The p-adic integers

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General introduction and visual representation

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The present document can be regarded as an introductory exploration to the p-adic number system as far as its metric and properties are concerned. The p-adic integers form a subset of the set of all p-adic numbers. The latter in turn constitute an extension of the field of rational numbers, analogous to the completion of the rationals by the real numbers with respect to the standard ordinary metric. In contrast, the p-adic extension arises from the use of the counter-intuitive p-adic metric. It's worthwhile noting that the interest of p-adic numbers relies upon the numerous applications they have, ranging from the resolution of Diophantine equations to pending problems in Quantum Mechanics.

This brief report will nonetheless limit the scope of its content to the presentation of the fundamental properties of p-adic numbers — and in particular those of p-adic integers. Proofs of the proposed choice of statements as well as devious propositions will therefore be omitted for the sake of simplicity, conferring more space to the discussion of the project itself. Departing from the definition of the p-adics, we will turn our focus to a feasible geometric approach to their visualisation.

1 P-adic norm and expansion

1.1 P-adic metric

Definition 1.1. Consider an arbitrary non-zero rational number x. We can express it as

$$x = p^{v_p(x)} \cdot \frac{m}{n} \tag{1}$$

with $m, n \in \mathbb{Z}$ and p the p-adic taking into account such that p, m and n are relatively prime.
The expression $v_p(y)$ is known as the p-valuation of $y \in \mathbb{Z}$, i.e. the mapping $v_p : \mathbb{Z} \to \mathbb{N}$ giving the power of the considered prime number p in the prime decomposition of y.
Since $x = \frac{a}{b} \in \mathbb{Q}\backslash\{0\}$, we can define the p-valuation of x as

$$v_p(x) = v_p(\frac{a}{b}) = v_p(a) - v_p(b).$$

Now we can define the p-adic absolute value of x as

$$|x|_p = \frac{1}{p^{v_p(x)}}. \tag{2}$$

It is clear that $|x|_p \geq 0$ for all $x \in \mathbb{Q}\backslash\{0\}$.
Using property (1) the p-adic absolute value can easily be determined for any p in each case after prime decomposition has been applied to the p-adic number being considered.

In addition we adopt the convention following which the p-adic absolute value of 0 will simply be 0 itself.

We will now introduce the general classification of p-adic numbers, which is discussed in more detail in the next heading of the present section.

Definition 1.2. We define $\mathbb{P}$ to be the set of representatives of the positive irreducible elements in $\mathbb{Z}$ up to association. The elements of $\mathbb{P} = \{2, 3, 5, 7, ...\}$ correspond precisely to prime numbers.

Definition 1.3. A p-adic number is a number that can be written as a p-adic expansion for some $p \in \mathbb{P}$ (see the next section "Expansion representation" for the definition and properties of a p-adic expansion).

Definition 1.4. A p-adic integer is a p-adic number with the property that its p-adic expansion satisfies $n \geq 0$, with $n \in \mathbb{Z}$ the starting term in the sum of the expansion (see (6) for a clear, precise definition).

Definition 1.5. The set of all p-adic integers will be noted as $\mathbb{Z}_p$ whereas that of all p-adic numbers will be noted as $\mathbb{Q}_p$ with $p \in \mathbb{P}$.

We will take for granted that $\mathbb{Z}_p$ is a ring and $\mathbb{Q}_p$ a field.

Corollary 1.6. *If $x \in \mathbb{Z}_p$, then $|x|_p \leq 1$ for all x and all $p \in \mathbb{P}$.*

Proof. Straightforward from the definition of p-adic absolute value. □

Example 1.7. Take $x = 162/13$. Suppose we want to find its 3-adic absolute value (hence p=3). Expressed in the form (1), we obtain

$$x = 81 \cdot \frac{2}{13} = 3^4 \cdot \frac{2}{13}$$

which means $|x|_3 = \frac{1}{3^4}$.

What about its 13-adic absolute value ? It will simply be $|x|_{13} = 13$ because

$$x = 13^{-1} \cdot 162$$

thus

$$|x|_{13} = \frac{1}{13^{-1}} = 13.$$

The p-adic absolute value where p divides neither the numerator nor the denominator is trivially 1 since $p^0 = 1$.

Definition 1.8. The p-adic norm is derived naturally from the p-adic absolute value as

$$\begin{array}{rccc} d_p & : & \mathbb{Q}^2 & \rightarrow & \mathbb{Q}_{\geq 0} \\ & & (x, y) & \mapsto & d_p(x, y) := |x - y|_p. \end{array}$$

The completion of $\mathbb{Q}$ by the p-adic norm gives rise to the p-adic numbers discussed all along, i.e. all number belonging to $\mathbb{Q}_p$.

As a matter of fact it is only with respect to the p-adic norm and no other that $\mathbb{Q}_p$ completes $\mathbb{Q}$. In clearer terms, every Cauchy sequence in $\mathbb{Q}_p$ converges in $\mathbb{Q}_p$ with respect to the p-adic norm but not every Cauchy sequence in $\mathbb{Q}_p$ will converge in $\mathbb{Q}_p$ with respect to any other norm.

Proposition 1.9. *The p-adic norm forms a norm in* $\mathbb{Q}$.

Proof. For all x and y in $\mathbb{Q}$, for all prime p, we must prove three statements, namely

$$|x|_p = 0 \Leftrightarrow x = 0, \tag{3}$$
$$|xy|_p = |x|_p|y|_p, \tag{4}$$
$$|x+y|_p \leq |x|_p + |y|_p. \tag{5}$$

The first statement (3) follows plainly by the definition of the p-adic norm. Some computations are required to prove (4) :

$$|xy|_p = \frac{1}{p^{v_p(xy)}} = \frac{1}{p^{v_p(x)+v_p(y)}} = \frac{1}{p^{v_p(x)} \cdot p^{v_p(y)}} = |x|_p|y|_p.$$

The statement holds trivially for $y = 0$ or $x = 0$.

Finally, to prove the triangle inequality in (5), we use another property of the p-adic norm, i.e. the strong triangle inequality

$$|x+y|_p \leq \max\{\, |x|_p, |y|_p\},$$

as follows

$$|x+y|_p = \frac{1}{p^{v_p(x+y)}} = p^{-v_p(x+y)}$$

$$\leq \max\{\, p^{-v_p(x)}, p^{-v_p(y)}\} = \max\{\, |x|_p, |y|_p\}$$

$$\leq |x|_p + |y|_p.$$

And it is trivially true for $x = 0$ or $y = 0$.

□

Remark 1.10. Due to the peculiar definition of distance of the p-adic norm, we have a remarkable property. Namely, numbers that find themselves "far apart" in terms of the standard metric become counter-intuitively very close to each other when seen with the p-adic metric.

