Visual Fractions

https://math.uni.lu/fractions

The website contains hundreds of illustrations to understand fractions and their basic arithmetic operations (addition, subtraction, multiplication, division).

We restrict to positive fractions not to mix the difficulty of fractions with that of the sign rules (a student who has understood the sign rules for the integers should be able to transfer them to fractions).

1 The meaning of fractions & equivalence of fractions

Introducing the numerator and the denominator

1.1 Cutting equal parts

We cut one object into equal slices, taking note of the number of slices that we make (the denominator).

In particular, we cut circular pizzas, linear baguettes, and rectangular chocolate bars. The chocolate bar is cut either horizontally, or vertically, or as a grid.

We also show a large circle divided into 100 parts, and a large rectangle divided into 1000 parts.

1.2 Taking some of the equal parts

We display some identical slices, taking note of how many slices we have (the numerator) and how many slices form the whole object (the denominator).

We show that it's only the number of slices that counts, and not their position.

We also consider the case of having zero slices.

1.3 Many of the equal parts

We display many identical slices, stemming from cutting more than one object.

A rectangular frame conveys the idea that all the slices within the frame must be counted.

Equivalence of fractions

1.4 Cutting further

We start with the illustration of a fraction, then make dashed cuts, and finally display the fully subdivided object. The intermediate step for the cutting procedure is omitted later.

Curved arrows show that the cutting involves a multiplication by a same number for the numerator and the denominator.

We do various types of cutting for rectangular objects.

An illustration displays that cutting into a slices and then cutting each of them into b slices is the same as cutting into $a \times b$ slices.

1.5 Bigger slices

We show how slices can be grouped together. The cutting process cannot be undone in real life, so we just imagine making larger slices in a different scenario.

The grouping is shown with an explicit framing.

Curved arrows show that the taking bigger slices involves a division by a same number for the numerator and the denominator.

1.6 Cutting entire pieces

As a special case, we cut several entire objects into one, two, or more equal pieces.

The idea of equivalent fractions is extended to natural numbers.

Comparison of fractions

1.7 Comparing equal slices

We compare the quantity of two collections of slices of the same type.

We make clear that it is only the number of slices (the numerator) that counts to obtain an equality or an inequality.

1.8 Comparing slices of different sizes

We compare the quantity of two collections of slices. There is the same number of slices in each collection, but the sizes differ.

We make clear that in this case it is only the size of slices (the denominator) that counts to obtain an equality or an inequality.

1.9 Comparing general amounts

We compare two collections of slices having different sizes. The procedure is cutting them further as to have all slices of the same size. Then the comparison of the quantity can be made by counting the number of slices.

1.10 Comparing to entire pieces

As a special case, we compare a collection of slices to a collection of whole objects.

1.11 Fractions on the number line

To illustrate fractions on the number line we consider a baguette aligned to it. The baguette is cut into pieces and there are appropriate markings.

Improper fractions are depicted with more baguettes placed end to end.

Coloured strips replace the baguettes for a more abstract illustration (to thicken the number line). The strips have a vertical gradient for the colors to convey that no horizontal cut is meaningful.

We chose to use the convention to use integers along the number line instead of the equivalent fraction.

1.12 Equivalent fractions on the number line

We show that equivalent fractions occupy the same point on the number line.

Two number lines are stacked on top of each other, one divided more coarsely than the other. The shaded strips or the baguettes end at exactly the same horizontal position on both number lines.

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2 Addition and subtraction of fractions

2.1 Collecting equal slices

These slides illustrate the simplest possible case of fraction addition, where both fractions share the same denominator and the results remain at or below 1. Two coloured shapes, one blue and one orange, appear on the left and right of a plus sign. The result is shown on the right with both colours visible in the combined shape.

The colours of the two addends are always preserved in the result, so pupils can trace each contribution. This establishes the foundational intuition that fraction addition is really just counting same-sized parts before any complexity is introduced.

