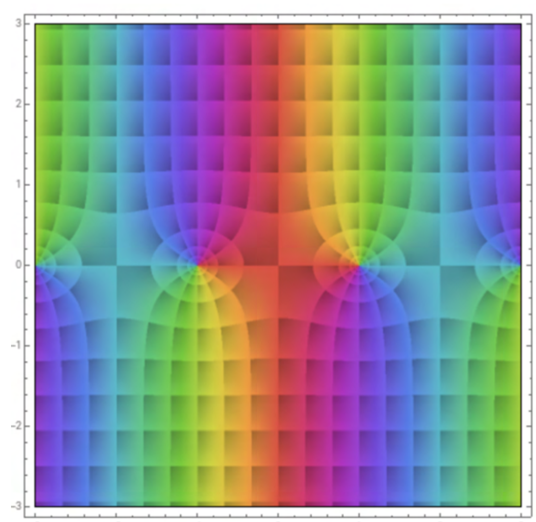
   (b) The chosen n is an underestimate. In that case, carry on with $(n; b)$ and define a new $n_2$ such that $n_2 = (n + b)/2$.

   (c) The chosen $n$ is equal to the desired $k$ and $q(z; n) = 0$. The algorithm ends here.

4. In case of (a) and (b), repeat steps 2. and 3. Either the algorithm ends at c) or shrinks the interval in question to arbitrary size, giving an approximate value for $k$ (in case of irrational k, for example).

Let's remark that if $b = +\infty$ in case 2.b), a new $n$ must be chosen such that $n$ overestimates $k$ (similar applies to 2.a) if $a = -\infty$). Alright. Now, in order to make progress in our initial task, lets represent $\mathcal{M}[h_1(t); z]$ (left) and $\mathcal{M}[h_2(t); z]$ (right):





What one sees is the following:

*The frequencies of the poles of $\mathcal{M}[h_1(t); z]$ and $\mathcal{M}[h_2(t); z]$ are identical. However, the whole plot of $\mathcal{M}[h_2(t); z]$ has been shifted by 1 to the left, as compared to $\mathcal{M}[h_1(t); z]$.*

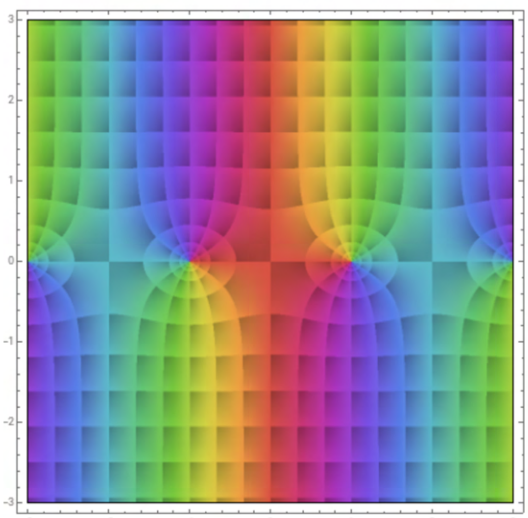
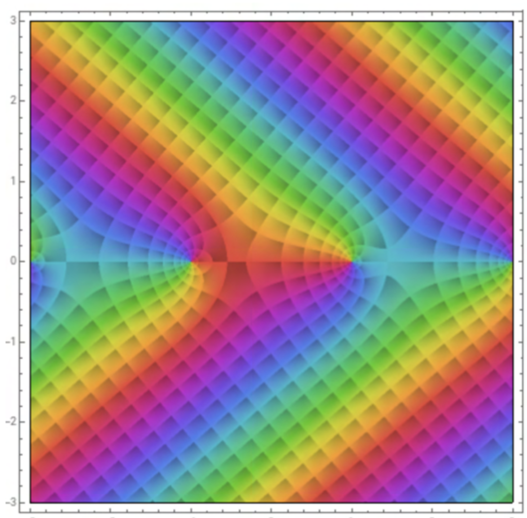
Or, this can be translated in mathematics by replacing $z$ in $\mathcal{M}[h_1(t); z]$ by $z + 1$, leading to:

$$\mathcal{M}[h_2(t); z] = k_2 \frac{1}{2} \frac{\pi}{\cos(\frac{\pi}{2} z)}$$

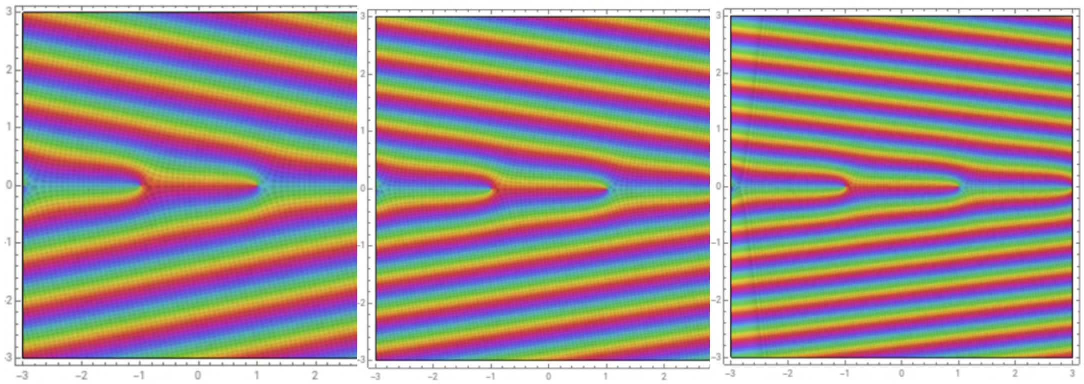
Again, $k_2$ must be real and positive. One can verify that by applying the k-algorithm above, $k_2 = 1$. Thus:

$$\mathcal{M}[h_2(t); z] = \frac{1}{2} \frac{\pi}{\cos(\frac{\pi}{2} z)}$$

As a last step, we will search for $\mathcal{M}[h_{3,n}(t); z]$. However, by considering that $h_{3,n}$ is given by $h_{3,n}(t) = \frac{t/(2n\pi)}{1+t^2/(2n\pi)^2}$ (depending therefore on $n$) and taking a glance at its phase plot (at the right, plotted for $n = 1$), one can already see that it considerably differs from $\mathcal{M}[h_2(t); z]$:

In order to get an impression of the behavior of $\mathcal{M}[h_{3,n}(t); z]$, below are depicted several $\mathcal{M}[h_{3,n}(t); z]$'s for increasing (from left to right) $n$ (n = 500, n = 5000 and n = 100000):



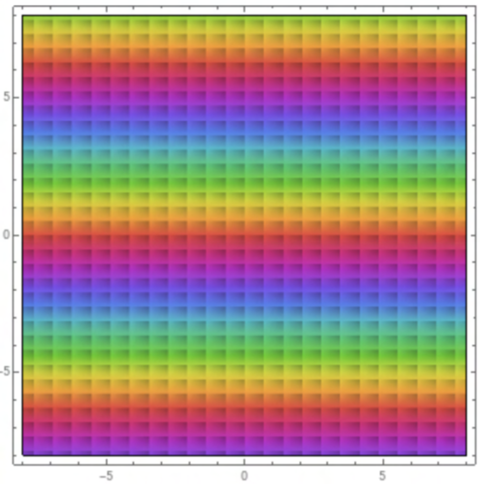

Even though the plots appear to be very similar at first, it can nevertheless be said that whenever $n$ increases, the isochromatic color lines are progressively becoming more *horizontal* (although they do so very slowly). However, as compared to $\mathcal{M}[h_2(t); z]$, the pole-properties (like number and position) are conserved. Now, whenever parallel lines of isochromatic colors are involved, it might be worth searching for a relation with an *exponential* function $E_n(z)$, such that

$$\mathcal{M}[h_{3,n}(t); z] = E_n(z) k_3 \frac{1}{2} \frac{\pi}{\cos(\frac{\pi}{2} z)}$$

and where $E_n(z)$ has not necessarily base $e$. Furthermore, since the number of such lines increases in the square |Re(z)|< 3 and |Im(z)|< 3 whenever $n$ increases (this is true for an *arbitrary* square, not just the one we considered above), the function $E_n(z)$ will also be dependent on $n$. But where does $n$ reside inside of $E(z)$? After all, it could be that the exponential function we are searching is similar to $(a/n)^z$, $(a)^{nz}$ or even $(an^2)^z$. To see that a reasonable guess would be