Let us take the numbers 2, 3 and the large number 28814 in the 7-adic number system as an example. It turns out 28814 is "closer" to 2 than 3 is. Indeed,

$$|2 - 28814|_7 = |-28812|_7 = |-7^4 \cdot 13|_7 = \frac{1}{7^4} < 1$$

whilst

$$|2 - 3|_7 = |-1|_7 = 1.$$

This can be pictorially appreciated in figure (2.4), which corresponds to a 7-adic representation. The numbers 28814 and 2 occupy almost the same place whereas 3 does not share that proximity.

Theorem 1.11 (Ostrowski)**.** *The standard absolute value and the p-adic absolute value are the only non-trivial absolute values in* $\mathbb{Q}$ *up to equivalence. (We understand by trivial absolute value the absolute value given by* 0 *when taking* 0 *as an argument and* 1 *otherwise.)*

For a formal proof of Ostrowski's theorem we refer the reader to the publication "p-adic Numbers: An Introduction", by Fernando Q. Gouvêa.

Proposition 1.12. *Let* $q \in \mathbb{Q}\backslash\{0\}$*. Then it follows that the product of the standard metric and the p-adic metric for any* $p \in \mathbb{P}$ *will equal* 1*, i.e.*

$$|q| \cdot \prod_{\substack{p \geq 2 \\ p \ prime}}^{\infty} |q|_p = 1$$

where $|\cdot|$ *designates the standard norm and* $|\cdot|_p$ *the p-adic norm.*

Proof. Let $|q| = \frac{|x|}{|y|}$ where $x, y \in \mathbb{Z}\backslash\{0\}$. As the prime decomposition of positive integers is unique, we can write

$$|x| = \prod_{\substack{p\geq 2 \\ p\ prime}}^{\infty} p^{v_p(x)} \quad \text{and} \quad |y| = \prod_{\substack{p\geq 2 \\ p\ prime}}^{\infty} p^{v_p(y)}$$

where v_p designates the p-valuation. Then

$$|q| = \prod_{\substack{p\geq 2 \\ p\ prime}}^{\infty} p^{v_p(x)-v_p(y)} = \prod_{\substack{p\geq 2 \\ p\ prime}}^{\infty} p^{v_p(q)}$$

By definition of the p-adic norm, we obtain

$$|q| \cdot \prod_{\substack{p\geq 2 \\ p\ prime}}^{\infty} |q|_p = \prod_{\substack{p\geq 2 \\ p\ prime}}^{\infty} p^{v_p(q)} \cdot \prod_{\substack{p\geq 2 \\ p\ prime}}^{\infty} \frac{1}{p^{v_p(q)}} = 1.$$

□

1.2 Expansion representation

Definition 1.13. Let x be an arbitrary rational number, $p \in \mathbb{P}$ and $n \in \mathbb{Z}$. A p-adic expansion of x is a series of terms with integral coefficients of the form

$$x = \sum_{k=n}^{\infty} a_k p^k \tag{6}$$

where $a_k \in \{0, 1, 2, ..., p-1\}$ for all k, as defined above, and with the fundamental property that any partial sum converges with respect to the p-adic norm (proved below). It is obtained from x by successively/inductively solving polynomial equations (determined only by the rational x being considered) modulo prime powers, though this will be explained in detail later on in the next section.

The p-adic number's expression through the use of this infinite series is called a p-adic expansion. Each a_k for $k \in \{n, n+1, n+2, \ldots\}$ is known as a p-adic digit.

Despite the fact that p-adic expansions are by definition infinite, they possess an important property, making the definition of $\mathbb{Z}_p$ concrete and precise. It is discussed in the next proposition.

Proposition 1.14. *Let $b_N \in \mathbb{Z}_p$ and $p \in \mathbb{P}$. The sequence of partial sums of $b_N := \sum_{k=n}^{N} a_k p^k$ where $a_k \in \{0, 1, 2, ..., p-1\}$ and $N, M, n \in \mathbb{N}$ forms a Cauchy sequence with respect to the p-adic norm.*

Proof. We need to prove that

$$\forall \varepsilon > 0, \exists L, \forall M, N \geq L : |b_M - b_N|_p < \varepsilon. \tag{7}$$

Without loss of generality, pick $M \geq N$. Then it follows that

$$\begin{aligned} |b_M - b_N|_p &= |\sum_{k=N+1}^{M} a_k p^k |_p \\ &\leq \max_{k \in [N+1, M]} |a_k p^k|_p \\ &= \max_{k \in [N+1, M]} |a_k|_p \cdot \frac{1}{p^k} \\ &\leq \max_{k \in [N+1, M]} \frac{1}{p^k} \\ &= \frac{1}{p^{N+1}}. \end{aligned}$$

Given $\varepsilon > 0$, choose L such that $\frac{1}{p^L+1} < \varepsilon$. This proves (7). □

1.3 Finding p-adic digits

With the definition of the p-adic norm we are now in a position to consider how to retrieve the coefficients of a p-adic expansion. In essence, a p-adic expansion is an approximation of the p-adic number being considered (with $p \in \mathbb{P}$) with respect to the p-adic norm.

Proposition 1.15. *First define $\mathbb{A}_p := \mathbb{Z}_p \cap \overline{\mathbb{Q}}$ where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. Now consider $\mathbb{Z}_p \cap \mathbb{Q} = \mathbb{Z}_{(p)}$ (with $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \,|\, gcd(a,b) = 1\}$).*

It follows that $\mathbb{Z}_{(p)} \subsetneq \mathbb{A}_p$.

An example of the above fact is observed when $p = 5$ and we look at the one of the zeroes of $f(X) = X^2 + 1$, namely $\sqrt{-1}$.

Lemma 1.16 (Hensel's Lemma)**.** *Consider a function $f(X) \in \mathbb{Z}[X]$ and $p \in \mathbb{P}$. If there exists an integer x_0 such that*

$$f(x_0) \equiv 0 \mod p \tag{8}$$

and

$$f'(x_0) \not\equiv 0 \mod p \tag{9}$$

are statisfied, then there exists $\alpha = \sum_{k=n}^{\infty} a_k p^k$ such that $f(\alpha) = 0$.

The proof of Hensel's Lemma stems from Number Theory, and requires the manipulation of some mathematical tools belonging to both Topology and Algebra. For this reason we will dismiss the proof to the publication "The p-adic completion of $\mathbb{Q}_p$ and Hensel's Lemma", by Theodor Christian Herwing. Yet we will use Hensel's Lemma in the following assertion.

Proposition 1.17. *Consider $\alpha \in \mathbb{Z}_p$ to be the p-adic integer we aim to expand as a p-adic series.*

Consider the map

$$\begin{array}{rcccc} Irr & : & \mathbb{A}_p & \to & \mathbb{Z}[X] \\ & & \alpha & \mapsto & Irr(X) \end{array}$$

mapping $\alpha \in \mathbb{A}_p$ to its minimal polynomial with integral coefficients.