"Why do we only add the numerators and not the denominators? What would it mean if we added the denominators too?"

2.2 Collecting many equal slices

This section keeps the same format as before, but the result now exceeds one whole. The result is shown inside a bordered rectangle.

This normalizes improper fractions as a natural outcome of addition rather than something unusual or wrong.

2.3 Collecting slices

Each illustration uses a two-row layout. The top row shows the original fractions with unlike denominators. The bottom row shows a modified representation of the addition. A downward arrow points to the modified representation of an addend, where dashed lines have been added to show the cutting further. A rightward arrow leads to the combined result.

A common error is to add both numerators and denominators separately. The illustration directly addresses this. The teacher can point out that the resulting slices are not the same size, so simply counting them makes no sense without conversion first.

2.4 Collecting horizontal and vertical slices

The same two-row layout is used, but now the fractions are cut in perpendicular directions: one horizontally and one vertically.

2.5 Collecting many slices

As in section 2.2, the results are now improper fractions. This section, however, covers the addition of fractions with unlike denominators. The format stays the same as in the two previous sections.

2.6 Addition on the number line

Three stacked number lines are shown in each example. The top two number lines represent the two addends as separate coloured gradient bars. The third line shows the combined bar end-to-end, with a label at the resulting position.

The three-row layout makes it explicit that the second bar starts exactly where the first one ends. For unlike-denominator examples, we show the cutting subtly with dashed lines.

See also: section 2.14 for the corresponding subtraction on the number line, where the second bar is lifted away rather than appended.

Properties of addition

2.7 Addition with equivalent fractions

Each slide shows two rows of the same addition. The top row uses one form of the fraction, while the bottom row uses an equivalent form. Dashed lines mark the finer subdivision in the equivalent version. In all cases, the result is identical.

2.8 Adding zero

Two circle diagrams are shown side by side, separated by a plus sign. In the first case, the blue fraction appears on the left and an empty, unshaded circle appears on the right. In the second case, the order is reversed.

2.9 Commutativity

Each slide shows two rows: the same addition with the two coloured addends swapped. The result in both rows is visually identical.

By seeing the two versions side by side consistently, pupils absorb this as a reliable pattern rather than a rule to be memorised in the abstract.

See also: section 3.11 for commutativity of multiplication.

2.10 Associativity

Three addends are shown in circles. The slide presents both possible groupings. In the first grouping, the first two addends are combined first (shown with a curly brace underneath), then the result is added to the third. In the second grouping, the last two are combined first. Both routes arrive at the same final result. A second layout uses a branching arrow structure rather than a linear sequence, showing both groupings fanning out from the three addends and converging on the same result.

Subtraction of fractions

2.11 Removing equal slices

These slides have the same format as the ones for the addition of fractions, but with a minus sign. The result shows only the remaining blue shading after the orange slices have been lifted away.

The orange shading in the subtrahend always matches the exact position of the pieces that have been removed to show the difference. This enforces the idea of the subtraction as the action of removing.

2.12 Removing many slices

The minuend is now an improper fraction, and slices are removed until a smaller improper fraction or proper fractions remain. The minuend and the subtrahend are now not in the same row. The left side represents the operation while the right side contains the result.

This section normalises working with improper fractions in subtraction and reinforces that improper fractions behave consistently under subtraction.

2.13 Removing slices

The two-row layout mirrors the layout from section 2.3 "collecting slices". The top row shows the original fractions, and the bottom row shows the modified representations where the operation is carried out. A downward arrow marks the replacement by an equivalent fraction, where dashed lines represent the cutting procedure employed to convert the fractions to a common denominator.

2.14 Subtraction on the number line

Three stacked number lines are shown. The top line shows the minuend as a shaded gradient bar, and the second shows the subtrahend as a shorter bar shaded in another colour. The third number line shows the result of the subtraction. The visual effect is that the second bar is lifted away from the first bar.

