Visualising modular forms

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General introduction and visual representation

Contents

Abstract

The modular forms have interested mathematicians for previous centuries. By their plural structure, they create a bridge between several fields of mathematics. One can describe modular forms using analysis tools, number theory constructions or Galois theory representations. However, their visualisation remains a challenging task. The difficulty lies in the fact that one can't imagine intuitively a 4-dimensional space. In this paper, we are trying to find a way to describe graphically how certain modular forms behave on the space. In order to do so, we will use Sage.

The objective in the end is to have an Escher-like representation of modular forms which could unite esthetical thinking with mathematical rigorous aspects.

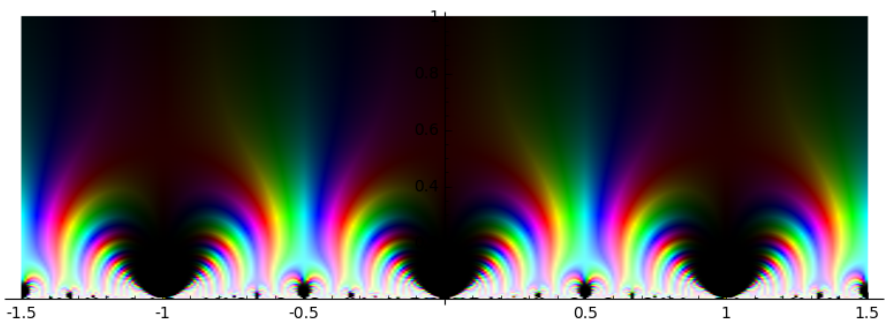


Figure 1: Modular discriminant (based on Dedekind Eta function)

1 Introduction

Modular forms are applications from a part of $\mathbb{C}$ (the upper half plane of $\mathbb{C}$) to $\mathbb{C}$. $\mathbb{C}$ being a 2-dimensional vector space, we end up having to represent 4-dimensional spaces. The question of the representation of 4-dimensional space has always been central for mathematicians. There exist two standard ways to represent those : the first one using colors, the second one using topological features associated to space (Here we used the modulus). We decided to concentrate on the first option as we considered it to be more intuitive and it allows us to represent the fundamental domain on a clearer way.

Example 1.1. We will show the difference between those two different methods by plotting some usual functions:

- Identity function;
- Exponential function;
- Modular forms;