We claim the following : if $Irr(\alpha) \equiv 0 \bmod p$ and $Irr'(\alpha) \not\equiv 0 \bmod p$, then Hensel's lemma applies and we can obtain the p-adic expansion by solving the polynomial equation modulo prime powers of p.

Assuming Hensel's lemma holds for $Irr(\alpha)$, we can use induction to illustrate how the algorithm functions.

By the definition given in (6), α can be written as $\sum_{k=0}^{\infty} a_k p^k$, which implies that

$$Irr(\alpha) \equiv 0 \mod p \;=\; Irr(\sum_{k=0}^{\infty} a_k p^k) \equiv 0 \mod p \;=\; Irr(a_0) \equiv 0 \mod p. \tag{10}$$

Solve for a_0. Now we turn towards the computation of a_1 by looking at $0 \bmod p^2$ such that

$$Irr(\alpha) \equiv 0 \mod p^2 \;=\; Irr(\sum_{k=0}^{\infty} a_k p^k) \equiv 0 \mod p^2 \;=\; Irr(a_0 + a_1 \cdot p) \equiv 0 \mod p^2. \tag{11}$$

By inspection, a_1 can be retrieved.

Suppose a_{n-1} *has been computed. Then* a_n *is given by solving*

$$\begin{aligned} &Irr(\alpha) \equiv 0 \mod p^{n+1} \;=\; Irr(\sum_{k=0}^{\infty} a_k p^k) \equiv 0 \mod p^{n+1} \;= \\ &Irr(a_0 + a_1 \cdot p + a_2 \cdot p^2 + \cdots + a_n \cdot p^n) \equiv 0 \mod p^{n+1}. \end{aligned} \tag{12}$$

A more efficient, though more intricate approach consists in computing a_0 *as above and making use of the Taylor expansion in the neighbourhood of* a_0 *to find the remaining coefficients.*

Again to apply the algorithm we must assume Hensel's lemma holds.

We proceed as follows. The Taylor expansion in the neighbourhood of a_0 *of (11) yields*

$$\begin{aligned} &Irr(a_0) + Irr'(a_0) \cdot (a_0 + a_1 \cdot p - a_0) + \frac{1}{2!} Irr''(a_0) \cdot (a_0 + a_1 \cdot p - a_0)^2 \\ &+ \frac{1}{3!} Irr'''(a_0) \cdot (a_0 + a_1 \cdot p - a_0)^3 + \cdots \equiv 0 \mod p^2 \\ &= Irr(a_0) + Irr'(a_0) \cdot (a_1 \cdot p) \equiv 0 \mod p^2 \end{aligned}$$

which under division by p *becomes*

$$\frac{Irr(a_0)}{p} + Irr'(a_0) \cdot a_1 \equiv 0 \mod p.$$

Assume $a_1 = \epsilon,\ Irr'(a_0) = \beta$ *and* $\gamma = \frac{Irr(a_0)}{p}$ *with* β *and* γ *known. We wind up with the simple linear relation*

$$\begin{aligned} &\gamma + \beta \cdot \epsilon \equiv 0 \mod p \\ \Leftrightarrow\ &\epsilon \equiv -\frac{\gamma}{\beta} \mod p \\ \Leftrightarrow\ &\epsilon \equiv -\gamma \cdot \beta^{-1} \mod p. \end{aligned}$$

Once a_1 *is known, we shall determine inductively the rest of the coefficients in a similar manner as with the inspection approach. That is, we solve repeatedly the Taylor expansion of (11) as above for each prime power* n *where* $n \in \{2, 3, \ldots\}$ *such that we obtain the matching* a_n *at each step.*

It is important to note that the usefulness of the Taylor expansion relies on the fact that, for arbitrarily large n, the congruence expression boils down to a straightforward linear equation that can be handled with relative ease.

As far as the uniqueness of p-adic expansions goes, some remarks have to be included. Let us place ourselves exclusively in $\mathbb{Z}_p$. A p-adic expansion is unique if $a_k \in \mathbb{Z}/p\mathbb{Z}$ — which is the case by the very definition of all p-adic integer. However, the polynomial whose root is the desired p-adic integer can furnish two or more p-adic expansions when the second condition (9) of Hensel's Lemma is not verified. This is the case when dealing with double root polynomials, viz. polynomials with a given root α satisfying

$$f(\alpha) \equiv 0 \mod p$$

and

$$f'(\alpha) \equiv 0 \mod p.$$

Moreover, Hensel's lemma (in particular the condition (9) is necessary since division by $Irr'(\alpha)$ is used to retrieve the next p-adic digit of the expansion at each inductive step. If this condition is not satisfied cases arise where an infinite number of p-adic expansions are possible or none are.

Example 1.18. A p-adic expansion can also be obtained by other means. Let $p = 3$, $x = \frac{2}{5}$ and $n = 0$. The p-adic expansion of x can be derived as follows (provided x be a rational number).

$$\frac{2}{5} = 1 - 3\frac{1}{5} \qquad -\frac{4}{5} = 1 - 3\frac{3}{5}$$

$$-\frac{1}{5} = 1 - 3\frac{2}{5} \qquad -\frac{3}{5} = 0 - 3\frac{1}{5}$$

$$-\frac{2}{5} = 2 - 3\frac{4}{5} \qquad -\frac{1}{5} = 1 - 3\frac{2}{5} \quad \cdots$$

Continuation of these elementary calculations indefinitely yields the infinite series

$$\frac{2}{5} = 1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + 1 \cdot 3^3 + 0 \cdot 3^4 + 1 \cdot 5^5 + \cdots$$

which consists of the 3-adic expansion of $\frac{2}{5}$.

Example 1.19. Suppose $\sqrt{-1}$ is the number we want to consider the 5-adic expansion of.
We define $Irr(x) = x^2 + 1$ as described above. Bringing Irr down to the form (10) it follows that

$$a_0^2 + 1 \equiv 0 \mod 5$$

and we can conclude by inspection that $a_0 = 2$ or $a_0 = 3$.

Case 1 : Assume $a_0 = 2$. Then we move on to step two and (11) takes the form

$$\begin{aligned} &(2 + 5 \cdot a_1)^2 + 1 \equiv 0 \mod 25 \\ \Leftrightarrow\ & 25 \cdot a_1^2 + 20 \cdot a_1 + 5 \equiv 0 \mod 25 \\ \Leftrightarrow\ & 20 \cdot a_1 + 5 \equiv 0 \mod 25. \end{aligned}$$

Notice $a_1 = 1$ is the sole solution standing.

