BOUNDED COHOMOLOGY AND VOLUME RIGIDITY

EXERCISE 1. Use the continuous maps

$$\mathrm{C_b}(G^n,\mathbb{R})^G \underset{\tau^n}{\overset{\rho^{n-1}}{\rightleftarrows}} \mathrm{C_b}(G^{n-1},\mathbb{R})$$

defined by

$$\begin{aligned}(\rho^{n-1}f)(g_1,\dots,g_{n-1}) &:= f(e,g_1,g_1g_2,\dots,g_1g_2\cdots g_{n-1})\\ (\tau^n f')(g_1,\dots,g_n) &:= \pi(g_1)f'(g_1^{-1}g_2,g_2^{-1}g_3,\dots,g_{n-1}^{-1}g_n)\end{aligned}$$

for $f\in \mathrm{C_b}(G^n,\mathbb{R})^G$ and $f'\in \mathrm{C_b}(G^{n-1},\mathbb{R})$ to give a non-homogeneous definition of continuous bounded cohomology.

EXERCISE 2. Compute

(1) $\mathrm{H}^0_{\mathrm{cb}}(G,\mathbb{R})$
(2) $\mathrm{H}^1_{\mathrm{cb}}(G,\mathbb{R})$

EXERCISE 3. Let $\mathcal{QM}(G,\mathbb{R})$ be the space of *quasimorphisms* of G, where $f:G\to\mathbb{R}$ is quasimorphism if it satisfies

$$\sup_{g_1,g_2\in G}|f(g_1g_2)-f(g_1)-f(g_2)|<+\infty\,.$$

Show that

$$\ker\{\mathrm{H}^2_{\mathrm{b}}(G,\mathbb{R})\to\mathrm{H}^2(G,\mathbb{R})\}\cong\mathcal{QM}(G,\mathbb{R})/(\ell^\infty(G)+\mathrm{Hom}(G,\mathbb{R}))$$

is an isomorphism of vector spaces.

EXERCISE 4. Let $\mathbb{F}_2$ be the free group in two generators.

(1) Show that $\mathrm{H}^2(\mathbb{F}_2,\mathbb{R})=0$;
(2) Let $w\in\mathbb{F}_2$ be a reduced word and let $\theta_w:\mathbb{F}_2\to\mathbb{Z}$ be the function that to $x\in\mathbb{F}_2$ associates the number of times the word w appears in x as a subword (with overlapping). Show that the function $f_w:\mathbb{F}_2\to\mathbb{Z}$ defined by

$$f_w(x):=\theta_w(x)-\theta_{w^{-1}}(x)$$

is a quasi-morphism.

(3) Try to use Part (2) to show that $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2,\mathbb{R})$ is infinite dimensional.

(4) Show that $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2,\mathbb{R})$ is infinite dimensional by showing that there is an injection

$$\ell^\infty(\mathbb{Z}) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2,\mathbb{R})\,.$$

[*Hint:* Let $\sigma \in \ell^\infty(\mathbb{Z})$ be an odd sequence, that is $\sigma(-n) = -\sigma(n)$. If $\mathbb{F}_2 = \langle a, b\rangle$ and $x = a^{k_1}b^{k_2}\cdots a^{k_{n-1}}b^{k_n} \in \mathbb{F}_2$ with $k_2\cdots k_{n-1} \neq 0$, show that the map $g_\sigma : \mathbb{F}_2 \to \mathbb{R}$ defined by

$$g_\sigma(x) = \sum_{j=1}^{n} \sigma(k_j)$$

is a quasi-morphism.]

EXERCISE 5.

(1) Let $x \in S^1$ be fixed. Show that the function $c_x : \mathrm{Homeo}^+(S^1)^3 \to \mathbb{Z}$ defined by

$$c_x(g_0,g_1,g_2) := \begin{cases} 1 & \text{if } (g_0x,g_1x,g_2x) \text{ is positively oriented} \\ -1 & \text{if } (g_0x,g_1x,g_2x) \text{ is positively oriented} \\ 0 & \text{otherwise} \end{cases}$$

is a cocycle.

(2) Show that if $y \in S^1$, the cocycles c_x and c_y are cohomologous.

(3) Show that $[c] = -2\mathrm{e}_{\mathrm{b}}$.

EXERCISE 6. Let G be a locally compact group and (X,μ) a G-space with a quasi-invariant measure such that G acts transitively on n-tuples of points in X. Let $a, b : X^{n+1} \to \mathbb{R}$ be G-invariant cocycles such that

$$a|_{X^{n+1}\setminus X^{(n+1)}} = b|_{X^{n+1}\setminus X^{(n+1)}} = 0\,,$$

where $X^{(n+1)}$ is the subset of X^{n+1} consisting of $(n+1)$-tuples with pairwise distinct elements. Show that if $a = b$ almost everywhere, then $a = b$ everywhere.

EXERCISE 7. Show that if G is amenable then $\mathrm{H}^n_{\mathrm{cb}}(G,\mathbb{R}) = 0$ for all $n \geq 1$.