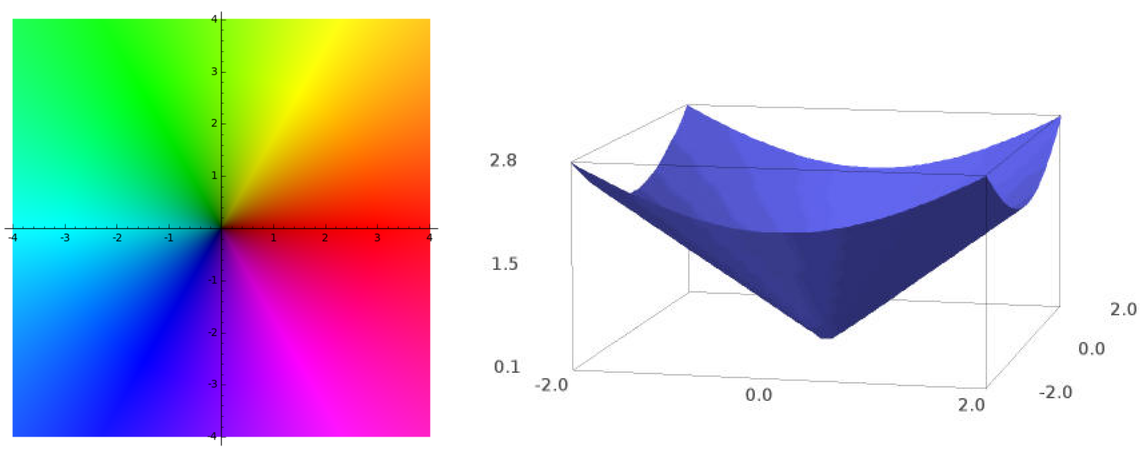


Figure 2: Identity function

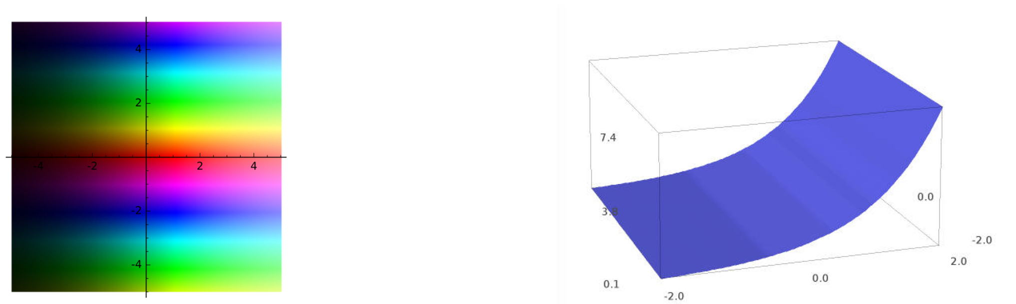


Figure 3: Exponential function

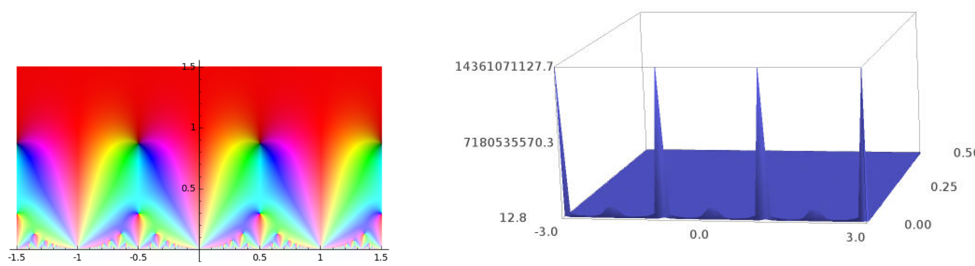


Figure 4: A modular form : Eisenstein series of weight 4

Remark 1.2. We see that the Eisenstein series, for example, is way clearer when we use the color convention.
Even within each of these methods, there exist different conventions. For instance there exist several methods for the color plotting. We will use the complex color wheel method which assigns a color to each point of the complex plane. The idea can be summarized as follows: the argument (angle) of the complex number defines the hue value while the modulus defines the light value of the color in the HLS (hue, lightness, saturation) color model; for a given (H,L) pair we choose the maximal saturation value. So as represented on the previous example, one could see the 6th roots of unity as: magenta, blue, cyan, green, yellow, and red.

Now let's define exactly what modular forms are.

2 A bit of theory

Definition 2.1. A *modular form* is a holomorphic application f from the upper half plane of $\mathbb{C}$ [1] to $\mathbb{C}$ with the following properties:

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{SL}_2$$

where k is a fixed constant usually called *weight*.

2.1 Complex analysis recaps

Definition 2.2. Let f be a complex application defined on $\Omega \to \mathbb{C}$ where Ω is a subset of $\mathbb{C}$, f is called holomorphic if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad exists \quad \forall z_0 \in \Omega$$

Proposition 2.3. *All modular forms are periodic.*

Proof. Let f be a modular form, one can easily see by taking the matrix T defined by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ that

$$f(z+1) = f(z) \quad \forall z \in \mathbb{H}$$

which proves the periodicity of f. □

Proposition 2.4. *A holomorphic function is also analytic. As it is periodic, it admits a Fourier power series extension.*

However, one can easily see that the existence of a Fourier power series for f doesn't necessarly imply that f is periodic.

Proof. A proof of this proposition can be found on Rudin's "Real and Complex Analysis" in the chapter about elementary properties of Fourier transform. □

[1] For the sake of simplicity, we shall call $\mathbb{H}$ the upper half plane of $\mathbb{C}$ (which equivalently can be described as the following set $\mathbb{H} := \{z \in C | Im(z) > 0\}$).

Example 2.5. Eisenstein series, which are the series we will visualize on this project, are modular forms of the following definition:
Let τ be a complex number in $\mathbb{H}$; we also define $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. We then have:

$$E_k(\Lambda) = \sum_{0 \neq \lambda \in \Lambda} \lambda^{-k}$$

Their expansion as Fourier series is of the following form:

$$E_k(\Lambda) = E_k(\tau) = \sum_{(0,0) \neq (m,n) \in \mathbf{Z}^2} \frac{1}{(m + n\tau)^k}$$

From here, we see that every modular form can be described not only as a complex power series (coming from the fact they are holomorphic) but also as Fourier transform. This will help us to represent them as we will use their definition as Fourier Series.