Case 2 : Assume $a_0 = 3$. Again we move on to step two and (11) translates into

$$\begin{aligned} &(3 + 5 \cdot a_1)^2 + 1 \equiv 0 \mod 25 \\ \Leftrightarrow\ & 25 \cdot a_1^2 + 30 \cdot a_1 + 10 \equiv 0 \mod 25 \\ \Leftrightarrow\ & 30 \cdot a_1 + 10 \equiv 0 \mod 25. \end{aligned}$$

And this time $a_1 = 3$ is the sole solution standing.

All $a_n,\ n \in \{2, 3, \ldots\}$ will be determined in a unique way hereafter. The choice of one expansion or the other has no consequences on the analytical properties of the p-adic number being considered.

The precise reason why in this specific example two expansions take form is further commented in (2.4).

1.4 Shorthand notation

There exists several widespread notations employed to represent p-adic digits in a concise manner. Most of them are written from right to left by convention. We will adopt this convention and represent the p-adic sequences accordingly.

Definition 1.20. Let $x \in \mathbb{Q}_p$ and let $x = \sum_{k=n}^{\infty} a_k p^k$ be its p-adic expansion. The p-adic notation of x is given by

$$x = \cdots a_{n+5}\ a_{n+4}\ a_{n+3}\ a_{n+2}\ a_{n+1}\ a_n\ {}_{\mathrm{p}}. \tag{13}$$

Furthermore, if x is also a p-adic integer, its expansion is replaced by

$$x = \sum_{\substack{\mathrm{k=n} \\ n \geq 0}}^{\infty} a_k p^k$$

from which we can always pick $n = 0$ (since any coefficient can simply equal 0). Doing thus its p-adic notation will be

$$x = \cdots a_5\ a_4\ a_3\ a_2\ a_1\ a_0\ {}_{\mathrm{p}}. \tag{14}$$

Example 1.21. Recall the p-adic integer in (1.18), with $p = 3$.

Its 3-adic expansion will be

$$\frac{2}{5} = \ \cdots 101211\ {}_{3}$$

with an infinite development on the left side of the shorthand notation.

2 Visualisation

The heart of the project will be resumed in this section. The central facet of the project consists of devising a program capable of graphically representing the p-adic metric (for any given p). The section serves also as guide to demonstrate how to set up and use the program. Furthermore, the relations between the norm of any two representable p-adic numbers can be seen through the proposed choice of graphic examples at the end of the section.

Note : The program was only tested on *Ubuntu* and *Lubuntu*. This guide should work identically on other *Debian*-based Linux OS's. Elsewhere, one needs different commands to achieve the same effect.

2.1 Prerequisites

The program was coded in SageMath, a mathematical software based on the Python programming language. It is intended to be launched through a Linux terminal, and uses LaTeX in order to generate a pdf file containing the visualisation of the desired p-adic integer and the corresponding norm (defined earlier). For the program to run properly, one needs the following.

- A Linux OS.
- The SageMath programming software (available at `http://www.sagemath.org/`).
- A distribution of the document preparation system LaTeX (the TeXLive distribution is available at `https://www.tug.org/texlive/`)

2.2 Setting up the program

Let "`Padic.sage`" be the file containing the code of the program, and let "`/home/F`" be the path of the folder containing that file.

1. Open your terminal and give the command "`cd /home/F`". Doing so will change the directory you are currently working in to the one you indicated.
2. Type "`sage`", and then hit "Enter" in order get into the environment where you can execute sage commands.
3. Using "`attach("Padic.sage")`" will then enable you to use the commands defined in the program and described in the next subsection.

2.3 Using the program

The program automatically creates a pdf file in its root folder. That file will contain a figure visualising p-adic integers and norms. The user can give instructions in the form of parameters, which will be respected during the program's execution. A user command must be of the form

```
padicPolygon(p, levels, rotation, labelling, size (optional), poly (optional))
```

- `p` : This represents the **prime number** considered in the definition of the p-adic integers, and the p-adic norm. It also works with non-prime numbers, but the geometric figure will lose its relation to p-adic integers as a consequence.
- `levels` : Corresponds to a **positive or zero integer**, with which the user can indicate how many levels of polygons they want. The program is subject to the convention that level 0 corresponds to a single polygon.

- rotation : Given by the user as an **angle in radians**, this parameter determines the orientation of the polygons. Consider the circumscribed circle C of the biggest polygon. Let O be its centre. Define a coordinate system such that C is the unit circle. Let A be the point $(1, 0)$, and P a summit of the biggest polygon. Then `rotation` is the angle between (OA) and (OP).

- labelling : Giving `true` as a value for this **boolean** will add the p-adic expansion of the corresponding integer to each point of the polygons. Giving `false` as a value will do no such thing.

- size : This optional **real number** will adjust the size of the polygon with respect to its labels. The default value is 10.

- poly : Let x be a root of this **polynomial**. The p-adic expansion of x will be calculated, and its location inside our figure will be approximated, and marked with a red dot (approximated meaning that the integer with the same p-adic expansion up to the level indicated by the user will be chosen). The size of the dot will also be affected by the value of the parameter `size`.

Note regarding the optionality of two arguments : There are three choices on how many of the two last arguments the user can indicate, they are listed below.

- If none of the last two arguments are given, their default values will be chosen. (10 for `size` and 0 for `poly`)

- If one of the last two arguments are given, that value will be assigned to `size`, and `poly` will receive its default value.

- If both arguments are indicated, they will be affected to `size` and `poly` respectively.

2.4 Some visualisations of p-adic norms

A common method to represent integers is by equidistant points on a line. In this way, the conventional norm in $\mathbb{Z}$ is visualised. The distance between two integers is equal to the number of those equidistant segments between these points. Our aim is to apply a similar visualisation with the p-adic norm.

Suppose the side of the biggest polygon is of length 1. Then the side of the second biggest polygon is of size $\frac{1}{p}$, that of the third biggest of size $\frac{1}{p^2}$, and so on.

Let $a, b \in \mathbb{R}$. The p-adic distance between a and b, $|a-b|_p$ is equal to the length of the side of the smallest polygon containing a and b.

Let $x \in \mathbb{N}_{\geq 0}$, and $x = \sum_{k=0}^{n} a_k p^k$ its p-adic expansion up to the order $n \in \mathbb{N}_{\geq 0}$. Then the labels in the following figures are of the form: $[a_0, a_1, ..., a_n]$

It is also possible to compute a p-adic expansion of some numbers that are not positive integers or positive numbers, for instance the 3-adic expansions of negative integers such as -1, quotients such as $-\frac{1}{2}$, irrational numbers such as $\sqrt{7}$ or $\sqrt[5]{5}$. Even the 5-adic expansion of complex numbers such as $\sqrt{-1}$ exist.