$$E_n(z) = e^{z \log(an)}$$

where $a$ is real and positive, one needs to take the *thickness* of the isochromatic color lines into account. In our case, the number of such lines is clearly increasing with growing $n$, meaning that they become gradually thinner. This already allows us to exclude entities like $(a/n)^z$, or in general anything that assigns a negative exponent to $n$, because this would allow the strips to grow thicker with increasing $n$, opposing our situation. Furthermore, we can rule out $(a)^{-nz}$ and alike since those function do not even preserve the characteristic horizontal strips. Also, since the "thinning" of the strips is pretty slow (remember, even for $n = 100000$, there were only a handful more that appeared in the square), this procedure would be considerably accelerated in the case of $(an^c)^z$ with $c > 0$ and even more with $(a)^{nz}$, so those cases are rather unlikely going to get us any further. What we'll do now is test our conjecture. In case of preposterous results, one can always go back and modify the initial assumptions with the aid of newly acquired information.

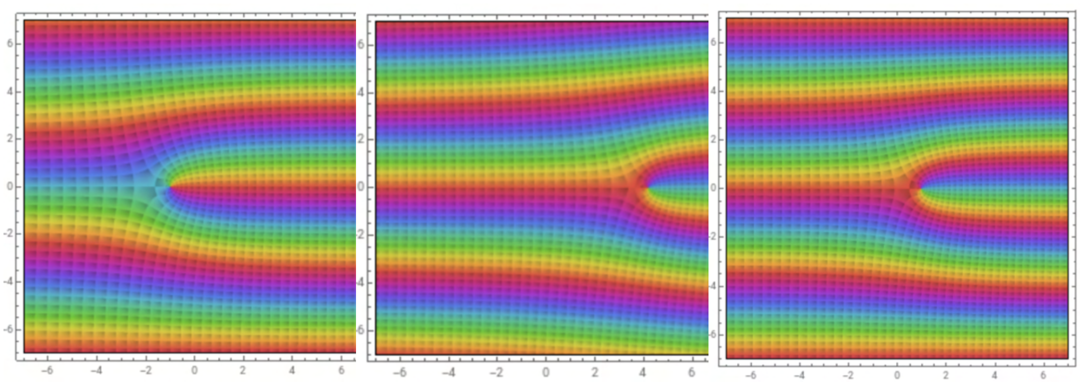


If we focus our attention now on

$$Q(z,a) = \mathcal{M}[h_{3,n}(t);z]\cos(\frac{\pi}{2}z) - e^{z\log(an)}$$

we'll just need to find an $a$ such that the plot of $Q(z;a)$ is also an exponential function, easily recognized by its characteristic horizontal isochromatic lines. For the moment, we can ignore the constants $k_3$ and $2/\pi$. At the left, you can see the plot of the classic function $e^z$.

Below are depicted several $Q(z;a)$'s for different values of $a$ (a = 4, a = 7 and a = 10, while keeping $n$ at n = 1):



A reasoning similar to the k-algorithm can be applied to firstly approximate $a$ and then find that it must indeed equal $2\pi$. With that in mind, we arrive at

$$E(z) = e^{z\log(2\pi n)} = (2\pi n)^z$$

and therefore

$$\mathcal{M}[h_3(t);z] = k_3(2\pi n)^z \frac{1}{2}\frac{\pi}{\cos(\frac{\pi}{2}z)}$$

By application of the k-algorithm, $k_3 = 1$. Or, since $g_n(t) = t/((2n\pi)^2+t^2) = 1/(2n\pi)h_{3,n}$, we can finally come back to $\zeta$:

$$\zeta(z)\Gamma(z) = 2\sum_{n=1}^{\infty}\int_0^{+\infty}\frac{t^z}{4\pi^2n^2+t^2}dt = 2\sum_{n=1}^{\infty}\mathcal{M}[1/(2n\pi)h_{3,n}(t);z]$$

$$= 2\sum_{n=1}^{\infty} 1/(2n\pi)\mathcal{M}[h_{3,n}(t);z] = 2\sum_{n=1}^{\infty} 1/(2n\pi)(2\pi n)^z \frac{1}{2}\frac{\pi}{\cos(\frac{\pi}{2}z)}$$