2.2 Transformation into the open unit disk of $\mathbb{C}$

Definition 2.6. The *open unit disk* around a point z_0 is the set of points such that their distance [2] to z_0 is less than 1:

$$D_1(z_0) = \{z : |z - z_0| < 1\}.$$

Proposition 2.7. *The open unit disk is homeomorphic to* $\mathbb{H}$*.*

Proof. The function $f(z) = \frac{z}{1-|z|^2}$ is an example of a real analytic and bijective function from the open unit disk to the plane; its inverse function is also analytic. This shows that the open unit disk is homeomorphic to the whole plane $\mathbb{C}$ but as $\mathbb{H}$ is homeomorphic to $\mathbb{C}$ and the homeomorphism is an equivalence relation, we have $\mathbb{H}$ is homeomorphic to the open unit disk.

[2] Here we work with the classical Euclidean distance.

Another way of proving this, is to realize that $g(z) = i\frac{1+z}{1-z}$ (called the inverse Cayley transformation) is a conformal bijective mapping from the unit circle to $\mathbb{H}$. $\square$

Remark 2.8. Actually the open unit disk and $\mathbb{H}$ are not only equivalent topologically, they are also similar as Riemann Surface.

From here, we see that we can take any modular forms on the upper plane and project it to the unit disk. This provides an evidence to prove that this Escher-like representation is possible to draw.

2.3 Representing modular forms on the open unit disk

We just saw that the open unit disk was homeomorphic to $\mathbb{H}$. By applying the inverse Cayley transformation on a modular form, we can have a representation of it in the open unit disk. Therefore one can transform the Eisenstein series we previously plotted in $\mathbb{H}$ into the following one :

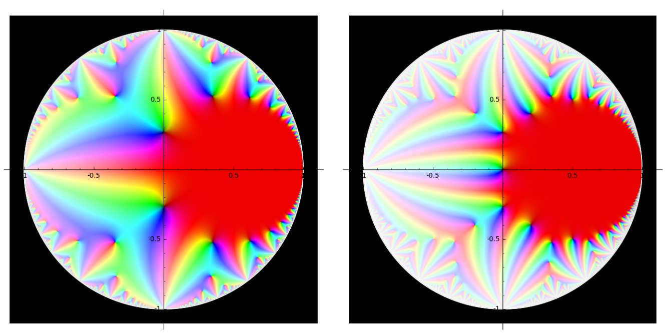


Figure 5: Eisenstein series of weight 4 and 8 in the open unit circle

Remark 2.9. This visualization allows us to easily see the periodic constitution of modular forms. However, by plotting directly from $\mathbb{H}$ to the open unit disk, we added a lot of unnecessary calculations, as we could restrict ourselves to calculate only one period. This impacts the quality of the picture and the time that was needed in order to plot it.

2.4 Möbius transformations

Definition 2.10. The *Möbius transformations* are applications such that $f(z) = \frac{az+b}{cz+d}$ are the projective transformations of the complex project line. They form a group called Möbius group, which is the projective linear $\mathbb{PGL}(2, C)$.

Proposition 2.11. *Möbius transformations are generated by the two following matrices :* $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Proof. Suppose in a first case that $c = 0$. Then the Möbius transformation is a linear transformation, which implies that it can be indeed represented as a translation. Now suppose that $c \neq 0$ and $a = 0$. Then the Möbius transformation is an inversion. Finally suppose $c \neq 0$ and $a \neq 0$. Then the Möbius transformation f can be written as:

$$f(z) = \frac{az+b}{cz+d}$$
$$\Leftrightarrow f(z) = \frac{a}{c}\left(1 + \frac{bc-ad}{a(cz+d)}\right)$$

which as a matter of fact is the composition of a linear transformation and inverse function.

□

This point is essential. We will see that in fact every Möbius transformation can be decomposed into translations and/or inversions. This allows us to have a systematic representation of them and to only get interested on how many translations and inversions are needed for a certain Möbius transformation. We will keep proving useful properties of Möbius transformations that we will use in our algorithms.

Proposition 2.12. *Möbius transformations are uniquely determined by three distincts points.*

Before proving this point, we will prove a fairly easy lemma.