The expansion of $a \in \mathbb{Z}_p$ can be determined by the simplest polynomial with integral coefficients of root a. Note that such a polynomial can have other roots too. Looking back to example (1.19), we notice that the polynomial x^2+1 has the roots $\sqrt{-1}$ and $-\sqrt{-1}$ (in $\mathbb{C}$). So from that polynomial, it is possible to obtain two different expansions, one of which will be the expansion of $\sqrt{-1}$, and the other that of $-\sqrt{-1}$. However, it is impossible to determine which expansion belongs to which number, it is simply a matter of convention.

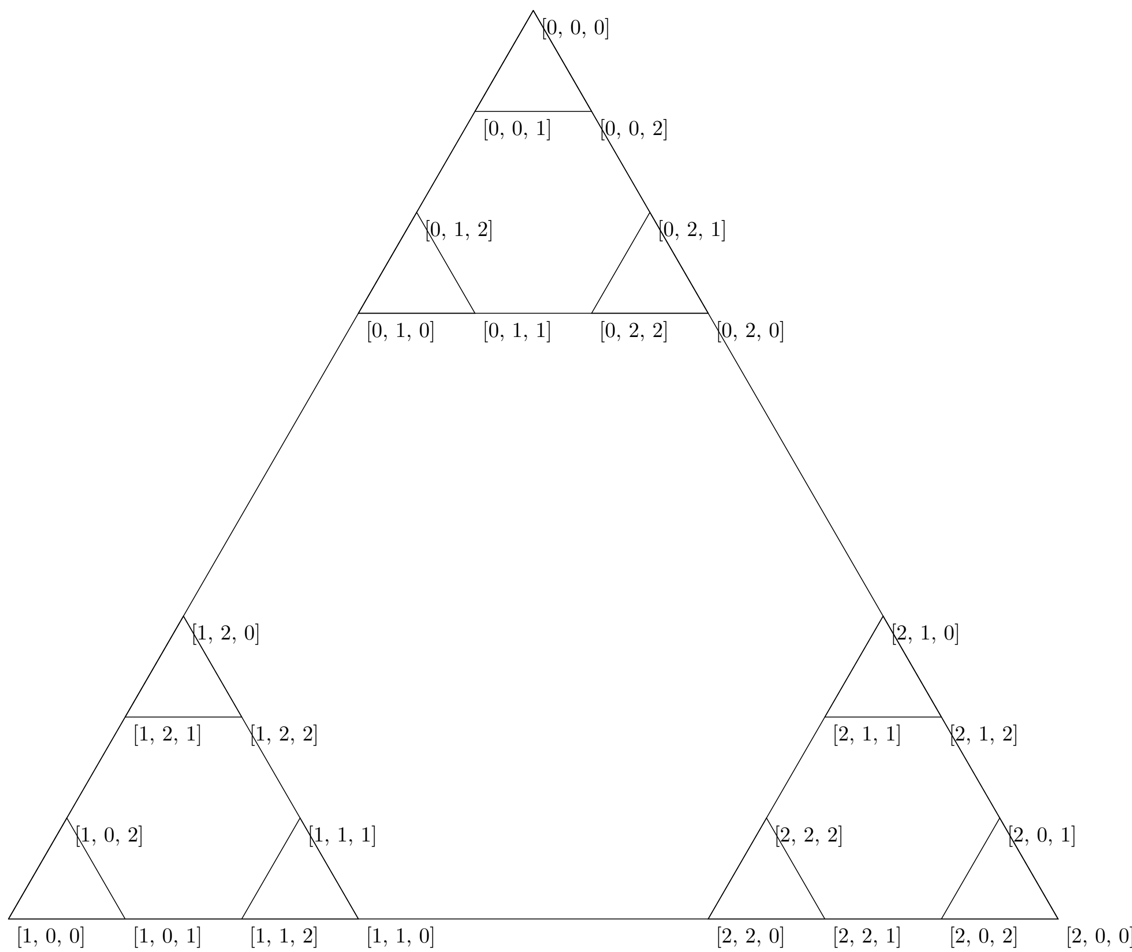

Figure 1: 3-adic integers, level 2

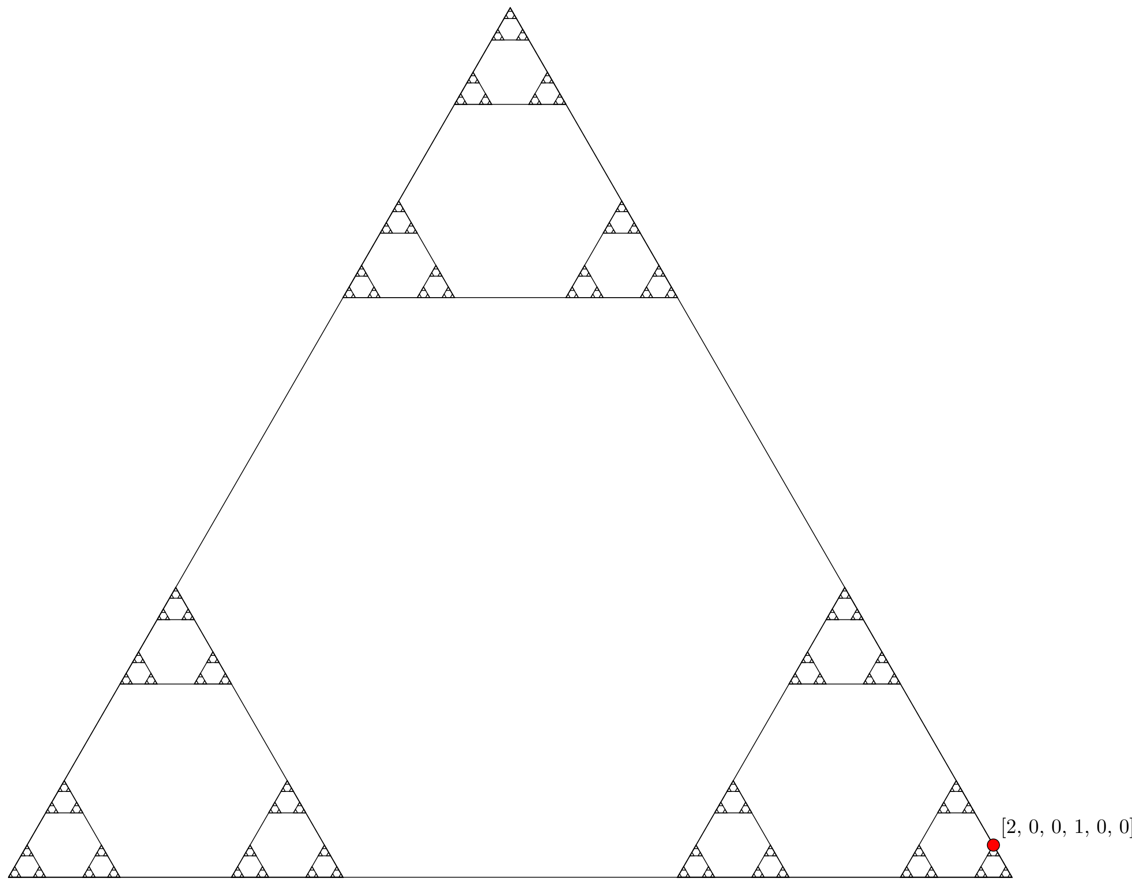

Figure 2: approximate location of $\sqrt[5]{5}$ in the representation of 3-adic integers, level 5