This illustration shows the contrast with the addition on the number line, where the second bar was appended to the first. For the subtraction, the second bar is lifted away from the first.

2.15 Subtraction with equivalent fractions

This section mirrors the section on addition with equivalent fractions (section 2.7). Two rows each illustrate a subtraction. In the second row, either the minuend or the subtrahend has been replaced by an equivalent fraction.

2.16 Subtracting zero

Two circle diagrams are shown side by side, separated by a minus sign. The second circle represents the zero fractions. It illustrates that removing no slices does not change the amount.

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3 Multiplication of fractions

3.1 Integer × Fraction

The pattern is straightforward: multiplying a fraction by a whole number n means copying the fraction n times. This pattern is illustrated for unit and non-unit fractions, including improper fractions.

"What does it mean to take three copies of a fraction? Is that the same as adding the fraction three times?"

3.2 Integer $\times$ Unit fraction (as a partition)

Here multiplying by a unit fraction is interpreted as a sharing operation: $\frac{1}{4} \times 4$ asks "if you share 4 bars equally among 4, how much does each person get?" The vertical line marks the boundary of one equal group, making the partition visually explicit.

This interpretation of multiplication by a unit fraction as sharing connects directly to the division work in section 4. Pointing out this link early prepares pupils for the reciprocal relationship.

3.3 Integer $\times$ Unit fraction (in general)

This section generalises the partition idea to cases where the result is not a whole number. Instead, each whole is partitioned separately into the same number of parts.

This section generalises that multiplying an integer n by a unit fraction $\frac{1}{m}$ produces $\frac{n}{m}$. It reinforces that the denominator is unchanged and that the numerators accumulate with each step.

3.4 Fraction $\times$ Unit fraction

This section illustrated a method that we will call "cracking". For instance, the representation of the fraction $\frac{2}{4}$ contains two coloured parts. To illustrate $\frac{1}{2} \times \frac{2}{4}$, we are going to crack away one half of those two parts, which is exactly one part. Hence, the cracking gives us the resulting product of $\frac{1}{4}$.

The "cracking" strategy is to divide the numerator of the given fraction by the denominator of the unit fraction. This works perfectly when the numerator is divisible by that denominator. In case of non-divisibility, the strategy is to cut the fraction into smaller parts in order to meet the divisibility criterion. As before, the cutting is done in an extra step.

3.5 Integer $\times$ Fraction (two steps)

Multiplying by a non-unit fraction like $\frac{2}{3}$ is broken into two successive operations. First, scale by the unit fraction $\frac{1}{3}$, which is illustrated by partitioning the whole into three and colouring one, and then multiply by the integer 2, which is illustrated by copying the intermediate step two times, or vice versa.

This two-step decomposition makes the process explicit and transparent. By showing both orderings, we plant the seed of commutativity.

3.6 Integer $\times$ Fraction (with an integer division)

Here, we use again the method of "cracking" the numerator of the second fraction by the denominator of the first fraction. However, as we crack by a non-unit fraction, we crack off more than one part.

This extends the cracking method to non-unit fractions. In cases where the division does not work out neatly, we transform our fraction first into an equivalent form by cutting further so that the number of parts matches. This transformation is shown by a downward arrow.

3.7 Alternative cracking styles

In this final section, we present the cracking step in different visual formats and orders. This demonstrates the flexibility of the cracking method.

Multiplication of fractions on the number line

3.8 Integer × Fraction (number line)

The fraction is shown as a coloured segment on a top number line, and repeated arcs of that same length hop along a bottom number line to show multiplication as repeated addition. Each arc is a different colour, making each copy of the fraction visually distinct. The final position of the last arc gives the result directly as a fraction on the number line.

3.9 Fraction × Unit fraction (number line)

This section uses either a two-step or three-step number line sequence to show that multiplying by a unit fraction $\frac{1}{n}$ means partitioning. A long bar is first shown, then divided into n equal arcs, and finally, a single arc of the resulting smaller size is highlighted as the answer. The three levels make the interpretation very concrete. If the division into n equal arcs can immediately be seen on the top number line by the way it is partitioned, the operation is done in two lines. If the division into n equal arcs is not clear given the partition, the bar is cut further into a finer partition.