Proposition 2.13. *Every Möbius transformation that isn't identity admits at most 2 fixed points.*

Proof. Let's suppose that f admits 3 points z_1, z_2 , z_3 such that $f(z_i) = z_i$ where $i = 1, 2, 3$. The previous equality can be rewritten as:

$$az + b = cz^2 + dz$$

But this admits more than 2 solutions if and only if $c = b = 0$ and $a = d$, which is equivalent to the identity mapping.

□

This lemma, which was in fact fairly direct, is interesting theoretically, as it allows us to understand more deeply how the Möbius transformation behave. Now we can prove Proposition 1.6.

Proof. Let's assume that we have two Möbius transformations S and T that map 0, 1 and ∞ to three given points $\alpha_1, \alpha_2, \alpha_3$. Since we know that Möbius transformations are invertible, we can form the map $S \circ T^{-1}$, which is also a Möbius transformation. This map preserves 0, 1 and ∞. This implies by the previous lemma that the Möbius transformation is the identity, which implies $T = S$. From here, we see that every Möbius transformation is uniquely determined by 3 points.

□

Proposition 2.14. *Möbius transformation are stable for circle and line.*

Proof. As we know, circles are uniquely determined up to 3 points, with the special case where the 3 points being collinear implies that the circle is a line. Let f be a Möbius transformation and $\alpha_1, \alpha_2, \alpha_3$ be 3 distinct points. We saw by Prop 2.11 that Möbius transformations are uniquely determined by 3 distinct points, which induces that $f(\alpha_i) = \beta_i$, $i \in 1, 2, 3$ defines a unique Möbius transformation. Every member of the set $\mathcal{F}$ which denotes the family consisting of all straight lines and straight circles of the $\mathbb{C}$ is the locus of the equation[3].

$$\alpha * z * \bar{z} + \beta * z + \bar{\beta} * \bar{z} + \gamma = 0$$

[3] This can be proved using elementary analytic geometry.

where α and γ are real constants and β is a complex constant, provided that $\beta\bar{\beta} > \alpha\gamma$. If $\alpha \neq 0$, this defines a circle; $\alpha = 0$ defines a straight line. Replacing z by $\frac{1}{z}$ transforms the previous equation into

$$\alpha + \beta * \bar{z} + \bar{\beta} * z + \gamma * z * \bar{z} = 0$$

which is an equation of the same form.

This concludes the proof. □

Now we shall define another key concept of this paper, *the fundamental domain.*

2.5 Fundamental domain

Definition 2.15. *The fundamental domain* is a closed subset $D \subset X$ such that X is the union of translates of D under the group action G:

$$X = \bigcup_{g \in G} gD$$

and such that $Int(gD \cap g'D) = \emptyset$ for any two distinct translates.

Example 2.16. Fundamental domain of a modular form:

$$F = \{z \in C : |Re(z)| \leq \frac{1}{2}, |z| \geq 1\}$$

The reason why we define such an object is because it allows us to only represent one period of our modular forms. Instead of losing time calculating the function on the whole space, we will only calculate it on its fundamental domain and then expand it to the whole upper plane of $\mathbb{C}$.

2.6 Representing the fundamental domain

Sparing operation and time are the core challenges when it comes to representation. Here in this project we tried our best to use the periodic nature of modular forms as a tool to represent them. In order to draw the fundamental domain associated to the Eisenstein series, we used the invariant nature of Möbius transformations[4].