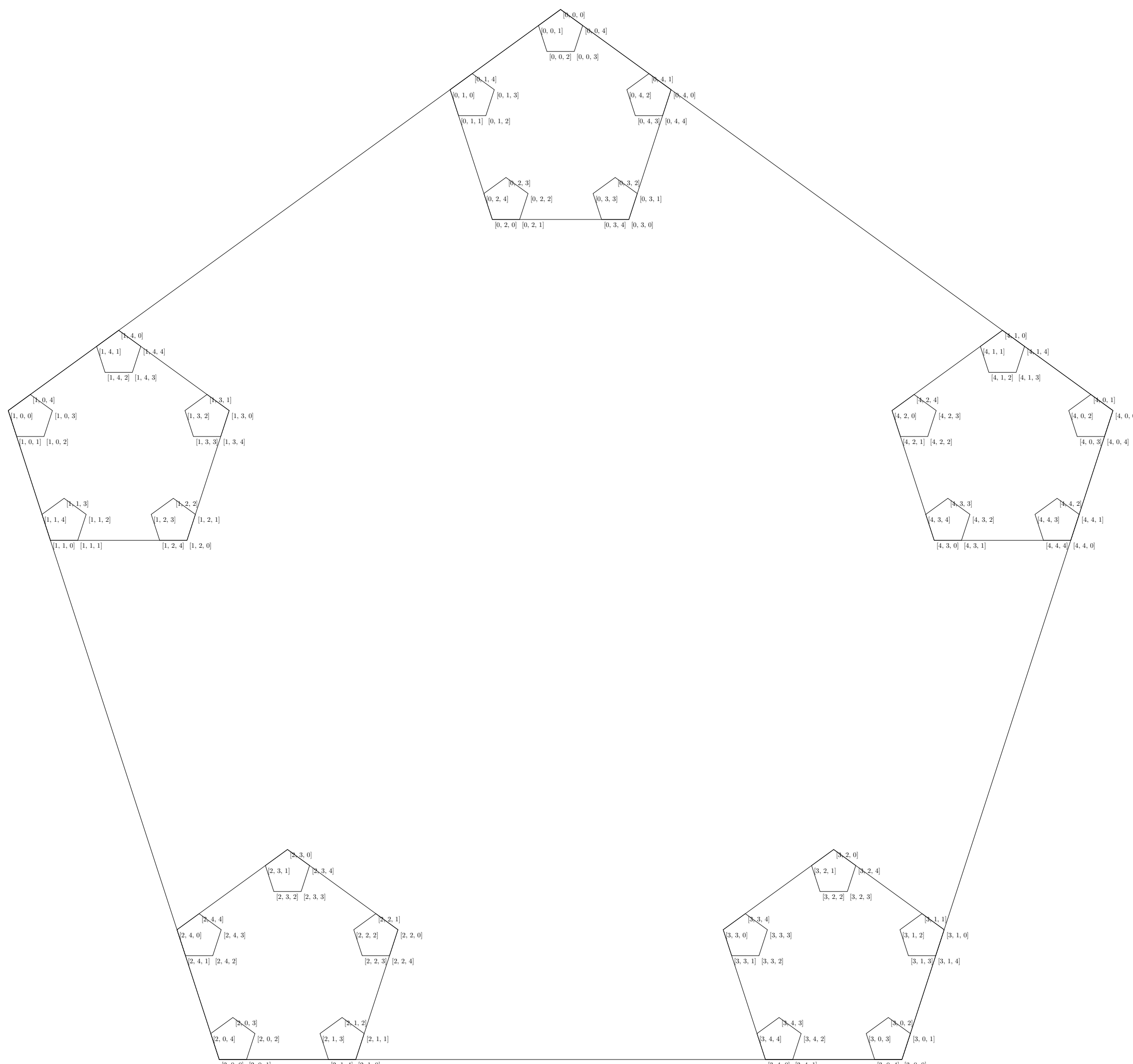

Figure 3: 5-adic integers, level 2

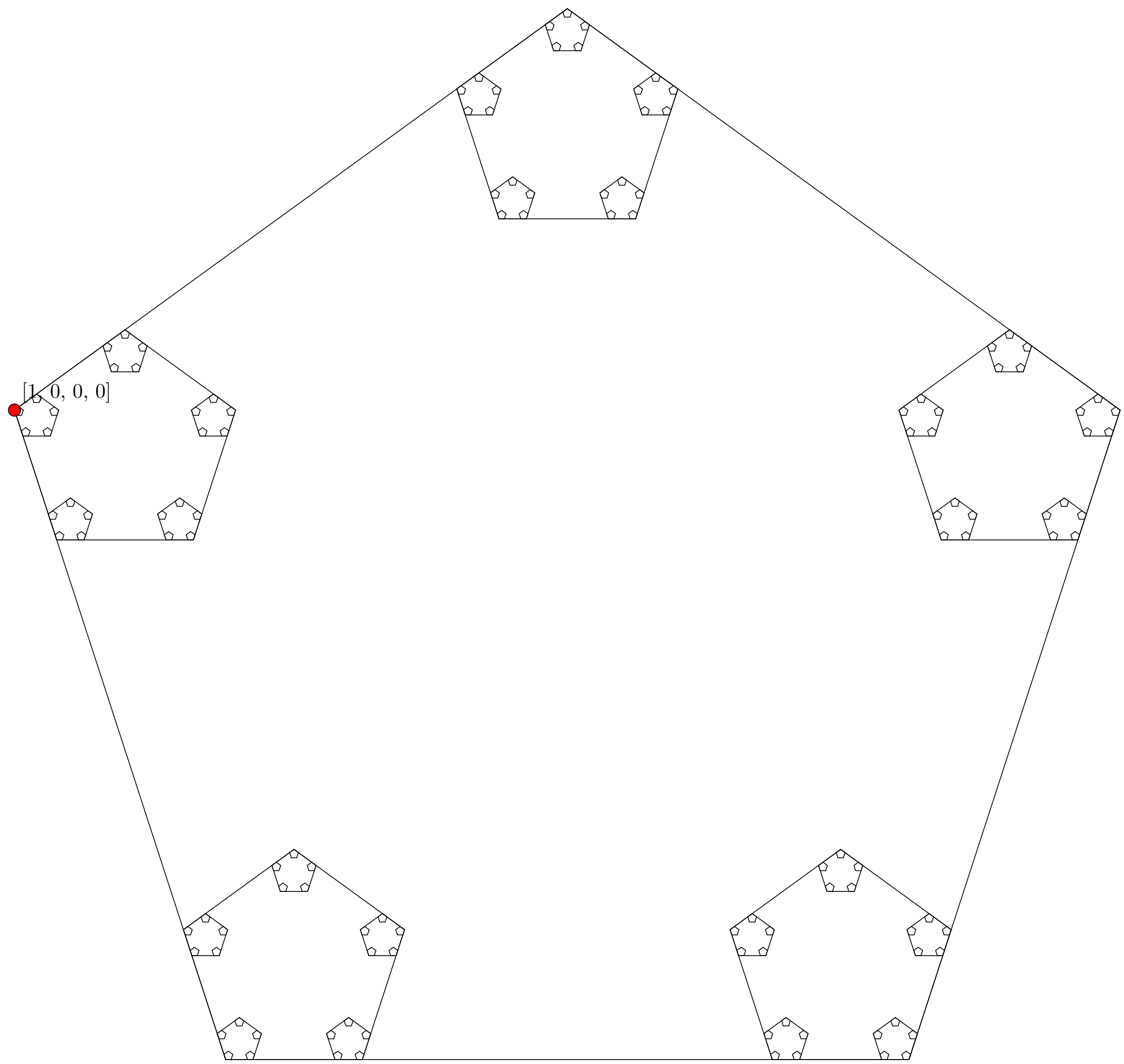

Figure 4: approximate location of $\sqrt{-1}$ or $-\sqrt{-1}$ in the representation of 5-adic integers, level 3