3.10 Fraction × Fraction (number line)

Multiplying two general fractions is handled in two successive steps on the number line, mirroring the two-step decomposition from the earlier section using the area model. The first step applies the unit fraction (one large arc), and the second step repeats that arc the appropriate number of times.

Properties of multiplication

3.11 Commutativity of multiplication

Each slide has two rows that illustrate the same multiplication with the two factors swapped. The result in both rows is visually identical.

By seeing the two versions side by side consistently, pupils absorb this as a reliable pattern rather than a rule to be memorised in the abstract.

3.12 Multiplication by 0

Taking copies of zeros or partitioning zero into a given number of parts does not change the fact that we still have nothing.

A brief but important reminder that the zero property applies universally. Any number, including any fraction, multiplied by zero equals zero.

3.13 Multiplication by 1

Multiplying any number by 1 is the same as copying this number one single time.

Multiplying by 1 always returns that same number unchanged. This identity property holds for fractions just as it does for whole numbers.

3.14 Distributivity

The figure is divided into two sides, separated by a dashed line. The left-hand side and the right-hand side show effectively the same result, however, they are represented differently.

When multiplying a sum or a difference by a number or a fraction, one can apply the multiplication to each term individually and the result will be the same.

See also: section 4.12 for the corresponding distributivity of division.

3.15 On the result (times a proper fraction)

A fraction is illustrated on the left. On the right-hand side, the illustration shows the result of multiplying this fraction by a proper fraction. The line underneath the illustration shows the conclusion algebraically: the product is smaller than the initial fraction.

By multiplying by a proper fraction, the result gets smaller than what we started with. Scaling by a proper fraction represents taking only a portion of the original value.

3.16 On the result (times an improper fraction)

A fraction is illustrated on the left. On the right-hand side, the illustration shows the result of multiplying this fraction by an improper fraction. The line underneath the illustration shows the conclusion algebraically: the product is bigger than the initial fraction.

Multiplying a number by a fraction that is greater than 1 will always produce a result larger than the original number. Scaling by an improper fraction means increasing beyond the original value.

This section is most effective when shown alongside section 3.15 so pupils can directly compare the effects of multiplying by proper fractions versus improper fractions.

3.17 Multiplication with equivalent fractions

Each slide shows two columns of the same multiplication, split by a dashed line one starting from the original fraction, one from an equivalent form. An orange line marks the cracking cut; where needed, a downward arrow first converts to the finer form before cracking. Both columns give the same product.

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4 Division of fractions

4.1 Dividing integers

Rule: $a : b = a \times \frac{1}{b}$

This section illustrates that dividing by a whole number is the same as sharing into equal amounts. They also show that dividing by a number is the same as multiplying by its reciprocal. Two types of illustrations are used: circles and rectangles. In each case, the top row shows the original objects, an arrow points downward to show these objects cut, and the results are shown on the right. A matching algebraic statement is found at the bottom or on the right-hand side.

4.2 Dividing a fraction by an integer

Rule: $\frac{a}{b} : n = \frac{a}{b} \times \frac{1}{n}$

These slides illustrate what happens when a fractional quantity is shared equally. The number of coloured parts is divided equally among a given number of groups. This is illustrated by the branched arrows that split up the coloured regions.

We chose not to use examples where the division requires the extra step of cutting further for simplicity.

4.3 Dividing an integer by a unit fraction

Rule: $a : \frac{1}{n} = a \times n$

Each illustration pairs two rows of rectangles. The top row shows the whole integers as solid coloured blocks, with a yellow "carpet" marking one whole block. A downward arrow leads to the second line, where each rectangle is divided into n equal parts. The yellow carpet stays unchanged. A final rightward arrow then points to the resulting collection of individual pieces. Here, an orange carpet highlights the size of the unit fraction by which we divided.