and with $\sum_{n=1}^{\infty} 1/n^{1-z} = \zeta(1-z)$ (still for $-1 < \mathrm{Re(z)} < 0$) we find

$$\zeta(z)\Gamma(z) = \frac{2^{z-1}\pi^z\zeta(1-z)}{\cos(\frac{\pi}{2}z)},$$

which can be transformed into the well known expression by remembering that $\sin(\pi z) = 2\cos(\frac{\pi z}{2})\sin(\frac{\pi z}{2})$and $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$:

$$\zeta(z) = \frac{2^{z-1}\pi^z\zeta(1-z)}{\Gamma(z)\cos(\frac{\pi}{2}z)} = \frac{\Gamma(1-z)\sin(\pi z)2^{z-1}\pi^z\zeta(1-z)}{\pi\cos(\frac{\pi}{2}z)}$$

And by just a little more effort:

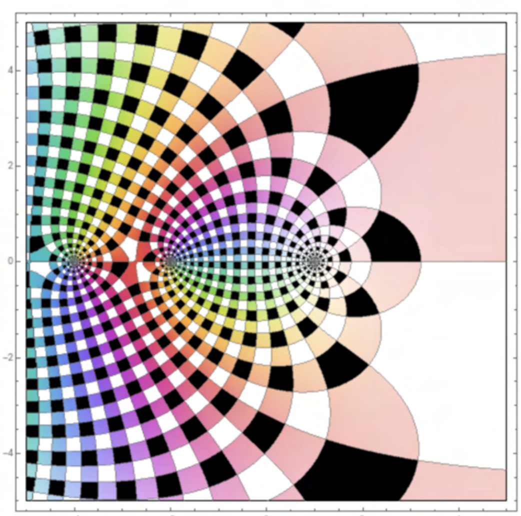
$$\zeta(z) = \frac{2\Gamma(1-z)\sin(\pi z)2^{z-1}\pi^z\zeta(1-z)\sin(\frac{\pi z}{2})}{\pi\sin(\pi z)}$$

we finally find:

$$\zeta(z) = \Gamma(1-z)2^z\pi^{z-1}\sin(\frac{\pi z}{2})\zeta(1-z)$$

# 4 The mystery of $\zeta(0)$

## 4.1 A tricky integral



Since we are now in the possession of a new and quite *powerful* representation for $\zeta$ ( certainly more handy than $\zeta(z) = \sum_{n=0}^{n=\infty} 1/n^z$ is) , let's investigate this just a little more. By modifying the phase plot command a little in order to introduce a colored (here, alternating black and white) mesh function, one obtains the following representation in the square $|\mathrm{Im(z)}| < 5$ and $|\mathrm{Re(z)}| < 5$, depicted on the right. What one sees on the horizontal line to the left of the origin are the first two of the so-called *trivial zeros*, all of order one and equally spaced. Surprisingly, this image seems to suggest that $\zeta(0) \neq 0$.

The striking nature of this observation is even more reinforced when inspecting the functional equation

$$\zeta(z) = 2^z\pi^{z-1}\sin(\frac{\pi z}{2})\Gamma(1-z)\zeta(1-z)$$

more closely and the fact that $\sin(0) = 0$. But before exploring this any further, lets stress upon the following observation:

**EO3**: $\zeta(z)$ has a simple pole at $z = 1$.

This is also easily seen to be true by examining the functional equation. However, there is even more that can be read off: since the region around $z = 0$ is light-blue (corresponding to complex numbers with a phase of $\pi$), it doesn't seem too unreasonable to assume that:

**EO4**: Whatever $\zeta(0)$ may be, it should satisfy $\text{Im}(\zeta(0)) = 0$ and $\text{Re}(\zeta(0)) < 0$.

Now, by multiplying the functional equation by $(1 - z)$ and using the property of the gamma function that $z\Gamma(z) = \Gamma(z + 1)$, one obtains:

$$\zeta(z)(1-z) = 2^z \pi^{z-1} \sin(\frac{\pi z}{2})\Gamma(1-z)(1-z)\zeta(1-z)$$

$$\zeta(z)(1-z) = 2^z \pi^{z-1} \sin(\frac{\pi z}{2})\Gamma(2-z)\zeta(1-z)$$

And letting $z$ tend to 1 yields:

$$\lim_{z\to 1} \zeta(z)(1-z) = \lim_{z\to 1} 2^z \pi^{z-1} \sin(\frac{\pi z}{2})\Gamma(2-z)\zeta(1-z)$$

$$\lim_{z\to 1} \zeta(z)(1-z) = 2\Gamma(1)\zeta(0)$$

Or, since $\Gamma(1) = 1$:

$$\lim_{z\to 1} \zeta(z)(1-z) = 2\zeta(0)$$

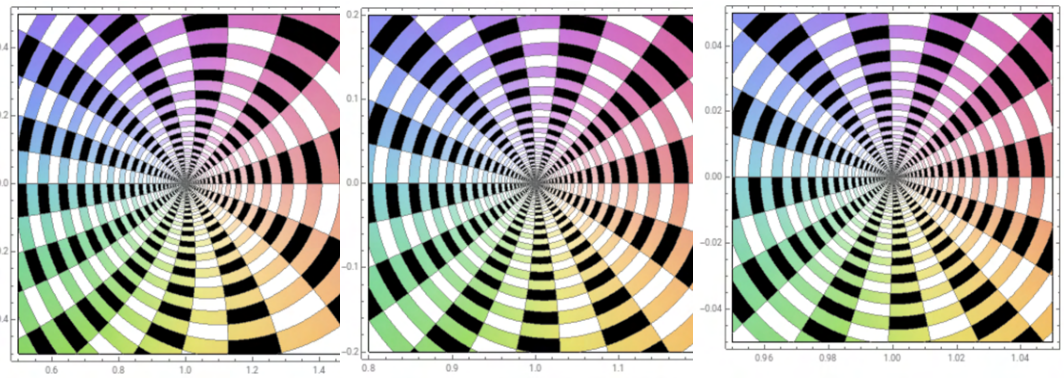
But wait a second, $\lim_{z\to 1} \zeta(z)(z-1)$ just happens to be the *residue* of $\zeta(z)$ at $z = 1$! One has now several ways to proceed. One possibility would be to use brute force calculation in order to evaluate the following integral

$$\text{Res}(\zeta, 1) = \frac{1}{2\pi i} \oint_C \zeta(z) dz$$

where C traces out a circle around $z = 1$ in the positive (counterclockwise) direction. Another option would however be to make use of *phase plots*.

## 4.2 An empirical approach for finding residues

With that in mind, the first thing to note is that the contour we are integrating around can have a radius that is as *small as we desire*. This on its own means that if we were to find some function, let's call it $g(z)$, that would approximate zeta in the neighborhood of $z = 1$, one could hope to simplify the calculations and perform the integration around this *new* function. Sounds promising, right? So, let's zoom onto the pole for it to reveal its secrets:



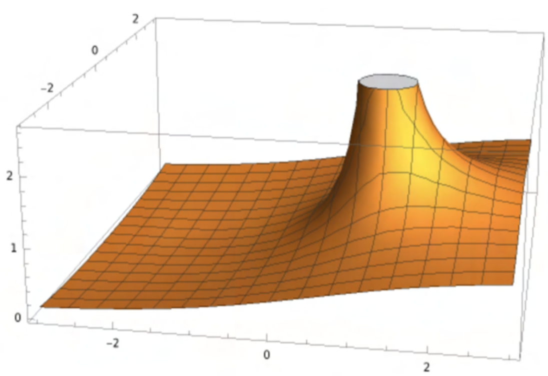
Well, doesn't it seem tempting to immediately state:

**EO5**: In the neighborhood of $z = 1$, $\zeta(z)$ behaves just like $g_k(z) := k/(z-1)$, where k is a nonzero constant that needs to be determined.