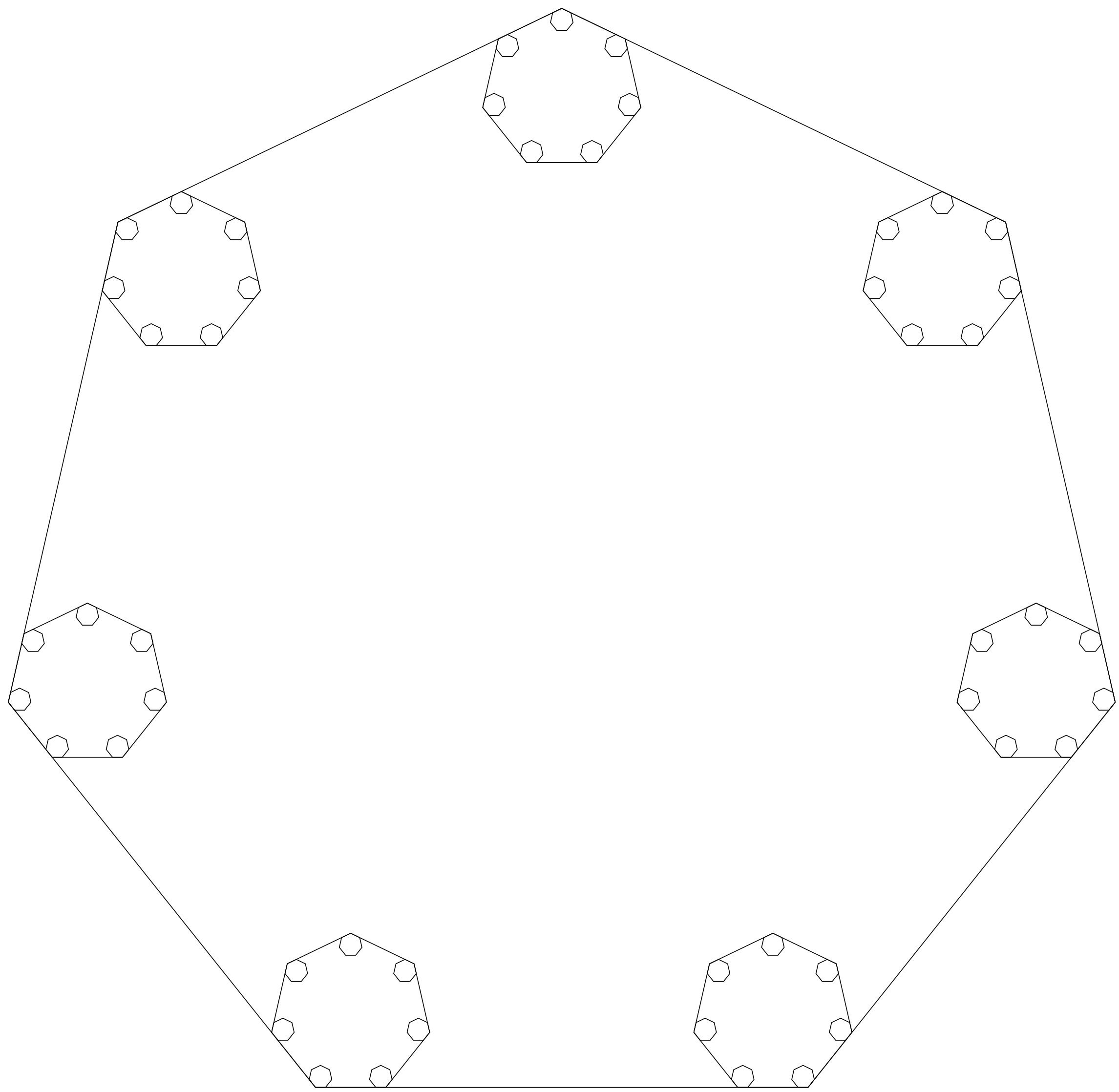

Figure 5: 7-adic integers, level 2

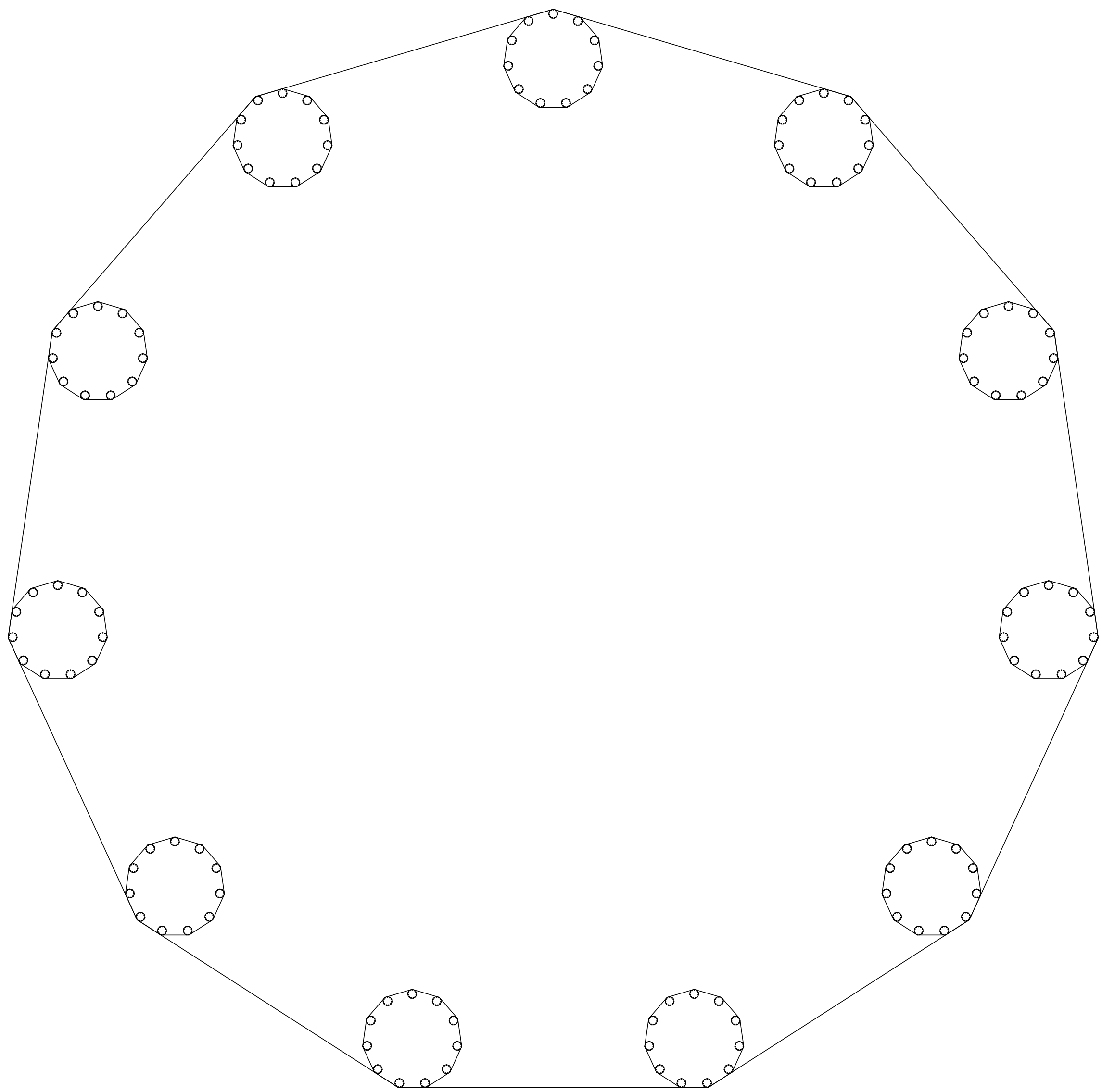