The key insight is counter-intuitive: dividing makes the answer bigger. Pose the question, "how many halves fit into three wholes?" and note how the yellow carpet shifts to the orange carpet as the unit changes from the whole to the divisor.

"How many quarters fit into one whole? How many fit into three wholes? Did the answer surprise you?"

See also: section 4.4, which extends this to dividing a fraction by a unit fraction, and section 4.9 for the number line version.

4.4 Dividing a fraction by a unit fraction

Rule: $\frac{a}{b} : \frac{1}{n} = \frac{a}{b} \times n$

A horizontal strip represents one whole, which is marked by the yellow carpet. The dividend fraction is represented by the horizontal strip and coloured accordingly. Then, the rightward arrow changes the unit from the whole to the divisor, which is marked by the orange carpet. By comparing the shaded region to the orange carpet, pupils can determine the quotient. Results may be whole numbers, proper fractions, or improper fractions.

In the case where the width of the orange carpet cannot be exactly determined by the partition of the rectangle, the representation is replaced by one that has been cut further into smaller parts whose widths correspond to the width of the divisor. This is marked by a downward arrow.

This is a good point to consolidate the "how many fit in" language established in section 4.3.

4.5 Dividing an integer by a fraction

Rule: $n : \frac{a}{b} = n \times \frac{b}{a}$

In this section, rectangles are fully coloured to represent whole integers. The unit is again highlighted by the yellow carpet. A downward arrow leads to a second row of the same rectangles, now subdivided into the divisor's denominator units. A rightward arrow then reveals the resulting groups. The changing of the unit is again illustrated by the orange carpet. Pupils can read off the number of the quotient of both fractions. The corresponding multiplication by the reciprocal is always stated alongside.

This section builds on section 4.3. The divisor is no longer a unit fraction. The orange carpet illustrates this quite well as it now highlights more than one sub-piece.

4.6 Dividing a fraction by a proper fraction

Rule: $\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, *where* $c < d$

A strip model is used throughout this section. The top strip shows the dividend fraction shaded in one colour against the total length of the strip, with a yellow carpet highlighting the whole as a unit. A rightward arrow leads to a second strip of the same length where both the dividend shading and the orange carpet representing the divisor are visible simultaneously. Pupils compare the shaded region to the orange carpet to determine how many times the divisor fits.

The extra step of cutting further is again used in order to represent the dividend in a way that allows shading the orange carpet accordingly. It is marked by a downward arrow.

4.7 Dividing a fraction by an improper fraction

Rule: $\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, *where* $c > d$

The left strip represents the dividend, fully shaded with a yellow carpet representing the whole. The right strip (or pair of strips) represents the division by an improper fraction. While the representation of the dividend is unchanged, the carpet changed width and colour. The orange carpet marks the full span of the divisor. A rightward arrow connects the comparison.

Improper fractions as divisors are often the most challenging case. Here, one can see that the orange carpet does not represent the length of the divisor. In fact, the orange carpet highlights the new number of parts the unit is divided into. It shows how many parts fit within the new unit. Its length is not proportional to the length of the unit.

Division of fractions on the number line

4.8 Dividing integers (number line)

Each slide shows two horizontal number lines stacked one above the other. The top has a yellow carpet highlighting the unit interval, and a cyan-to-violet gradient bar showing the dividend. The bottom line replaces the yellow carpet with the orange carpet. The orange carpet represents the divisor. The key visual idea is that the orange carpet is the "measuring stick": it counts the total number of parts. The gradient bar counts the number of selected parts.

4.9 Dividing by a unit fraction (number line)

The same structure is used for the slides illustrating the division by a unit fraction. Here, there may be two number lines or three number lines stacked on top of each other. The top shows the dividend in coarser units, the middle number line refines the subdivision to match the unit fraction's denominator (if needed), and the bottom represents the division. The cyan-to-violet gradient consistently marks the dividend across all two or three rows. Here, the orange carpet is narrower than before, as it highlights the width of the divisor, which is a unit fraction in this case.