Looking a little back, **EO4** allows us to restrict [6] the set of possible values for $k$:

**EO6**: The constant $k$ must be real and strictly greater than zero.

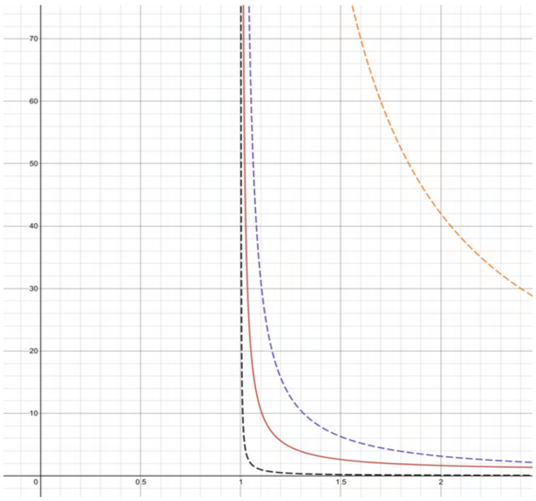
Alright, but how does one find the constant $k$? Since all of the $g_k(z)$'s have the same phase for all $k$ (meaning, they all display the same colors in their corresponding phase plots), it's only the *modulus* that is being modified and we are therefore seeking for a convenient method allowing us to compare the modulus of $\zeta(z)$ in the neighborhood of $z = 1$ with the one of the $g_k(z)$'s.



Even though working with the modular surface of $\zeta(z)$ (depicted on the right) is not the most convenient way to proceed here (because comparing it to the one of other functions is rather problematic), it turns out that at least *some* useful information can be extracted. First of all, remember that we are demanding that one of the $g_k(z)$'s approximates $\zeta(z)$ to arbitrarily high precision around $z = 1$.

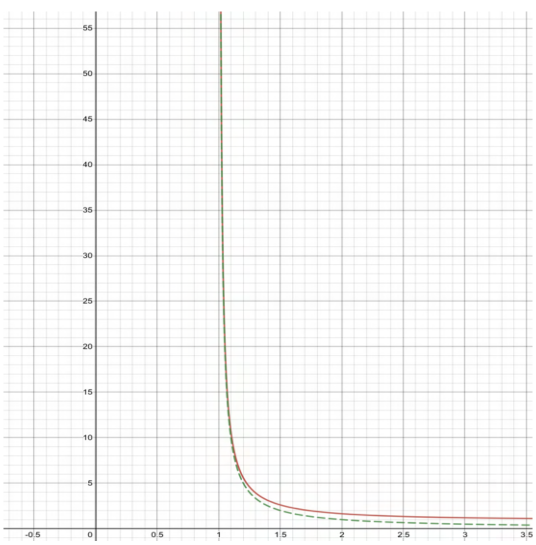
[6] This follows from the uniqueness principle for phase plots

Now, get around the ambiguities of the 2-dimensional modular surface, it suffices to jump a dimension down and only consider the curve obtained when slicing the surface with a vertical plane containing the point $z = 1$. Due to the symmetric nature of the pole-tower, one might as well choose the set of points such that Im(z)= 0.



Therefore, the complexity (pun intended) of the problem has been decreased and comparing the different $g_k(x)$'s to $\zeta(x)$ (all that for $x > 1$) will meet the requirement. At the side are illustrated $\zeta(x)$ in red, and a few of the $g_k(x)$'s as dashed curves (with $k = 0.1$ in black, $k = \pi$ in purple and $k = 42$ in orange). A useful fact to note is that two different $g_k(x)$ will *never* coincide, meaning that $k_i \neq k_j$ implies $g_{k_i}(x) \neq g_{k_j}(x)$ for all $x > 1$. Obviously, $k = \pi$ is an overestimate, whereby $k = 0.1$ is insufficient. Considering that, a reasonable guess would be $k = 1$.

And lo and behold, $g_1(x)$ conveys just right [7] (dashed green curve).



Having concluded that the desired $g_k(z)$ is $g_1(z) = g(z) = 1/(z-1)$, it follows that $\text{Res}(g(z), 1) = 1$ and that

$$\text{Res}(\zeta(z), 1) = 1$$

as well. Coming back to the long awaited $\zeta(0)$:

$$-\lim_{z\to 1} \zeta(z)(z-1) = 2\zeta(0)$$
$$-1 = 2\zeta(0)$$

we find that:

$$\zeta(0) = -\frac{1}{2}$$

[7] admittedly, using this method we can't conclude that the residue is *exactly* 1, only that it lies somewhere in the $\epsilon$-neighborhood of $k = 1$