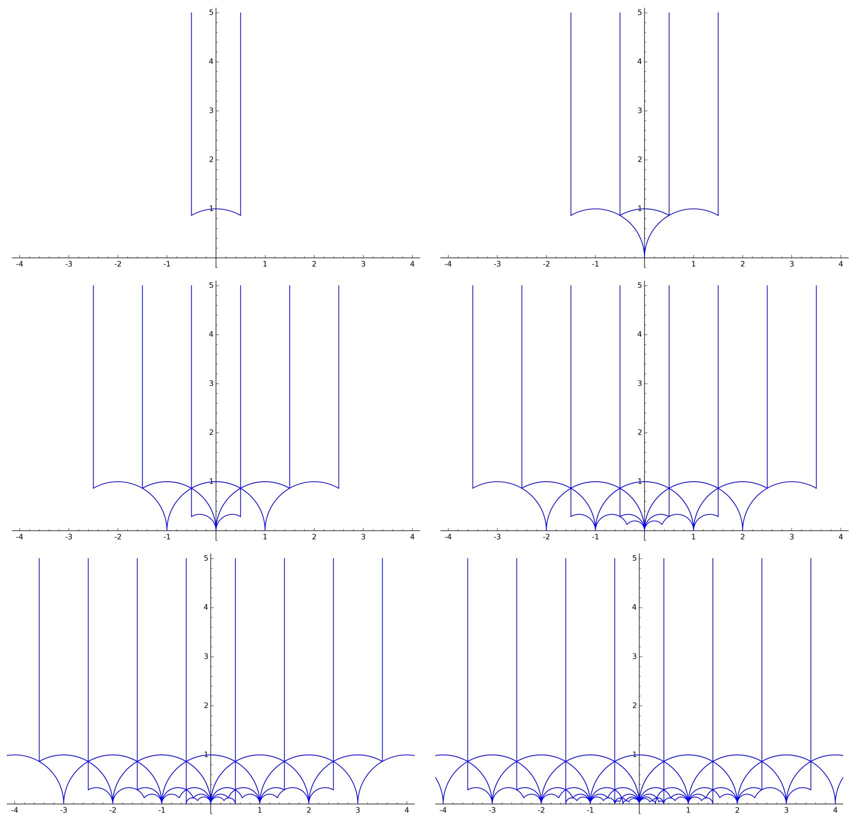


Figure 6: Expansion of the fundamental domain step by step

Remark 2.17. The idea of the algorithm to construct the fundamental domain is to apply the recursivity and the fact that the Möbius transformations are generated by T and S. So we have

$$A_{n+1} = \{A_n \times T, A_n \times S, A_n \times T^{-1}, A_n \times S^{-1}\}$$

with A_0 being equal to the original fundamental domain.

[4] By invariant, we mean that line and circles are conserved by transformations.

The same algorithm can give us the fundamental domain of the open unit circle.

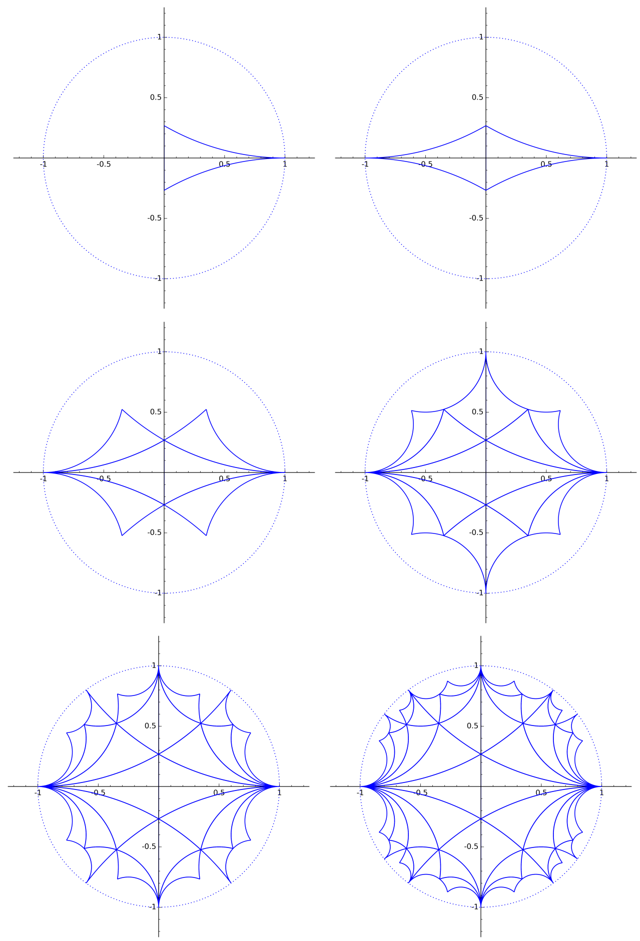


Figure 7: Expansion of the fundamental domain on the open unit circle step by step

2.7 Fundamental domain applied on modular forms

In the section, we will first implement the fundamental domain on a plot of the Eisenstein series and then only plot this Eisenstein series on the fundamental domain and see how it evolves step by step.