The three-level presentation (coarse unit → fine subdivision → result) mirrors the thought process pupils should internalise: first identify the whole, then establish the new unit, and then count.

4.10 Dividing by a fraction (number line)

As we now illustrate the division by a non-unit fraction, the orange carpet spans multiple subdivision ticks, representing a proper or improper fraction divisor. Because the divisor is a non-unit fraction, the orange band is wider than a single tick, making the comparison between dividend length and divisor length the central visual act. Examples include cases where the divisor is smaller than, equal to, and greater than the dividend, producing results greater than 1, equal to 1, and less than 1, respectively.

Properties of division

4.11 Division by 1

Each slide shows two pairs of rectangles side by side, connected by a rightward arrow. The left pair has a yellow carpet that highlights the whole as a unit. Both rectangles are shaded to represent the dividend. The right rectangle has an orange carpet which highlights the divisor. Here, the divisor is 1, so the orange carpet spans the same width as a full unit. Since the orange carpet matches the yellow carpet exactly, the result equals the dividend itself. Three cases are shown: a whole number, a proper fraction, and an improper fraction.

4.12 Distributivity

These slides demonstrate that division distributes over addition and subtraction, paralleling the distributive law for multiplication. The left side of each slide shows the combined quantity, and the right side, separated by a dashed vertical line, shows the two colour groups split apart. In every case, the algebraic statement is written below in two lines. First, in the division form, then the equivalent multiplication by the reciprocal.

4.13 Dividing by a proper fraction

Each slide shows the same strip model used in the main slides for division. However, the equation below is replaced by an inequality. The dividend is shown to be less than the result of dividing it by a proper fraction.

This is a conceptually important slide. Many pupils expect division to always make things smaller. The visual makes the counter-intuitive result concrete: the orange carpet is smaller than the yellow carpet. This means that the measuring unit is smaller, so it fits multiple times, pushing the answer above the original value.

4.14 Dividing by an improper fraction

This section is the mirror image of the previous section. The same strip model is used, but now the orange carpet representing the divisor is wider than one whole unit, extending beyond the bordered rectangle. The inequality is reversed. As the measuring unit is larger than the dividend, less than one full copy fits, so the result is smaller than the original.

This section is most effective when shown alongside section 4.13 so pupils can directly compare the effects of dividing by proper fractions versus improper fractions.

4.15 Taking the reciprocal

Each slide has the same three-part structure. At the top, a fully shaded rectangle represents one whole. This is underlined by the yellow carpet representing the unit. A downward arrow leads to a second row where the same rectangle is subdivided by the denominator of the divisor fraction. A rightward arrow then shows the division. The orange carpet shows the new unit defined by the divisor. The algebraic statement can be found at the top right. Below the diagram, the original fractions and their reciprocals are written side by side, with two curved arrows connecting them in order to show how the numerator and the denominator are flipped.

The double curved arrow is the centerpiece of the illustration. It shows that the numerator and denominator simply swap and that the relationship is symmetric. Pupils often memorise “flip the fraction” without understanding why. This serves as the conceptual bridge to the general rule used in all the earlier division categories.

4.16 Division with equivalent fractions

Each slide contrasts two columns: the left converts the dividend into an equivalent fraction (downward arrow, dashed cuts) before applying the orange carpet; the right starts from that equivalent form directly. Both columns yield the same quotient, closing the pattern from sections 2.7, 2.15, and 3.17 for division.

To illustrate division we often make use of *the magic carpet*. The quantity inside the carpet will be our unit: there is an initial yellow carpet and a final orange carpet after the rescaling. This is not the same as working on the number line. Indeed, our pieces may also be disjoint: important is then not the width of the carpet but the amount of pieces that it contains.