## 5 The code

Every image you encountered throughout this text has been generated with the *Mathematica* software. It is indisputable that one of the best parts of experimentation with complex functions is that those kind of investigations can easily be reproduced and only require a few lines of code, as we will show. For instance, to generate the modular surface of a complex valued function it suffices to feed *Mathematica* with the following:

```
h[x_, y_] := With[{z = x + Iy}, Abs[z/(1 + z + z^4)]]
Plot3D[h[x, y], {x, −1.5, 1.5}, {y, −1.5, 1.5}]
```

This can be trivially modified to plot custom functions of (almost) any kind, try it on your own! Now, with emphasis on the main theme, phase portraits are generated with the subsequent command:

```
f[z_] := Gamma[z]
ComplexPlot[f[z], z, −10 − 10I, 10 + 10I,
ColorFunction → None , PlotLegends → Automatic]
```

In order to obtain the greyscaled contour lines of constant modulus and phase, one only needs to tell *Mathematica* that

```
ColorFunction → "CyclicLogAbsArg"
```

instead of ColorFunction → None, like before. In order to visualize essential singularities, it is necessary to modify the raster size a little:

```
ComplexPlot[ Exp[1/z^4], {z, −3 − 3I, 3 + 3I}, Sequence[
PlotLegends → Automatic, ColorFunction → "CyclicLogAbsArg",
Method →{"RasterSize" → 800}, ImageSize → 200]]
```

And to introduce the black and white mesh function one needs to do the following:

```
ComplexPlot[Zeta[z], {z, −5 − 5I,
5 + 5I}, Mesh → {40, Range[−Pi, Pi, Pi/12]}, PlotPoints− > 100,
MeshFunctions → {Log[Abs[#2]]&, Arg[#2]&},
MeshShading → {{Automatic, Black}, {White, Automatic}}]
```

The representation of the Riemann sphere requires a little more code:

```
f[z_] := Exp[z]
```

$textureSphere = Texture[Rasterize[ComplexPlot[$
$f[Cot[Im[w]]Exp[IRe[w]]], \{w, -[\pi], [\pi] + [\pi]/2I\},$
$Sequence[$
$ColorFunction \rightarrow "CyclicLogAbsArg", Frame \rightarrow False,$
$PlotRangePadding \rightarrow 0, BoundaryStyle \rightarrow None]],$
$ImageResolution \rightarrow 400]];$
$sphere = SphericalPlot3D[$
$Cos[\phi]], \{\phi], 0, \pi]/2\}, \{\theta], -\pi], \pi]\},$
$TextureCoordinateFunction \rightarrow (\#5, \#4\&),$
$PlotStyle \rightarrow Directive[textureSphere], Sequence[$
$Mesh \rightarrow \{\{0.5\}\}, MeshFunctions \rightarrow \{\#3\&\},$
$MeshStyle \rightarrow Directive[Thick, Black], PlotPoints- > 40,$
$Lighting \rightarrow "Neutral", ViewPoint \rightarrow \{-2, -2, 1.5\},$
$BoundaryStyle \rightarrow None]]$

Just as rendering the sphere *and* the complex plane does:

$f[z\_] := Exp[z]$
$texturePlane =$
$Texture[Rasterize[$
$ComplexPlot[f[z], \{z, -(3/2) - (3I)/2, 3/2 + (3I)/2\}, Sequence[$
$ColorFunction \rightarrow "CyclicLogAbsArg", Epilog \rightarrow \{Thick,$
$Circle[]\}, Frame \rightarrow False, PlotRangePadding \rightarrow 0,$
$BoundaryStyle \rightarrow None]], ImageResolution \rightarrow 400]];$
$plane = Graphics3D[\{texturePlane, Polygon[$
$Sequence[\{\{(-3)/2, (-3)/2, 0\}, \{3/2, (-3)/2, 0\}, \{$
$3/2, 3/2, 0\}, \{(-3)/2, 3/2, 0\}\},$
$VertexTextureCoordinates \rightarrow \{\{0, 0\}, \{1, 0\}, \{1, 1\}, \{0, 1\}\}]]\},$
$Sequence[ViewPoint \rightarrow \{-2, -2, 1.5\}, Axes \rightarrow True]];$
$Show[sphere, plane, BoxRatios \rightarrow Automatic,$
$PlotRange \rightarrow \{\{-3/2, 3/2\}, \{-3/2, 3/2\}, \{0, 1\}\}]$

The remaining graphs were created with *Desmos*.

## References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