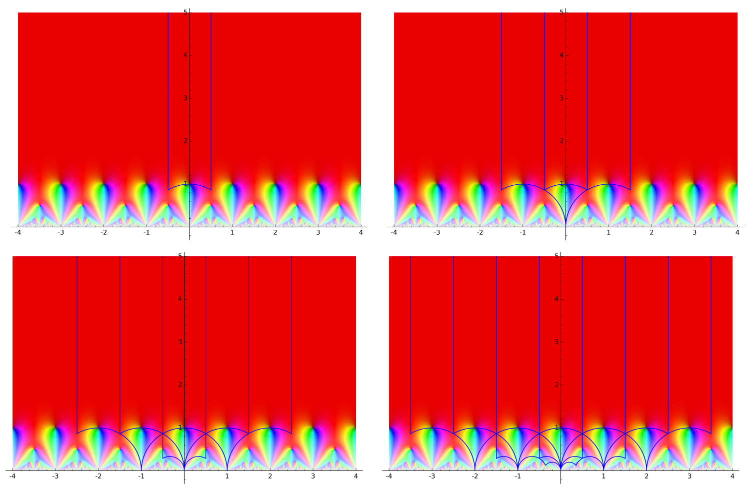


Figure 8: Eisenstein series of weight 2 and its fundamental domain expanding

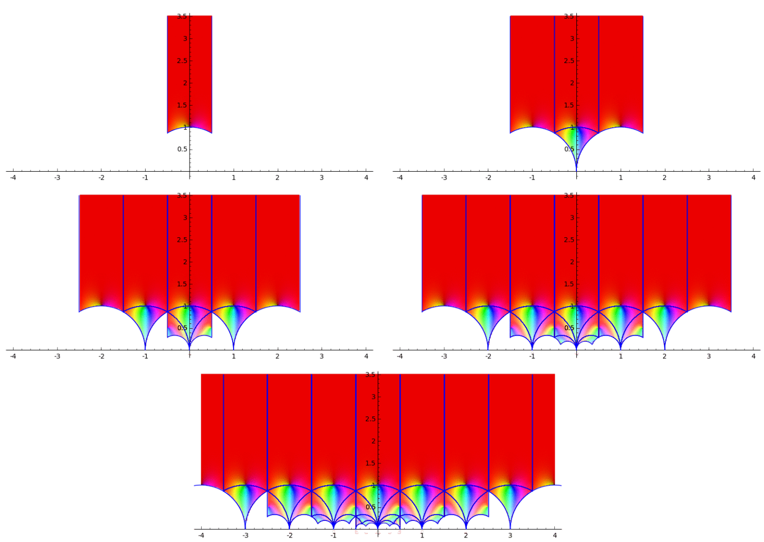


Figure 9: Expansion of the Eisenstein series of weight 2 on its fundamental domain

2.8 A bit more of algorithm

Here we will propose an algorithm that translates every point of the upper half plane of $\mathbb{C}$ to the fundamental domain F only using Möbius transformation. The idea behind it is fairly simple. Let z be a random point in $\mathbb{H}$. We first check if the norm $|z| \geq 1$. If it is not the case, we apply T until it is the case then we apply S such that the $|Re(z)| \leq \frac{1}{2}$. Before doing this, we will prove that this process indeed allows us to arrive at F.

Proof. Let $z(x, y)$ be a point on $\mathbb{H}$ where x defines $Re(z)$ and $y = Im(z)$. We suppose without loss of generality that $|z|^2 = x^2 + y^2 \ngeq 1$ and $|Re(z)| \leq \frac{1}{2}$, as one can easily obtain the second condition by applying S on the point (which means translating the point compared to the real numbers axis). Applying T changes the norm, one can indeed see that $|Tz| = \frac{|z|}{|z^2|}$, but as $|z| < 1$, $|Tz| > |z|$. As the fixed points are defined such that $|z| = 1$, which happens to be in F, this concludes the proof.

□

Remark 2.18. This is obviously not the uniquely possible algorithm. This one is the most intuitive, although one may improve it. The decomposition of the Möbius transformation to land on the fundamental domain is also not unique as we can apply S and S^{-1} as many times as we want (it is equivalent by definition of $\mathbb{I}$d). There exists an infinite amount of these decompositions.

We are done with the theory for now.

3 Representing modular forms as an Escher work

In this section, we explain how we transformed our periodic modular forms into pictures similar to Escher litography. Our interest for Escher comes primarly from the fact that the huge majority of his work is constructed on periodicity and symmetries. From here, making an analogy to our work becomes fairly straightforward. A second reason of this interest is the aesthetic research behind Escher's work. We believe that beauty is deeply connected to intuition, which implies that the creation of an intuition of 4-dimensional spaces and modular forms should be followed with a perpetual research of beauty. The latter reason explains the whole purpose of this project.