Figure 6: 11-adic integers, level 3

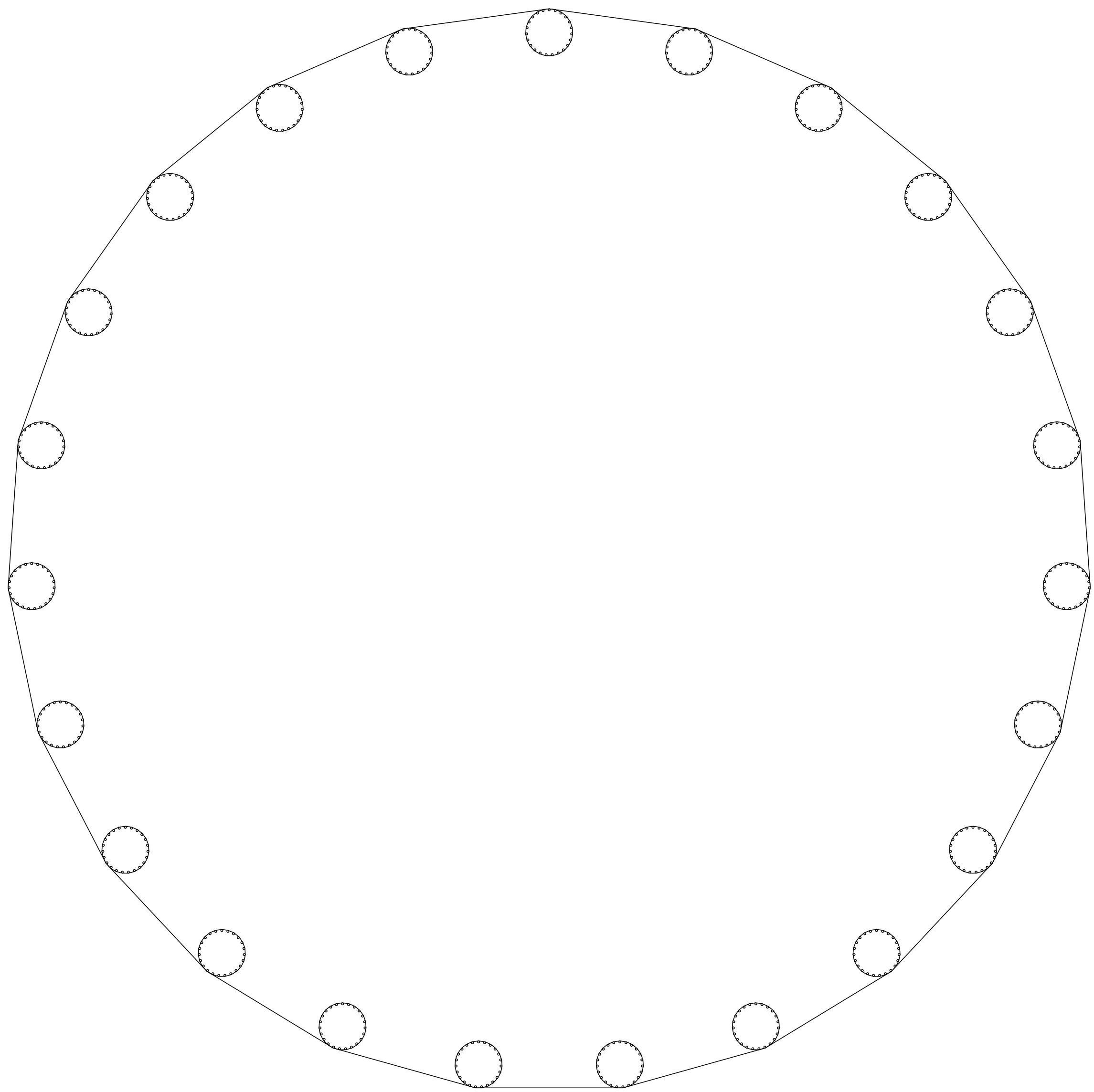

Figure 7: 23-adic integers, level 2