Figure 10: Escher - Circle Limit V

As we can observe, there is a lot of symmetry and repetition in this image. Moreover, those symmetries resemble a lot the ones of modular forms; that is no coincidence.

The image above is a tessellation on the open unit disk, seen as hyperbolic space with the Poincaré disk model. The shapes of the tiles are identical to the ones of the copies of the fundamental domain in the open unit disk by the Möbius transformations.

To create such an image is a challenge. But it's possible to create a more simple example with the help of Sage. The idea is that we start with the fundamental domain, which we paint one color, for instance black. Then the copies of this domain by one transformation ($S, T,\ and T^{-1}$) are painted black. The next copies of those copies will be painted white again, and so forth.

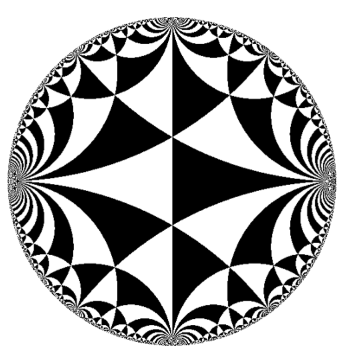
To do this we will define an easy function that we will then plot on the open unit disk. To this goal we define the following concept.

Definition 3.1. Let be $z \in \mathbb{H}$. We call the *order* of z, and we denote $\mathrm{ord}(z)$, the smallest number of transformations (among S,T,T^{-1}) needed to transform z into a complex number in the fundamental domain.
Equivalently, $\mathrm{ord}(z)$ is the minimal number of Möbius transformations needed to transform the fundamental domain into the copy of itself that includes z.

With this definition it's easy to create a function that paints us a two-color Escher-like picture:

$$f : \mathbb{H} \longrightarrow \mathbb{C} \cup \{\infty\} : z \longmapsto \begin{cases} 0, & \text{if} \quad 2 \mid \mathrm{ord(z)} \\ \infty & \text{else} \end{cases}$$

Here is what the result looks like :

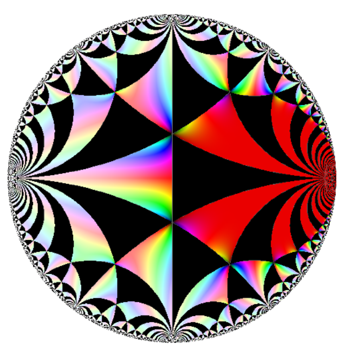


By modifying the outcoming values 0 and ∞ to other values, we can modify the colours too. For instance, if we change ∞ to 1, we get a black and red picture like this:

Another possibility is that if we have a modular form m, we can define the function as follows:

$$f : \mathbb{H} \longrightarrow \mathbb{C} : z \longmapsto \begin{cases} 0, & \text{if} \quad 2 \mid \text{ord(z)} \\ m(z) & \text{else} \end{cases}$$

That way we get a picture where some tiles show a part of the modular form, while the others are all black, like this:

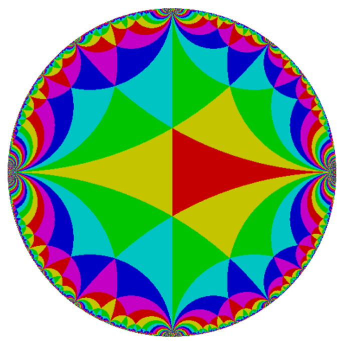


But wait, there's more! We can have f to send z to more than only two complex numbers, creating a picture with more than two colours, other than the one with the modular form.

Let be $n \in \mathbb{N}$. Let $\{z_k \in \mathbb{C} \,|\, k = 1, ..., n\}$ be a family of distinct complex numbers. Then we define f as follows:

$$f : \mathbb{H} \longrightarrow \mathbb{C} : z \longmapsto z_k, \text{ if } ord(z) \equiv k - 1 \bmod n$$

That way we obtain an Escher-like picture with n colours. Here is a picture with 6 colours for instance:



Remark 3.2. The functions above are all defined on $\mathbb{H}$ and not on the open unit circle, as we treated the modular forms in $\mathbb{H}$ during the whole report. To get the correct picture, one has to simply transform z from $\mathbb{H}$ into the open unit disk by using the Cayley transformation.

A Annexe

A.1 eisensteinSeries.sage

- `Eisenstein(k,N=200)`: k is half the weight of the Eisenstein series and N is the number of terms of the Fourier series. By default, the function takes an exactitude of $N = 200$ coefficients. The function can be associated to a variable `f` by the command `f(z)=Eisenstein(k,N)` for a certain $(k, N) \in \mathbb{N}^2$.

A.2 modularDiscriminant.sage

- `modularDiscriminant(N=200)`: This function returns the modular discriminant, which is a modular form based on the *Dedekind Eta function*. N is the numbers of factors. By default, the function takes an exactitude $N = 200$ factors. The function can be associated to a variable `f` by the command `f(z)=modularDiscriminant(N)` for a certain $N \in \mathbb{N}$.

A.3 functionPainter.sage

- `paintFunction(f,Xext=2,Yext=1.5,DPI=300,function_name="function")`: This function paints the complex plot of a function on the upper half plane, preferably a modular form, and saves it as a .png file. f is the function, $Xext$ and $Yext$ are the extremities of the axes and $function\_name$ is the name of the function and will also be used as the name of the saved .png file.

- `paintFunctionDisk(f,DPI=300,function_name="function")`: This function paints the complex plot of a function on the open unit disk, preferably a modular form and saves it as a .png file. f is the function, $Xext$ and $Yext$ are the extremities of the axes and $function\_name$ is the name of the function and will also be used as the name of the saved .png file.

A.4 fdBorderAnimator.sage

- `transform(M)`: This function returns the transformation of the border of the fundamental domain in the upper half plane by the matrix M, which is presumably a product of Möbius transformations.
- `transformDisk(M)`: This function returns the transformation of the border of the fundamental domain in the open unit disk by the matrix M, which is presumably a product of Möbius transformations.
- `drawFDBorder()`: This function returns a plot of the border of the fundamental domain in the upper half plane.
- `drawFDBorderDisk()`: This function returns a plot of the border of the fundamental domain in the open unit disk.
- `animateFDtransformations(N)`: This function returns a list of the plots of all the transformations of the border of the fundamental domain in the upper half plane based on up to N Möbius transformations.
- `animateFDtransformationsDisk(N)`: This function returns a list of the plots of all the transformations of the border of the fundamental domain in the open unit disk based on up to N Möbius transformations.

A.5 heckeTriangleFunctions.sage

- `getOrder(z)`: This function returns the order of a complex number z of the upper half plane. The order of a complex number z is the minimal number of Möbius transformations needed to transform z into a complex number in the fundamental domain of the upper half plane.
- `getOrderDisk(z)`: This function returns the order of a complex number z of the open unit disk. The order of a complex number z is the minimal number of Möbius transformations needed to transform z into a complex number in the fundamental domain of the open unit disk.

A.6 modularFormAnimator.sage

All of the following functions create an animation that visualises any modular form f on the fundamental domain first and then gradually its Möbius transformations.

- `animateModularForm(f,N,Xext = 4, Yext = 3.5, DPI = 200)`: This function creates the animation on the upper half plane without borders. f is the function, N the number of frames and $Xext$ and $Yext$ the extremities of the axes.
- `animateModularFormDisk(f,N, DPI = 200)`: This function creates the animation on the open unit disk without borders. f is the function and N the number of frames.
- `animateModularFormWithBorder(f,N,Xext = 4, Yext = 3.5, DPI = 200)`: This function creates the animation on the upper half plane with borders. f is the function, N the number of frames and $Xext$ and $Yext$ the extremities of the axes.
- `animateModularFormDiskWithBorder(f,N, DPI = 200)`: This function creates the animation on the open unit disk with borders. f is the function and N the number of frames.

A.7 escherPainter.sage

- `EscherDisk(dpi=300)`: This method defines the function needed to create a black and white Escher-like picture.
- `paintEscher(N,lightness=1,dpi=200)`: This function paints an Escher-like picture with N differents colors and saves it.

References

Wikipedia, https://en.wikipedia.org/wiki/Dedekind_eta_function, Dedeking Eta Functions.

Sage Math Documentation, http://doc.sagemath.org/, Documentation Sage.

Wikipedia, https://en.wikipedia.org/wiki/Eisenstein_series, Eisenstein Series.

Michel Serre, Cours d'Arithmetique, Formes modulaires.

Walter Rudin, Real and Complex Analysis, McGraw-Hill International Editions, 1993.